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THE RANGE OF r-MAXIMUM INDEX OF GRAPHS

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Abstract. For a connected graph G, an r-maximum edge-coloring is an edge-coloring f defined on $E(G)$ such that at every vertex v with $d_G(v) \geq r$ exactly r incident edges to v receive the maximum color. The r-maximum index $\chi'_r(G)$ is the least number of required colors to have an r-maximum edge coloring of G. In this paper, we show how the r-maximum index is affected by adding an edge or a vertex. As a main result, we show that for each $r \geq 3$ the r-maximum index function over the graphs admitting an r-maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r,k)$ with $\chi'_r(G(r,k)) = k$. Also, we construct a family of graphs not admitting an r-maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r+1\}$, where for each $k \geq r+1$ there is no r-maximum edge-coloring of G_k and $\Delta(G_k) = k$.

1. Introduction

In this paper we consider only simple graphs. The *edge-connectivity* of a connected graph G is the least integer k such that there exist k edges whose deletion increases the number of components of the remaining graph. We denote the edge-connectivity of G by $\kappa'(G)$. In particular, we call an edge a *cut-edge* of a nontrivial connected graph G if $G - e$ is not connected. A *k-factor* of a graph G is a k-regular spanning subgraph of G. Given an edge-coloring f of G, not necessarily proper, a *maximum color* of a vertex $v \in V(G)$ is $\max_{e \in E_v(G)} f(e)$, where $E_v(G)$ is the set of incident edges to v. A color c *appears l times at v under f* if exactly l incident edges to v receive the color c.

For hypergraphs a vertex coloring is called a *unique-maximum coloring* if the maximum color in each edge appears exactly once on the vertices of the edge. There are results on unique-maximum coloring of hypergraphs in [1]. For a graph this coloring is just a proper coloring of the vertices. For a planar graph G, we can define an associated hypergraph with $V(G)$ as the vertex set and the

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faces as hyperedges. This unique-maximum coloring is called *unique-maximum coloring of a planar graph with respect to faces.* In other words, it is a vertex coloring such that for every face the maximum color occurs exactly once on the vertices of the face. The *unique-maximum index* of a planar graph G is the minimum number of colors for G to have a unique-maximum coloring. The parameter is studied in [2, 3].

Without a restriction on planar graphs, a similar concept on edges can be applied: a *unique-maximum edge-coloring* of a graph is an edge-coloring such that at every vertex the maximum color appears exactly once [4]. In [4] S. Jendrol' and Vrbjarová generalized the concept by fixing the number of maximum colors appearing at each vertex. For any natural number r, we say that a graph G has an r-maximum edge-coloring if there is an edge-coloring f defined on $E(G)$ such that at every vertex v with $d_G(v) \geq r$ the maximum color appears r times at v under f. If $\Delta(G)$ is smaller than r, then r-maximum coloring problems become trivial by using one color for all the edges. In fact, every edge-coloring is an r-maximum edge-coloring. On the other hand, if $\Delta(G)$ is larger than r, properties on r-maximum coloring reflect various aspects of structure of graphs. For example, the existence of an r-factor of a graph implies the existence of an r-maximum coloring of the graph. Note that the converse is not true. (See also [6], [7].)

The *r-maximum index* of a graph G, denoted by $\chi'_r(G)$, is the smallest number of required colors to have an r-maximum edge coloring of G. A natural question on r-maximum index is what the possible numbers for $\chi'_r(G)$ will be. When $r \leq 2$, every graph admits an r-maximum edge coloring. Moreover, $\chi'_r(G) \leq 3$ for $r = 1, 2$.

Theorem 1.1 ([4])**.** *For $r = 1, 2$, $\chi'_r(G) \leq 3$ for all connected nontrivial graph G.*

In addition, they characterized graphs according to the value of $\chi'_r(G) = i$, $i = 1, 2, 3$. On the other hand, not every graph has an r-maximum edge-coloring when $r \geq 3$. For $r \geq 3$ they expected $\chi'_r(G) \leq 3$ for every graph G admitting an r-maximum edge-coloring.

As answering this question, we show that for each $r \geq 3$ the r-maximum index function over the graphs admitting an r-maximum edge-coloring is unbounded and the range is the set of natural numbers. In other words, for each $r \geq 3$ and $k \geq 1$ there is a family of graphs $G(r, k)$ with $\chi'_r(G(r, k)) = k$. Also, we construct a family of graphs not admitting an r-maximum edge-coloring with arbitrary maximum degrees: for any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r + 1\}$, where for each $k \geq r + 1$ there is no r-maximum edge-coloring of G_k and $\Delta(G_k) = k$.

We make a remark that there is a similar name for the parameter called a *maximum edge coloring* or known as the Maximum Edge Coloring (MEC) problem. This maximum edge coloring is a generalization of a proper edge coloring. (For this topic, see [5].) The r-maximum edge-coloring is not related to a proper edge-coloring, and it is completely different from the maximum edge coloring. In this paper, we only discuss on r-maximum edge-coloring.

In the next section, we first explore effects of an r-maximum edge-coloring and the r-maximum index as adding an edge in many different ways.

2. Preliminaries

The maximum edge-coloring of a graph is related to r-regular subgraphs. It is trivial to see the following results.

Proposition 2.1. *If G has an r-factor, then $\chi'_r(G) \leq 2$.*

Proposition 2.2. *If G has an r-maximum edge-coloring f, then all the incident edges to a vertex of degree r must get the same color under f.*

Proposition 2.3. *Suppose that $\delta(G) \geq r$ and G has an r-maximum edge-coloring f. Let c be the maximum value of f. Then the induced subgraph by the edges receiving c is r-regular.*

The r-maximum index of a graph is not inherited from its supergraph. All four combinations are possible. For example, $\chi'_r(K_r) = 1$ but $\chi'_r(K_{2r}) = 2$. But there is a $2r$-vertex graph whose r-maximum index does not exist [4]. We provide sufficient conditions to make an r-maximum index as an invariant under some operations on a graph.

Proposition 2.4. *Given a graph H admitting an r-maximum edge-coloring with $\chi'_r(H) > 1$, let H' be a graph obtained by adding a new leaf to a vertex v with $d_H(v) > r$. Then $\chi'_r(H') \leq \chi'_r(H)$.*

Proof. For any given r-maximum edge-coloring of H, the number of colors that all incident edges to v received is at least two because $d_H(v) \geq r$. If we assign a smaller color than the maximum color at v to the newly added edge, then it is still an r-maximum edge-coloring. Therefore, $\chi'_r(H') \leq \chi'_r(H)$. □

By arguing similarly we obtain the following fact.

Proposition 2.5. *Given a graph H admitting an r-maximum edge-coloring with $\chi'_r(H) > 1$, let H' be a graph obtained by adding a new edge connecting two vertices u and v with $d_H(u), d_H(v) > r$. Then $\chi'_r(H') \leq \chi'_r(H)$.*

Proof. Since both of $d_H(u)$ and $d_H(v)$ are strictly greater than r, for each of u and v there is an incident edge that has a non-maximum color among the incident edges to the vertex. Let c_u and c_v be a non-maximum color at u and v, respectively. Without loss of generality we assume that $c_u \leq c_v$. We assign c_u for the color of the newly added edge. The edge-coloring of H' is now an r-maximum edge-coloring. Therefore, $\chi'_r(H') \leq \chi'_r(H)$. □

For example, note that the star $K_{m,1}$ with $m \geq r + 1$ has an r-maximum edge-coloring. For a graph G admitting an r-maximum edge-coloring assume that G has a vertex v of degree at least $r + 1$. Let u be the vertex of degree

m in $K_{m,1}$. We denote $G_1 + G_2$ the disjoint union of two graphs G_1 and G_2. If $G' = (G + K_{m,1}) \cup \{uv\}$, then $\chi'_r(G') \leq \max\{\chi'_r(G), 2\}$ by Proposition 2.5. We generalize this observation as follows.

Corollary 2.6. *For G_i with $i = 1, 2$ suppose that each graph G_i admits an r-maximum edge-coloring. Also suppose that $d_{G_1}(u) > r$ and $d_{G_2}(v) > r$ for some vertices $u \in V(G_1)$ and $v \in V(G_2)$. Let $G' = (G_1 + G_2) \cup \{uv\}$. Then $\chi'_r(G') \leq \max\{\chi'_r(G_1), \chi'_r(G_2)\}$.*

Theorem 2.7. *If a connected n-vertex graph G has only one vertex of degree larger than r which is not a cut-vertex and all the rest of the vertices have degree r, then G does not have an r-maximum edge-coloring.*

Proof. Suppose to the contrary that G has an r-maximum edge-coloring, say f. Let u be the vertex of maximum degree. Then $\chi'_r(G) > 1$. Otherwise, the unique color appears more than r times at u. Therefore, there are at least two edges e_1 and e_2 incident to u such that $f(e_1) \neq f(e_2)$. Let x be the end vertex of e_1 different from u and y be the end vertex of e_2 different from u. Since u is not a cut-vertex, $G - u$ is connected. Therefore, there is an x, y-path in $G - u$, say this path $xv_1v_2 \cdots v_ty$ for some $t \geq 0$. Then by applying Proposition 2.2 to f on G, the edge xv_1 must get the same color as $f(e_1)$, and we repeat this argument until we reach the edge v_ty. If $t = 0$, xy must get the same color as $f(e_1)$ and $f(e_2)$. Now we have a contradiction: $f(e_1) = f(v_ty) = f(e_2)$. □

There is a relation between the edge-connectivity and r on the maximum edge-coloring. If $r \leq \kappa'(G)$, then G admits an r-maximum edge-coloring if and only if G has an r-factor since $r \leq \kappa'(G) \leq \delta(G)$.

3. Main results

In this section we show that for each $r \geq 3$, $\chi'_r(G)$ is arbitrary.

It is easy to see that we can characterize all graphs admitting an r-maximum edge-coloring according to the values 1, 2, and at least 3 for r-maximum index in the following way.

Theorem 3.1. *Let G be a connected graph. Then*

(1) $\chi'_r(G) = 1$ *if and only if* $\Delta(G) \leq r$.
(2) $\chi'_r(G) = 2$ *if and only if* $\Delta(G) > r$ *and G has a spanning subgraph H such that* $d_H(v) = r$ *if* $d_G(v) \geq r$ *and* $d_H(u) = t$ *for some* $t < r$ *if* $d_G(v) < r$.
(3) *For G admitting an r-maximum edge-coloring,* $\chi'_r(G) \geq 3$ *if and only if G does not satisfies any of the conditions above.*

Note that in particular, if $\delta(G) \geq r$, a spanning subgraph H satisfying the condition in (2) in Theorem 3.1 is an r-factor of G.

It is trivial to see that $\chi'_r(G) = 1$ if $r \geq \Delta(G)$. However, $\chi'_r(G)$ may not exist if $r < \Delta(G)$. In fact, it is known that for every $r \geq 3$ there is a graph

$G(r)$ with $\Delta = 2r-2$ that has no r-maximum edge-coloring [4]. We show that there is no correlation between the maximum degree of a graph and existence of r-maximum edge-coloring. For any fixed $r \geq 3$ there are graphs with arbitrary maximum degree at least $r+1$ that do not admit an r-maximum edge-coloring.

Theorem 3.2. *For any fixed $r \geq 3$, there is an infinite family of graphs $\mathcal{F}_r = \{G_k : k \geq r+1\}$, where for each $k \geq r+1$ there is no r-maximum edge-coloring of G_k and $\Delta(G_k) = k$.*

Proof. Let H be an r-regular graph such that $\alpha'(H) \geq 2$ and $\kappa'(H) \geq 3$, where $\alpha'(H)$ is the size of a maximum matching in H. For any $r \geq 3$, such H exists, for example $H = K_{r+1}$. We pick two independent edges $e = xy$ and $e' = uv$. Let H_i be an i-th copy of H. In this case we rename the vertices and the edges by putting subscript i. For example, the corresponding vertex to x in H_i is denoted by x_i.

We construct G_k as follows:

(1) We take k copies of H, and take disjoint union of them: we start with $H_1 + H_2 + \cdots + H_k$.
(2) We delete the edges: $\{e_1, e_2, \ldots, e_k, e'_1, \ldots, e'_k\}$ from (1).
(3) Add a new vertex z to (2).
(4) Add new edges: $\{y_1x_2, y_2x_3, \ldots, y_{k-1}x_k, y_kx_1\}$ to (2).
(5) Add new edges: $\{u_iz, v_iz : i = 1, 2, \ldots, k\}$ to (2).

(Case 1) *r* is odd.

Let $G = G_{\frac{r+1}{2}}$. Then $\Delta(G) = r+1 = d_G(z)$. All the rest of the vertices have degree r.

Claim 1. G does not admit an r-maximum edge-coloring.

Proof. Suppose that there is an r-maximum edge-coloring of G, say f. Let $c = f(u_1z)$. Note that all the other incident edges to u_1 appear in H_1 and must receive the color c since the degree of u_1 in G is r. Moreover, by Proposition 2.2, all the incident edges to each vertex except for z must receive only one (maximum) color. For every $i = 1, 2, \ldots, \frac{r+1}{2}$, the graph $H_i - \{e_i, e'_i\}$ is connected since $\kappa'(H_i) \geq 3$. Therefore, for any vertex $w \in V(H_1) - \{u_1\}$, there is a w, u_1-path in $H_1 - \{e_1, e'_1\}$. Applying the same argument as in the proof of Theorem 2.7, we conclude that in G the edges from $H_1 - \{e_1, e'_1\}$ must receive the color c under f. In fact, the incident edges of a and the incident edges of b must get the same color under f whenever there is an a, b-path in G for vertices a and b with $d_G(a) = d_G(b) = r$. Hence, we get $f(u_iz) = f(v_jz)$ and $f(u_iz) = f(u_jz)$ for all i, j. Then c appears $r+1$ times at z under f, which is contradiction. □

If we add arbitrarily many leaves making z as their neighbor, then we can make the degree of z become an arbitrary size at least $r+2$. In this case, by repeating the proof of Claim 1, the color of the previously existing edges must get the same color anyway. This forces that all $r+1$ incident edges to

z must get the maximum color. Therefore this new graph does not have an r-maximum edge-coloring.

(Case 2) r is even.

We let $G = G_{\frac{r}{2}+1}$. Then $\Delta(G) = r + 2$. For larger maximum degree we add leaves to z as in r odd case. In this case we can make $\Delta(G) \geq r + 2$. Then the proof is exactly the same as above.

For $\Delta(G) = r + 1$ when r is even, we make the following construction: from the above graph G with $\Delta(G) = r + 2$, let G' be the graph adding a new vertex w and a new edge v_1w after deleting the edge v_1z. Then $\Delta(G') = r + 1$. Applying the same argument from above a color assigned on u_1z forces the rest of the edges of G' to have the same color. Therefore, it is not possible to admit an r-maximum edge-coloring. □

In a graph admitting an r-maximum edge-coloring, a structure that forces a certain color on certain edges may require many colors to be used. In [4] the authors proposed an open problem asking whether $\chi'_r(G) \leq 3$ for every $r \geq 3$ and for every graph G admitting an r-maximum edge-coloring. But unlike the case $r = 2$, from $r \geq 3$ we show that some graphs require many colors to allow an r-maximum coloring. Our construction requires $r \geq 3$ and with $r = 2$ the construction does not contain a structure forcing many colors anymore. We will construct a family of graphs using H_n^m, which is described below. (Figure 1 presents H_3^3.)

Construction of H_n^m:

- Let $V = \{v_1, v_2, \dots, v_n\}$.
- Let $W = \{w_1^1, \dots, w_1^m, w_2^1, \dots, w_2^m, \dots, w_n^1, \dots, w_n^m\}$.
- Let $X = \{x_1^1, \dots, x_1^m, x_2^1, \dots, x_2^m, \dots, x_n^1, \dots, x_n^m\}$.
- Let $U = \{u_1^1, \dots, u_1^m, u_2^1, \dots, u_2^m, \dots, u_n^1, \dots, u_n^m\}$.
- $V(H_n^m) = V \cup W \cup X \cup U$.
- There are three types of edges.
  - **Edges between V and W:** Each v_i has exactly m neighbors in W: $w_i^1, \dots, w_i^m$.
  - **Edges between W and X:** Each x_i^j has exactly $m-1$ neighbors in W: starting from w_i^j, take the first consecutive $m-1$ vertices in order in W as neighbors of x_i^j reading subscripts and superscripts modulo m.
  - **Edges between X and U:** Each x_i^j in X has only one neighbor u_i^j in U.

We see that every vertex in U is a leaf. All other vertices have degree m.

Lemma 3.3. *For any $m \geq 3$ and $n \geq 1$, $\chi'_m(H_n^m) = 1$. In any m-maximum edge-coloring of H_n^m, any color c assigned on the edge incident to a leaf forces c as the color for the rest of the edges.*

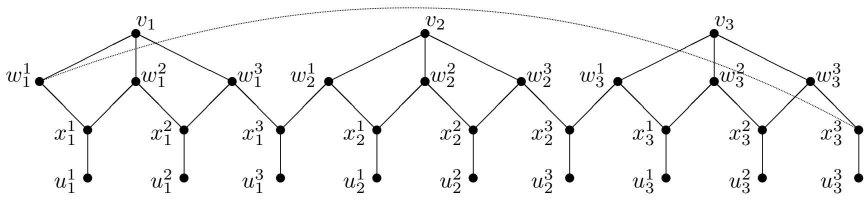

Figure 1: H_3^3

Two vertices w_1^1 and x_1^1 are adjacent in H_3^3.
There are 9 leaves and 21 vertices of degree 3.

Proof. Assigning the same color to all edges satisfies the condition for an m-maximum edge-coloring. Now we will show that edge-coloring with one color is the only possible way.

Consider an m-maximum edge-coloring of H_n^m. Let c be the color that was assigned on an edge $e = x_1^1 u_1^1$. Since the degree of x_1^1 is m, all incident edges to this vertex must receive the same color. In other words, if a color appears at a vertex of degree m, then that color is the only color appearing at the vertex. Note that for any two vertices y, y' with degree m, if there is an y, y'-path via the vertices of degree m, then y and y' must get the same color for their incident edges. Note that H_n^m is connected. In fact, we can see that for any x_i^j and $w_{i'}^{j'}$, there is an $x_i^j, w_{i'}^{j'}$-path using vertices only in $W \cup X$. To see this it is sufficient to show that there is an $x_1^j, w_1^{j'}$-path using vertices only in $W \cup X$. It is because that x_1^i and w_1^i are always adjacent for every i. Once the vertices in $W \cup X$ allow only one color c, then the vertices in $V \cup U$ are forced to allow color c only. $\square$

Given an integer $L \geq 2$, let $H_n^m(k)$ be the k-th copy of H_n^m for $k = 1, 2, \ldots, L$. Let $G(m, L)$ be the graph constructed in the following way:

- Concatenate $H_{m^k}^m(k)$ and $H_{m^{k+1}}^m(k+1)$ for $k = 1, 2, \ldots, L-1$.
  - Identify u_i^t in $V(H_{m^k}^m(k))$ and $v_{m(i-1)+t}$ in $V(H_{m^{k+1}}^m(k+1))$ for $1 \leq i \leq m^k$ and $1 \leq t \leq m$.

Note that there are only three values for the degrees: $1, m$, and $m+1$. The leaves of $G(m, L)$ are the leaves of $H_{m^L}^m(L)$.

Theorem 3.4. *$\chi_r'(G(r, L)) = L$ for $r \geq 3$. In other words, for any fixed integer $r \geq 3$, the r-maximum index function (over the graphs admitting an r-maximum edge-coloring) is unbounded and the range is the set of natural numbers.*

Proof. Let $r \geq 3$ and consider $G(r, L)$. First of all, if we assign color k for the edges of $E(H^r_{r^k}(k))$ in $G(r, L)$, then this edge-coloring becomes an r-maximum edge-coloring. Therefore, $\chi'_r(G(r, L)) \leq L$.

Let f be an optimal r-maximum edge-coloring of $G(r, L)$. Let $c = f(e)$, where e is the edge incident to a leaf in $H^r_{r^L}(L)$. By the proof of Lemma 3.3 all the edges of $H^r_{r^L}(L)$ must receive color c. Now in $G(r, L)$ the color c already appeared r times at every vertex of degree $m + 1$. Therefore, the uncolored incident edge at every such vertex must receive a smaller color, say c_1. Applying the same argument, all the edges of $H^r_{r^{L-1}}(L - 1)$ will get color c_1. We inductively apply this argument to conclude that all the edges of $H^r_{r^{L-i}}(L - i)$ will get color c_i, where $c > c_1 > \cdots > c_{i-1} > c_i > \cdots > c_{L-1}$ and $i = 0, 1, \ldots, L-1$. Therefore, f requires at least L distinct values, and $\chi'_r(G(r, L)) \geq L$. □

In [4] the family of trees and the family of complete graphs were characterized in terms of an r-maximum index. They are bounded above by 3 for every r. Note that $G(r, L)$ is bipartite for $r \geq 3$. Therefore, the r-maximum index function is not bounded over the family of bipartite graphs. It would be an interesting question whether there is any well-known family of graphs that has a bounded range for the r-maximum index function when $r \geq 3$.

Remark 3.5. Note that the above construction with $G(2, L)$ does not give the same conclusion as in Theorem 3.4. The graph $H^2_{2^k}(k)$ is disconnected for each k, and $G(2, L)$ is a tree. Therefore, in $G(2, L)$, at any level k a color assigned on an edge incident to a leaf does not necessarily force the same color for the rest of the edges in the copy $H^2_{2^k}(k)$.

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