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DIFFERENT VOLUME COMPUTATIONAL METHODS OF GRAPH POLYTOPES

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Abstract. The aim of this work is to introduce several different volume computational methods of graph polytopes associated with various types of finite simple graphs. Among them, we obtained the recursive volume formula (RVF) that is fundamental and most useful to compute the volume of the graph polytope for an arbitrary finite simple graph.

1. Introduction

Bóna, Ju and Yoshida [2] enumerated certain weighted graphs with the following conditions: For a given positive integer k, a nonnegative integer n and a simple graph $G = (VG, EG)$ with $VG = [n]$, where $[n] := \{1, 2, \ldots, n\}$ and $[n]_* := [n] \cup \{0\}$, we consider the set

$$W(k; G) := \{\alpha = (k_1, \ldots, k_n) \in ([k]_*)^n \mid ij \in EG \Rightarrow k_i + k_j \leq k\}.$$

An element in $W(k; G)$ is called a (*vertex-*) *weighted graph*. In fact, the number of weighted graphs is given by the Ehrhart polynomial of some convex polytope in a unit n-hypercube. Such a convex polytope is determined uniquely by the given finite simple graph as follows: Let $G = (VG, EG)$ be a simple graph with $VG = [n]$. Then the *graph polytope* $P(G)$ associated with the graph G is defined as

$$P(G) := \{(x_1, x_2, \ldots, x_n) \in [0, 1]^n \mid ij \in EG \Rightarrow x_i + x_j \leq 1\}.$$

Our main concerns in this article are the computational results on the volumes of graph polytopes associated with many types of graphs using several different methods. In order to obtain the volume of a graph polytope we need a certain kernel function $K : [0, 1]^2 \to \mathbb{R}$ defined by the following:

$$K(s, t) := \begin{cases} 1, & s + t \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

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Then the volume $vol(P(G))$ of the polytope $P(G)$ is

$$vol(P(G)) = \int_{Q_n} \prod_{ij \in EG} K(x_i, x_j)\ dx_1 dx_2 \cdots dx_n,$$

where $Q_n = [0,1]^n$ is the n-dimensional unit hypercube. For simplicity, we use the notion $dx = dx_1 dx_2 \cdots dx_n$ if there is no confusion. The volume of a graph polytope $P(G)$ will be denoted by $vol(G)$ rather than $vol(P(G))$. From now on, all graphs we mention are finite simple graphs and all polytopes are convex.

This paper is organized as follows: Section 2 introduces a recursive volume formula for the volume of a graph polytope and volume formulae for graph polytopes associated with various types of graphs. Section 3 describes the graph joins and the corresponding volume formula. The volume of the graph polytope associated with a bipartite graph with certain symmetry is dealt in Section 4. In Section 5 we use the operator theory to find values for interesting series. In the last section we mention another way to compute the volumes of graph polytopes, which uses the Ehrhart polynomial and series of a graph polytope.

2. Recursive volume formula

The recursive volume formula (RVF for short), which will be introduced in Theorem 2.3, is a fundamental technique since it is effectively used in the volume computation of the graph polytope corresponding various graphs, as shown in the succeeding examples. It is, however, NP-hard problem on the number of vertices, like TSP algorithm, from the point of view of computational complexity.

Next two lemmas will be used to prove the RVF.

Lemma 2.1 (**polytope partitioning**). *Let P be an n-dimensional polytope in $\mathbb{R}^n$ containing a point x. For a facet F of P, let $d(x,F)$ be the shortest distance from x to a point in the affine span of F and $vol(F)$ be the $(n-1)$-dimensional volume of F. Then*

$$vol(G) = \frac{1}{n} \sum_F d(x,F) vol(F),$$

where the sum runs over all facets F of $P(G)$.

Proof. Since P can be decomposed as a union of $\mathrm{conv}\{x, F\}$ where the union is over all facets F of $P(G)$, the result follows. □

Lemma 2.2. *If a graph G has no isolated vertex, then the graph polytope $P(G)$ has no facet of the form $x_i = 1$. In other words, $P(G)$ is only composed of facets of form $x_k = 0$ or $x_i + x_j = 1$ for $ij \in E$.*

Proof. Since G has no isolated vertex, each vertex has at least one adjacent vertex. The result follows from the definition of the graph polytope. □

Theorem 2.3 (RVF). *Let $G = (VG, EG)$ be a graph with the vertex set $VG = [n]$ and having no isolated vertex. Then*

$$vol(G) = \frac{1}{2n} \sum_{i=1}^{n} vol(G - i),$$

where $G - i$ is the graph with the vertex set $[n] \setminus \{i\}$ and, accordingly, with the inherited edge set in the original edge set EG.

Proof. Let $x = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in P(G)$. Then $d(x, F) = \frac{1}{2}$ for the facet F in the hyperplane $x_i = 0$ (hence $F = P(G-i)$ in this hyperplane) and $d(x, F) = 0$ for all other facets F since $x \in F$. Note that $vol(G - i)$ is the $(n-1)$-dimensional volume of the graph polytope $P(G - i)$. By Lemmas 2.1 and 2.2 we have the desired formula. ☐

The RVF can be used to obtain the volume formulas for several classes of graphs. We provide them with ideas of the proof. For detail, see [5].

Corollary 2.4. *Let $L_n = ([n], EL_n)$ be the path with $EL_n := \{i(i+1) \mid i \in [n-1]\}$. Then*

$$vol(L_n) = \frac{E_n}{n!},$$

where E_n is the n-th Euler number. Hence its generating function is

$$\sum_{n \geq 0} vol(L_n) x^n = \sec x + \tan x.$$

Proof. Since $L_n - i$ is a path or a disjoint union of two paths, the RVF can be used to see that $n! vol(L_n)$ satisfies the same recurrence relation as E_n. ☐

Corollary 2.5. *Let $C_n = ([n], EC_n)$ be the cycle with $EC_n := \{i(i+1) \mid i \in [n]\}$ where $n + 1 := 1$. Then*

$$vol(C_n) = \frac{1}{2} \frac{E_{n-1}}{(n-1)!}.$$

Hence its generating function is

$$\sum_{n \geq 1} vol(C_n) x^n = \frac{x(\sec x + \tan x)}{2}.$$

Proof. Removing a vertex in the cycle C_n results in a path L_{n-1}. Hence RVF and Corollary 2.4 imply both of conclusions. ☐

Corollary 2.6. *Let $K_n = ([n], EK_n)$ be the complete graph with the edge set $EK_n := \{ij \mid i, j \in [n]\}$. Then*

$$vol(K_n) = 2^{1-n}.$$

Hence its generating function is

$$\sum_{n \geq 1} vol(K_n) x^n = \frac{2x}{2 - x}.$$

Proof. By RVF, $vol(K_n) = \frac{n \cdot vol(K_{n-1})}{2n} = \frac{vol(K_{n-1})}{2}$. Since $vol(K_1) = 1$, the conclusion follows easily. □

Corollary 2.7. *Let $K_{s,t} = ([s+t], EK_{s,t})$ be the complete bipartite graph with $EK_{s,t} := \{ij \mid i \in [s], j \in \{s+1, s+2, \ldots, s+t\}\}$. Then*

$$vol(K_{s,t}) = \frac{1}{\binom{s+t}{s}}. \tag{1}$$

Proof. Since $vol(K_{s,0}) = 0 = vol(K_{0,t})$ and

$$vol(K_{s,t}) = \frac{s \cdot vol(K_{s-1,t}) + t \cdot vol(K_{s,t-1})}{2(s+t)},$$

we have the required formula by induction. □

3. Volumes of the graph joins

In this section, we calculate the volume of the graph polytope of the join of graphs. The *join* of graphs H and K is the graph $G = (VG, EG)$ where $VG = VH \cup VK$ and $EG = EH \cup EK \cup \{xy \mid x \in VH, y \in VK\}$. We denote the join of graphs H and K simply by $H + K$. We also provide another way to compute the volumes of the graph polytopes for complete graphs and complete bipartite graphs.

Definition (sliced volume)**.** Let $G = ([n], EG)$ be a graph and $r \in [0,1]$. The *sliced volume* $vol(G, r)$ is defined by

$$vol(G, r) := \int_{[0,r]^n} \prod_{ij \in EG} K(x_i, x_j) dx,$$

where $K(\cdot, \cdot)$ is defined as

$$K(s,t) := \begin{cases} 1, & s + t \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

It is obvious that $vol(G) = vol(G, 1)$. Note that $vol(G, r) = r^n$ if $0 \leq r \leq \frac{1}{2}$.

Lemma 3.1. *Let $f : [0,c]^n \to \mathbb{R}$ and $g : [0,c]^m \to \mathbb{R}$ be continuous functions such that $F(s) = \int_{[0,s]^n} f(x_1, \ldots, x_n) dx$ and $G(s) = \int_{[0,s]^m} g(y_1, \ldots, y_m) dy$ are continuously differentiable for any $0 \leq s \leq c$. For any continuous function $k : [0,c]^2 \to \mathbb{R}$, we have*

$$\iint_{[0,c]^{n+m}} f(x_1, \ldots, x_n) g(y_1, \ldots, y_m) k(\max(x_1, \ldots, x_n), \max(y_1, \ldots, y_m)) dx dy$$
$$= \int_0^c \int_0^c \frac{d}{ds} F(s) \frac{d}{dt} G(t) k(s,t) ds dt.$$

Proof. For each $N \in \mathbb{N}$, get partitions of interval $0 = s_0 < s_1 < \cdots < s_N = c$ and $0 = t_0 < t_1 < \cdots < t_N = c$ such that

$$\lim_{N \to \infty} \max\{s_i - s_{i-1} \mid 1 \leq i \leq N\} = \lim_{N \to \infty} \max\{t_i - t_{i-1} \mid 1 \leq i \leq N\} = 0,$$

and sample points $x_i^* \in [0, s_i]^n \setminus [0, s_{i-1}]^n, y_i^* \in [0, t_i]^m \setminus [0, t_{i-1}]^m (1 \leq i \leq n)$. We can divide the sum as follows:

$$I = \sum_{i=1}^{N} \sum_{j=1}^{N} \iint_{([0,s_i]^n \setminus [0,s_{i-1}]^n) \times ([0,t_j]^m \setminus [0,t_{j-1}]^m)} f(x)g(y)k(\|x\|_\infty, \|y\|_\infty) dxdy.$$

Let $\delta_N = \max\{s_i - s_{i-1} \,|\, 1 \leq i \leq N\}$ and $\delta'_N = \max\{t_i - t_{i-1} \,|\, 1 \leq i \leq N\}$. Note that $|\,\|x_i^*\|_\infty - x| < \sqrt{n}\delta_N$ for all $x \in [0, s_i]^n \setminus [0, s_{i-1}]^n$ and $|\,\|y_j^*\|_\infty - y| < \sqrt{m}\delta'_N$ for all $y \in [0, t_j]^m \setminus [0, t_{j-1}]^m$. By the uniform continuity of the continuous function k, for any $\epsilon > 0$, we can take N large enough so that $|x - x'| < \sqrt{n}\delta_N$ and $|y - y'| < \sqrt{m}\delta'_N$ implies $|k(x, y) - k(x', y')| < \epsilon$. Let $M = \int\int_{[0,c]^n \times [0,c]^m} |f(x)||g(y)| dxdy$. Then we get

$$|I - \sum_{i=1}^{N} \sum_{j=1}^{N} k(\|x_i^*\|_\infty, \|y_j^*\|_\infty)(F(s_i) - F(s_{i-1})(G(t_j) - G(t_{j-1}))| < \epsilon c^{n+m} M.$$

By the mean value theorem for F and G, we have

$$|I - \sum_{i=1}^{N} \sum_{j=1}^{N} k(\|x_i^*\|_\infty, \|y_j^*\|_\infty) \frac{d}{ds} F(s_i^*) \frac{d}{dt} G(t_j^*)| < \epsilon c^{n+m} M.$$

Since we assumed $\frac{dF}{ds}$ and $\frac{dG}{dt}$ to be continuous, $M' = \int_0^c \int_0^c |\frac{d}{ds} F(s) \frac{d}{dt} G(t)| dsdt$ exists. Note that $\|x_i^*\|_\infty \in [s_{i-1}, s_i]$ and $\|y_j^*\|_\infty \in [t_{j-1}, t_j]$. Applying the uniform continuity of k again, we have

$$|I - \sum_{i=1}^{N} \sum_{j=1}^{N} k(s_i^*, t_j^*) \frac{d}{ds} F(s_i^*) \frac{d}{dt} G(t_j^*)| < \epsilon c^{n+m} (M + M').$$

Taking the limit $N \to \infty$, we have

$$|I - \int_0^c \int_0^c \frac{d}{ds} F(s) \frac{d}{dt} G(t) k(s, t) dsdt| < \epsilon c^{n+m} (M + M')$$

for arbitrary $\epsilon > 0$, which concludes the proof. □

Lemma 3.2. *For $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m \in [0, 1]$, we have*

$$\prod_{i=1}^{n} \prod_{j=1}^{m} K(x_i, y_j) = K(\max\{x_i | 1 \leq i \leq n\}, \max\{y_j | 1 \leq j \leq m\}).$$

Proof. Divide the cases.

- When there exist $1 \leq k \leq n$ and $1 \leq l \leq m$ such that $x_k + y_l > 1$:
  The left hand side becomes 0 since $K(x_k, y_l) = 0$ have been multiplied. Since $\max\{x_i\} + \max\{y_j\} \geq x_k + y_l > 1$, the right hand side is also 0.
- Otherwise:
  The left hand side is 1 since $x_i + y_j \leq 1$ for all i, j. On the other hand, the right hand side is 1 since there exists k, l such that $x_k = \max\{x_i\}$, $y_l = \max\{y_j\}$. □

The next theorem gives us a volume formula for the graph polytope associated with the joined graph.

Theorem 3.3. *For any graphs G and H, the sliced volume of the graph polytope associated with the graph $G+H$ is*

$$vol(G+H,r) = \int_0^r \int_0^r \frac{d}{ds}vol(G,s)\frac{d}{dt}vol(H,t)K(s,t)dsdt.$$

Proof. The result follows simply from Lemmas 3.1 and 3.2. □

Theorem 3.4. *Let $\frac{1}{2} \le r \le 1$. Then the sliced volume of the graph polytope $P(K_{m,n})$ is given by the formula*

$$vol(P(K_{m,n}),r) = r^n(1-r)^m + m\sum_{i=0}^{i}\binom{n}{i}(-1)^n\frac{r^{m+i}-(1-r)^{m+i}}{m+i}.$$

Proof. Note that $K_{m,n} = \overline{K}_m + \overline{K}_n$, where $\overline{K}_n$ is the null graph with n vertices. Since $vol(\overline{K}_m, r) = r^m$,

$$\begin{aligned}
vol(K_{m,n},r) &= \int_0^r \int_0^r \Big(\frac{d}{ds}vol(\overline{K}_m,s)\Big)\Big(\frac{d}{dt}vol(\overline{K}_n,t)\Big)K(s,t)dsdt \\
&= \int_0^r \int_0^r \frac{ds^m}{ds}\frac{dt^n}{dt}K(s,t)dsdt \\
&= \int_0^r \Big(\frac{ds^m}{ds}\int_0^r K(s,t)\frac{dt^n}{dt}dt\Big)ds \\
&= \int_0^r \Big(\frac{ds^m}{ds}\int_0^{\min(1-s,r)} \frac{dt^n}{dt}dt\Big)ds \\
&= \int_0^r \Big(\frac{ds^m}{ds}(\min(1-s,r))^n\Big)ds \\
&= \int_0^{1-r} r^n\frac{ds^m}{ds}ds + \int_{1-r}^r (1-s)^n\frac{ds^m}{ds}ds \\
&= r^n(1-r)^m + m\sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{r^{m+i}-(1-r)^{m+i}}{m+i}.
\end{aligned}$$

□

Corollary 3.5. *The volume of the graph polytope associated with the complete bipartite graph $K_{m,n}$ is*

$$vol(K_{m,n}) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{m}{m+i}. \tag{2}$$

Proof. Substitute $r=1$ in Theorem 3.4. □

The following result is immediate, comparing Corollaries 2.7 and 3.5.

Corollary 3.6. *For positive integers m and n,*

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{m}{m+i} = \frac{1}{\binom{m+n}{n}} = \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{n}{n+i}.$$

Remark 3.7. According to Rudin [6], the *beta function* $B(r,s)$ is defined as

$$B(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1}dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)},$$

where $\Gamma(x)$ is the gamma function. Note that

$$vol(K_{m,n}) = mB(m, n+1) = nB(m+1, n)$$

from the formula (1) and the definition of the beta function involving the gamma function.

Since the associative law holds for joins, one can define $kG := G+G+\cdots+G$ (add k times).

Theorem 3.8 (multiple join of a graph)**.** *Let G be a graph with n vertices. Then, for any positive integer k and $\frac{1}{2} \le r \le 1$,*

$$\frac{d}{dr} vol(kG, r) = k(1-r)^{n(k-1)} \frac{d}{dr} vol(G, r).$$

Proof. We have

$$\begin{aligned} vol(kG, r) &= \int_0^r \int_0^r \Big(\frac{d}{ds} vol(G,s)\Big)\Big(\frac{d}{dt} vol((k-1)G, t)\Big) K(s,t)dsdt \\ &= \int_0^r \frac{d}{ds} vol(G,s)\Big(\int_0^{\min(1-s,r)} \frac{d}{dt} vol((k-1)G, t)dt\Big)ds \\ &= \int_0^r \frac{d}{ds}\Big[vol(G,s) \cdot vol((k-1)G, \min(1-s,r))\Big]ds \\ &= (i) + (ii) + (iii), \end{aligned}$$

where

$$\begin{aligned} (i) &:= \int_0^{1-r} vol((k-1)G, r)\frac{d}{ds} vol(G,s)ds \\ &= vol((k-1)G, r)(1-r)^n \qquad \left(0 \le 1-r \le \frac{1}{2}\right), \\ (ii) &:= \int_{1-r}^{1/2} vol((k-1)G, 1-s)\frac{d}{ds} vol(G,s)ds \\ &= \int_{1-r}^{1/2} vol((k-1)G, 1-s)|VG|s^{|VG|-1}ds, \\ (iii) &:= \int_{1/2}^{r} vol((k-1)G, 1-s)\frac{d}{ds} vol(G,s)ds \end{aligned}$$

$$= \int_{1/2}^{r} (1-s)^{(k-1)|VG|} \frac{d}{ds} vol(G,s) ds.$$

Now, we denote $\frac{d}{dr} vol(kG, r)$ simply by a_k. Then

$$\begin{aligned} a_k &= \frac{d}{dr}((i) + (ii) + (iii)) \\ &= (1-r)^n a_{k-1} - n(1-r)^{n-1} vol((k-1)G, r) \\ &\qquad + n(1-r)^{n-1} vol((k-1)G, r) + a_1 (1-r)^{(k-1)n} \\ &= (1-r)^n a_{k-1} + (1-r)^{(k-1)n} a_1. \end{aligned}$$

Let $F(x, r) = \sum_{k \geq 1} a_k x^k$. Then, from the previous recursion formula we get

$$F(x,r) = \frac{x a_1}{(1-(1-r)^n x)^2} = \sum_{k \geq 1} k a_1 (1-r)^{(k-1)n} x^k.$$

Hence

$$\frac{d}{dr} vol(kG, r) = k(1-r)^{(k-1)n} \frac{d}{dr} vol(G, r). \qquad \square$$

Corollary 3.9. *For the value* $\frac{1}{2} \leq r \leq 1$*, we have*

$$vol(K_n, r) = 2^{1-n} - (1-r)^n \text{ and } vol(K_n) = 2^{1-n}.$$

Proof. Since $K_n = nK_1$, the results follow from Theorem 3.8. $\square$

The *Turán graph* $T(nk, k)$ is a complete multipartite graph formed by partitioning a set of nk vertices into k subsets of size n and connecting two vertices by an edge if and only if they belong to different subsets. The following theorem provides the volume of the graph polytope for a Turán graph $T(nk, k)$.

Corollary 3.10. *For the value* $\frac{1}{2} \leq r \leq 1$*, we have*

$$vol(T(nk,k)) = 2^{-nk} + k 2^{-nk} \frac{1}{\binom{nk}{n}} \sum_{i=0}^{n-1} \binom{nk}{i}.$$

Proof. Since $T(nk, k) = k\overline{K}_n$, Theorem 3.8 implies

$$\frac{d}{dr} vol(T(nk,k), r) = k(1-r)^{n(k-1)} \frac{d}{dr} vol(\overline{K}_n, r) = nkr^{n-1}(1-r)^{n(k-1)}.$$

Thus

$$\begin{aligned} vol(T(nk,k)) &= 2^{-nk} + nk \int_{1/2}^{1} r^{n-1} (1-r)^{n(k-1)} dr \\ &= 2^{-nk} + nk \int_{1}^{0} \Big(-\frac{1}{2}\Big) \Big(1 - \frac{t}{2}\Big)^{n-1} \Big(\frac{t}{2}\Big)^{n(k-1)} dt \\ &= 2^{-nk} + nk 2^{-nk} \int_{0}^{1} t^{n(k-1)} (2-t)^{n-1} dt \end{aligned}$$

$$
\begin{aligned}
&= 2^{-nk} + nk2^{-nk} \sum_{i=0}^{n-1} \binom{n-1}{i} \int_0^1 t^{n(k-1)}(1-t)^i dt \\
&= 2^{-nk} + nk2^{-nk} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{(n(k-1))! \cdot i!}{(n(k-1)+i+1)!} \\
&= 2^{-nk} + nk2^{-nk} \sum_{i=0}^{n-1} \frac{1}{n\binom{nk}{n}} \binom{nk}{n-1-i} \\
&= 2^{-nk} + k2^{-nk} \frac{1}{\binom{nk}{n}} \sum_{i=0}^{n-1} \binom{nk}{i}.
\end{aligned}
$$

□

4. Volume of bipartite graphs

In this section, we decompose the unit cube using permutations to obtain the volume of the graph polytope associated with a bipartite graph. Let $\mathfrak{S}_n$ be the set of all permutations of $[n]$. We use the one-line notation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ for a permutation $\sigma \in \mathfrak{S}_n$ defined by $\sigma(i) = \sigma_i$.

Definition. For a permutation $\sigma \in \mathfrak{S}_n$, let

$$[0,1]^n_\sigma := \{x = (x_1, x_2, \ldots, x_n) \in [0,1]^n | x_{\sigma_1} \le x_{\sigma_2} \le \cdots \le x_{\sigma_n}\}.$$

Note that

$$[0,1]^n = \bigcup_{\sigma \in \mathfrak{S}_n} [0,1]^n_\sigma$$

and each intersection of two distinct n-simplices $[0,1]^n_\sigma$ has measure 0 so that for any measurable function f,

$$\int_{[0,1]^n} f dx = \sum_{\sigma \in \mathfrak{S}_n} \int_{[0,1]^n_\sigma} f dx$$

and

$$\int_{[0,1]^n_\sigma} f dx = \int_0^1 \Big(\int_0^{x_{\sigma_n}} \Big(\int_0^{x_{\sigma_{n-1}}} \cdots \Big(\int_0^{x_{\sigma_2}} f dx_{\sigma_1} \Big) \cdots dx_{\sigma_{n-2}} \Big) dx_{\sigma_{n-1}} \Big) dx_{\sigma_n}$$

by Fubini's theorem.

Let $B = (VB, EB)$ be a bipartite graph with $VB = V_1 \sqcup V_2$, where $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{v_1, v_2, \ldots, v_m\}$, and let $N_i := \{j \in V_1 \,|\, jv_i \in EB\}$.

Theorem 4.1. *The volume of the graph polytope associated with the bipartite graph B mentioned above is*

$$vol(B) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^n \frac{1}{i + \sum_{j=1}^i \alpha_{j,\sigma}},$$

where $\alpha_{i,\sigma}$ is the number of vertices in

$$\{v_k \in V_2 \,|\, \sigma(i) \in N_k\} \setminus (\cup_{j=1}^{i-1} \{v_k \in V_2 \,|\, \sigma(j) \in N_k\}),$$

which means the number of vertices in V_2 where the smallest among σ^{-1} values of its neighbors is i.

Proof. We have

$$\begin{aligned}
vol(B) &= \int_{[0,1]^n} \prod_{j=1}^{m} (1-\max\{x_i \mid i \in N_j\}) dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_{[0,1]^n_\sigma} \prod_{j=1}^{m} (1-\max\{x_i \mid i \in N_j\}) dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_{[0,1]^n_\sigma} \prod_{j=1}^{m} (\min\{1-x_i \mid i \in N_j\}) dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_{[0,1]^n_{rev(\sigma)}} \prod_{j=1}^{m} \min\{x_i \mid i \in N_j\} dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_{[0,1]^n_\sigma} \prod_{j=1}^{m} \min\{x_i \mid i \in N_j\} dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_{[0,1]^n_\sigma} \prod_{i=1}^{n} x_{\sigma(i)}^{\alpha_{i,\sigma}} dx \\
&= \sum_{\sigma\in\mathfrak{S}_n} \int_0^1 (x_{\sigma_n}^{\alpha_{n,\sigma}} \int_0^{x_{\sigma_n}} [x_{\sigma_{n-1}}^{\alpha_{n-1,\sigma}} \cdots \\
&\qquad \int_0^{x_{\sigma_3}} (x_{\sigma_2}^{\alpha_{2,\sigma}} \int_0^{x_{\sigma_2}} (x_{\sigma_1}^{\alpha_{1,\sigma}}) dx_{\sigma_1}) dx_{\sigma_2} \cdots] dx_{\sigma_{n-1}}) dx_{\sigma_n} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} \frac{1}{i+\sum_{j=1}^{i} \alpha_{j,\sigma}},
\end{aligned}$$

where $rev(\sigma) = (\sigma_n, \sigma_{n-1}, \ldots, \sigma_2, \sigma_1)$. □

An *automorphism* of a simple graph $G = (VG, EG)$ is a permutation π of VG which has the property that uv is an edge of G if and only if $\pi(u)\pi(v)$ is an edge of G.

Theorem 4.2. *Let $B = (VB, EB)$ be a bipartite graph with $VB = V_1 \sqcup V_2$. Suppose that for any permutation π on V_1, there exists a permutation σ on V_2 such that the combination of π and σ induces an automorphism on G. Then*

$$vol(P(B)) = n! \prod_{i=1}^{n} \frac{1}{i+\sum_{j=1}^{i} \alpha_j},$$

where $\alpha_j = \alpha_{j,\sigma}$ when σ is the identity.

Proof. The symmetry of the graph B implies that all $\alpha_{i,\sigma}$'s are the same for different $\sigma \in \mathfrak{S}_n$. The conclusion follows from Theorem 4.1. □

Corollary 4.3. *Let B_n be the graph that is obtained from the complete bipartite graph $K_{n,n}$ by deleting n disjoint edges. Then,*

$$vol(B_n) = \left(1 + \frac{1}{n}\right) \frac{1}{\binom{2n}{n}}.$$

Proof. Since $\alpha_1 = n - 1, \alpha_2 = 1$, and $\alpha_i = 0$ for $i \geq 3$, the result follows from Theorem 4.2. □

Example 4.4. In particular, $vol(B_3) = \frac{1}{15}$. Note that the bipartite graph B_3 is the graph obtained from the 1-skeleton of the 3-cube.

5. An application related to the operator theory

We introduce here another interesting fact that uses the linear operator theory to obtain the value of a series described in the theorem below. One of the results related with the operator theory is the computation of the $vol(C_n)$, which is referred from Elkies [3]. We will restate a lemma regarding this.

We define $\mathcal{K}_n$ inductively as in the following:

$$\mathcal{K}_1(t, s) := K(t, s),$$

and

$$\mathcal{K}_n(t, s) := \int_0^1 \mathcal{K}_1(t, x)\mathcal{K}_{n-1}(x, s)dx \qquad (n \geq 2).$$

Let $T : L^2(0, 1) \to L^2(0, 1)$ be the linear operator with the kernel $\mathcal{K}_1(\cdot, \cdot)$ on $L^2(0, 1)$ defined by

$$(Tg)(t) = \int_0^1 \mathcal{K}_1(t, s)g(s)ds = \int_0^{1-t} g(s)ds. \tag{3}$$

From the definition of $\mathcal{K}_n$ we see that $\mathcal{K}_n(\cdot, \cdot)$ is the kernel function of the linear operator T^n as follows:

$$(T^n g)(t) = \int_0^1 \mathcal{K}_n(t, s)g(s)ds. \tag{4}$$

The next lemma gives the spectral decomposition of the linear operator T, and also of T^n. Its proof is immediate from the standard linear operator theory. (See Elkies [3] or Hutson et al. [4].)

Lemma 5.1. *The linear operator T is compact and self-adjoint on $L^2(0, 1)$. Its eigenvalues are $\frac{2}{\pi(4k+1)}(k \in \mathbf{Z})$ and the corresponding eigenfunctions are $\cos(\pi(4k+1)/2)$. Moreover, The linear operator T^n is compact and self-adjoint on $L^2(0, 1)$. Its eigenvalues are $(\frac{2}{\pi(4k+1)})^n$ with same corresponding eigenfunctions $\cos(\pi(4k + 1)/2)$. Each of the eigenvalues for T and T^n is simple.*

Our main goal here is to find the value of certain formula using the operator theory. In fact, it is the $vol(P(C_n))$ which is obtained from the RVF. By the simple calculations we can get the following formula from the definition of $\mathcal{K}_n$ (see [3]):

$$vol(C_n) = \int_0^1 \mathcal{K}_n(t,t)dt. \tag{5}$$

It turns out that the right hand side of the formula (5) is the trace of a trace-class operator T^n over the diagonal, and is equal to

$$\sum_{k=-\infty}^{\infty} \frac{2^n}{(\pi(4k+1))^n}.$$

Note that this series is absolutely convergent for $n \geq 2$. As a summary we have the following theorem.

Theorem 5.2. *For any integer $n \geq 2$, the following holds*:

$$\sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} = \frac{\pi^n vol(C^n)}{2^n}.$$

Example 5.3. For the case $n = 3$,

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots + \frac{(-1)^m}{(2m+1)^3} + \cdots = \frac{\pi^3 vol(C^3)}{8} = \frac{\pi^3 vol(K_3)}{8} = \frac{\pi^3}{32},$$

meanwhile, for the case $n = 4$,

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots + \frac{1}{(2m+1)^4} + \cdots = \frac{\pi^4 vol(C^4)}{16} = \frac{\pi^4 vol(K_{2,2})}{16} = \frac{\pi^4}{96}.$$

6. Concluding remarks

In fact, we have another volume computational method which comes from the Ehrhart polynomial of $P(G)$. Let P be an integral convex polytope in $\mathbb{R}^d$. Then we call $L_P(t) = |tP \cap \mathbb{Z}^d|$ the *Ehrhart polynomial* of P. A 0/1-*polytope* is the convex hull of a certain subset of the vertices of the regular cube $C^d = [0,1]^d$. It is known that, for a given 0/1-polytope P,

$$vol(P) = \lim_{t\to\infty} \frac{L_P(t)}{t^d}, \text{ where } d = \dim(P),$$

or

$$\frac{f(1)}{d!}, \text{ where } \sum_{t=0}^{\infty} L_P(t)x^t = \frac{f(x)}{(1-x)^{d+1}}.$$

(Refer [1] or [7] about this.) If G is a bipartite graph with n vertices, then its graph polytope $P(G)$ is a 0/1-polytope of dimension n. Hence, we can get the volume $vol(G)$ from the Ehrhart polynomial $L_{P(G)}(t)$, which we can get by using divided difference technique. (See Bóna et al. [2] for details.)

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