Bull. Korean Math. Soc. **55** (2018), No. 5, pp. 1389–1396
https://doi.org/10.4134/BKMS.b170799
pISSN: 1015-8634 / eISSN: 2234-3016

UNITS, NILPOTENT ELEMENTS, AND UNIT-IFP RINGS

Sangwon Park and Sang Jo Yun

Abstract. We observe the structure of a kind of unit-IFP ring that is constructed by Antoine, in relation with units and nilpotent elements. This article concerns the same argument in a more general situation, and study the structure of one-sided zero divisors in such rings. We also provide another kind of unit-IFP ring.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The group of all units and the set of all nilpotent elements in R are denoted by $U(R)$ and $N(R)$, respectively. The polynomial ring with an indeterminate x over R is denoted by $R[x]$.

Due to Bell [2], a ring R is said to be *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. A ring is usually called *reduced* if it has no nonzero nilpotent elements. A ring is usually called *Abelian* if every idempotent is central. It is easily checked that commutative rings and reduced rings are contained in the class of IFP rings. IFP rings are shown easily to be Abelian. It is also easily shown that if R is an IFP ring, then RaR is nilpotent for all $a \in N(R)$, entailing $N_*(R) = N^*(R) = N(R)$, where $N^*(R)$ and $N_*(R)$ mean the upper nilradical (i.e., the sum of all nil ideals) and the lower nilradical (i.e., the intersection of all prime ideals) of R.

Following Kim et al. [5], a ring R is said to be *unit-IFP* if $ab = 0$ for $a, b \in R$ implies $aU(R)b = 0$. IFP rings are clearly unit-IFP, and the converse need not hold by [5, Example 1.1]. Kim et al. provide various results for units and nilpotent elements which are useful to the research of related topics. For example, they show that Köthe's conjecture (i.e., the upper nilradical contains every nil left ideal) holds for unit-IFP rings in [5, Theorem 1.3(1)]. An element u of R is called *right regular* if $ur = 0$ for $r \in R$ implies $r = 0$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular.

Let K be a field, $n \geq 2$, and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Following Antoine's ring construction in [1, Theorem 4.7], let I be the ideal of A generated by b^n and

Received September 14, 2017; Revised November 15, 2017; Accepted December 4, 2017.
2010 *Mathematics Subject Classification.* 16U60, 16U80, 16N40.
Key words and phrases. unit-IFP ring, unit, nilpotent element, Antoine's construction.

set $R = A/I$. Kim et al. showed that R is a unit-IFP ring for the case of $n = 2$ in [5, Example 1.1], and Lee showed that R is a unit-IFP ring for the case of $n \geq 3$ in [7, Theorem 1.2].

In this article we can obtain these results in a more general situation as in [1, Theorem 4.7].

Theorem. *Let K be a field, $n \geq 2$, and D be a set of noncommuting indeterminates of cardinality ≥ 2. Set $A = K\langle D\rangle$ be the free algebra generated by D over K. Let $b \in D$ and I be the ideal of A generated by b^n. Set $R = A/I$ and identify elements of A with their images in R for simplicity. Then R is a unit-IFP ring such that*

$$U(R) = \{k + g + b^p f b^q \mid k \in K\backslash\{0\},\ g \in bK[b],\ f \in R,\ \text{and}\ p, q \geq 1 \text{ with } p + q \geq n\}.$$

Moreover R is a prime ring.

We can obtain [5, Example 1.1] and [7, Theorem 1.2] as corollaries of Theorem.

Corollary. *Let K be a field, $n \geq 2$, and D be a set of noncommuting indeterminates of cardinality ≥ 2. Set $A = K\langle D\rangle$ be the free algebra generated by D over K. Let $b \in D$ and I be the ideal of A generated by b^n. Set $R = A/I$ and identify elements of A with their images in R for simplicity. Then*

$$N(R) = \{g + b^p f b^q \mid g \in bK[b],\ f \in R,\ \text{and}\ p, q \geq 1 \text{ with } p + q \geq n\}.$$

Moreover $N(R)^n = 0$.

Proof. The proof is done by Theorem and a similar argument to the proof of [7, Theorem 1.3]. □

In Section 1, we prove the theorem for the case of $n = 2$; and in Section 2, we prove the theorem for the case of $n \geq 3$. In what follows, we apply the arguments in [1], [6], and [7] to the situation of this article. Given a set S, we denote the cardinality of S by $|S|$. Let K be a field and R_1, R_2 be K-algebras. $R_1 *_K R_2$ denotes the ring coproduct of R_1 and R_2 over K.

The following lemma is a restatement of [3, Corollary 2.16] which does an important role in this article.

Lemma ([1, Lemma 4.4]). *Let S_1 and S_2 be D-algebras over a field D such that any one-sided invertible element of either S_1 or S_2 is two-sided invertible, and let $S = S_1 *_D S_2$. Then:*

(a) *The group of units of S is generated by the units of S_1 and S_2 together with elements of the form $1 - \gamma\delta\epsilon$, where $\delta \in S$ and $\epsilon, \gamma \in S_i$ for some i, such that $\epsilon\gamma = 0$.*

(b) *If $xy = 0$ in S, then there exist a unit $\alpha \in S$ and sets U, V in some S_i with $UV = 0$ such that $x \in SU\alpha$ and $y \in \alpha^{-1}VS$.*

1. The proof of Theorem for the case of $n = 2$

Suppose $n = 2$. The case of $|D| = 2$ is proved by [5, Example 1.1] and [7, Theorem 1.2]. So we assume $|D| \geq 3$. Let $D_1 = D\backslash\{b\}$ and $R_1 = K\langle D_1\rangle$ be the free algebra generated by D_1 over K. Then R is isomorphic to $K\langle D_1\rangle *_K \frac{K[b]}{b^2K[b]}$ that is the coproduct of R_1 and $R_2 = \frac{K[b]}{b^2K[b]}$ over K. We apply the argument in [6]. The procedure is similar, but it is proceeded with writing in details for completeness.

By Lemma (a), every unit in R is generated by the units of R_1 and R_2 together with elements of the form $1-\gamma\delta\epsilon$, where $\delta \in R$ and $\epsilon, \gamma \in R_i$ for some i, such that $\epsilon\gamma = 0$. Note that $U(R_1) = K\backslash\{0\}$ and $U(R_2) = \{k_1 + k_2 b \mid k \in K\backslash\{0\}$ and $k_2 \in K\}$.

Suppose $\epsilon, \gamma \in R\backslash\{0\}$. Then ϵ, γ are contained in R_2 because $\epsilon\gamma = 0$; hence $\epsilon, \gamma \in Kb$. Thus every unit is of the form $k_1 + k_2b + brb$ with $k_1 \in K\backslash\{0\}$, $k_2 \in K$ and $r \in R$; that is,

$$U(R) = \{k_1 + k_2b + brb \mid k_i \in K,\ k_1 \neq 0,\ \text{and } r \in R\}.$$

Let $\alpha\beta = 0$ for $\alpha, \beta \in R\backslash\{0\}$. Then, by Lemma (b), $\alpha = r_1 f_1 u$ and $\beta = u^{-1} f_2 r_2$ for some $u \in U(R)$, $f_1 \in U$, $f_2 \in V$, and $r_1, r_2 \in R$, where U, V are sets in some R_i with $UV = 0$. Here U, V are nonzero subsets; hence these must be contained in R_2 because $UV = 0$. It then follows $U, V \subseteq Kb$. This enables us to write $\alpha = r_1 bu$ and $\beta = u^{-1} br_2$.

Now, letting $u = k_1 + (k_2b + brb)$, we obtain $u^{-1} = k_1^{-1} - k_1^{-2}(k_2b + brb)$ and furthermore

$$\alpha = r_1b(k_1 + k_2b + brb) = r_1k_1b \text{ and } \beta = (k_1^{-1} - k_1^{-2}(k_2b + brb))br_2 = bk_1^{-1}r_2,$$

implying that $\alpha \in Rb$ and $\beta \in bR$.

Therefore R is a unit-IFP ring as can be seen by

$$\alpha u\beta = r_1k_1b(k_1 + k_2b + brb)bk_1^{-1}r_2 = 0 \text{ for all } u = k_1 + k_2b + brb \in U(R),$$

but R is not IFP because $bab \neq 0$ for $b^2 = 0$.

We show next that R is prime. Suppose that $\alpha R\beta = 0$ for $\alpha, \beta \in R$. Assume on the contrary that α and β are both nonzero. Since $\alpha\beta = 0$, $\alpha = rb$ and $\beta = bs$ for some $r, s \in R$ by the argument above. But $0 \neq (rb)a(bs) = \alpha a\beta \in \alpha R\beta = 0$, a contradiction. Consequently $\alpha = 0$ or $\beta = 0$, showing that R is prime.

2. The proof of Theorem for the case of $n \geq 3$

Suppose $n \geq 3$. The case of $|D| = 2$ is proved by [5, Example 1.1] and [7, Theorem 1.2]. So we assume $|D| \geq 3$. Let $D_1 = D\backslash\{b\}$ and $R_1 = K\langle D_1\rangle$ be the free algebra generated by D_1 over K. Then R is isomorphic to $K\langle D_1\rangle *_K \frac{K[b]}{b^nK[b]}$ that is the coproduct of R_1 and $R_2 = \frac{K[b]}{b^nK[b]}$ over K. We apply the argument in the proof of [7, Theorem 1.2]. The procedure is similar in parts, but it is proceeded with writing in details for completeness.

By Lemma (a), $U(R)$ is generated by the units of $R_1 = K\langle D_1\rangle$ and $R_2 = K[b]/b^nK[b]$, together with with elements of the form $1-\gamma\delta\epsilon$, where $\delta \in R$ and $\gamma, \epsilon \in R_i$ for some i, such that $\epsilon\gamma = 0$.

Note that

$$U(R_1) = U(K\langle D_1\rangle) = K\backslash\{0\},\ N(R_1) = \{0\},\ N(R_2) = bK[b],$$

and

$$U(R_2) = \{k_1 + f \mid k \in K\backslash\{0\} \text{ and } f \in bK[b]\}.$$

Thus, if both γ and ϵ are nonzero, then they are contained in R_2 because $\epsilon\gamma = 0$. Hence we have that $\epsilon = b^pf(b)$ and $\gamma = b^qg(b)$ with $p+q \geq n$. But $\gamma\delta\epsilon = b^qg(b)\delta b^pf(b) = b^q[g(b)\delta f(b)]b^p$ with $g(b)\delta f(b) \in R$. Therefore

$$U(R) = \{k + g + b^pfb^q \mid\ k \in K\backslash\{0\},\ g \in bK[b],\ f \in R, \text{ and } p, q \geq 1 \text{ with } p+q \geq n\}.$$

Suppose $\alpha\beta = 0$ for $\alpha, \beta \in R\backslash\{0\}$. Then, by Lemma (b), there exist $u \in U(R)$ and nonzero subsets U, V in some R_i with $UV = 0$ such that $\alpha \in RUu$ and $\beta \in u^{-1}VR$. We must have $U, V \subseteq R_2$ because $UV = 0$. Furthermore, $UV = 0$ implies that

$$U \subseteq b^lR_2 \text{ and } V \subseteq b^mR_2 \text{ for some } l, m \geq 1 \text{ with } l+m \geq n.$$

Consider the shapes of α and β. Since $\alpha = (r_1b^lg_1 + \cdots + r_nb^lg_k)u$ and $\beta = u^{-1}(b^mh_1s_1 + \cdots + b^mh_ps_p)$ with $g_j, h_q \in K[b]$ and $r_j, s_q \in R$, we have

$$\alpha = (r_1g_1+\cdots+r_ng_k)b^lu \in Rb^lu \text{ and } \beta = u^{-1}b^m(h_1s_1+\cdots+h_ps_p) \in u^{-1}b^mR.$$

Now say $\alpha = rb^lu$ and $\beta = u^{-1}b^ms$ with $r, s \in R$.

We will show $\alpha v\beta = [rb^lu]v[u^{-1}b^ms] = 0$ for all $v \in U(R)$. Then $\alpha U(R)\beta = 0$ and hence R is unit-IFP. Here $uvu^{-1} \in U(R)$, say $w = uvu^{-1}$. By the argument above, $w = k_1 + bg(b) + b^pfb^q$ with $k_1 \in K\backslash\{0\}$, $g(b) \in K[b]$, and $f \in R$, where $p+q \geq n$.

Note that

$$[rb^l][k_1 + bg(b)][b^ms] = [rb^l][k_1][b^ms] + [rb^l][bg(b)][b^ms] = 0$$

because $l+m \geq n$. Consequently we obtain

$$\begin{aligned}[rb^lu]v[u^{-1}b^ms] &= [rb^l]w[b^ms] = [rb^l][k_1 + bg(b) + b^pfb^q][b^ms]\\ &= [rb^l][k_1 + bg(b)][b^ms] + [rb^l][b^pfb^q][b^ms]\\ &= [rb^l][b^pfb^q][b^ms] = rb^{l+p}fb^{q+m}s.\end{aligned}$$

But $l+m \geq n$ and $p+q \geq n$, so $l+p \geq n$ or $q+m \geq n$. Thus $[rb^l][b^pfb^q][b^ms] = 0$, and so R is a unit-IFP ring. But R is not IFP since $b^n = 0$ and $b^{n-1}ab \neq 0$.

We show next that R is prime. Suppose that $\alpha R\beta = 0$ for $\alpha, \beta \in R$. Assume on the contrary that α and β are both nonzero. Note $\alpha\beta = 0$. So, by the argument above, $\alpha = rb^l u$ and $\beta = u^{-1}b^m s$ for some nonzero $r, s \in R$.

By the argument above, $u = k + g + b^c h b^d$ with $k \in K\backslash\{0\}$, $g \in bK[b]$, $h \in R$, and $c + d \geq n$. We can obtain $(g + b^c h b^d)^n = 0$ by applying the proof of [7, Theorem 1.3], hence

$$u^{-1} = k^{-1}(1 - [k^{-1}(g + b^c h b^d)] + \cdots + (-1)^{n-1}[k^{-1}(g + b^c h b^d)]^{n-1}).$$

This yields

$$\begin{aligned}
\alpha a\beta &= (rb^l u)a(u^{-1}b^m s) \\
&= rb^l[k + (g + b^c h b^d)]a[k^{-1} - k^{-2}(g + b^c h b^d) \\
&\qquad + \cdots + (-1)^{n-1}k^{-(n-1)}(g + b^c h b^d)^{n-1}]b^m s \\
&= r[kb^l + b^l g'(b)]a[k^{-1}b^m - k^{-2}g'(b)b^m \\
&\qquad + \cdots + (-1)^{n-1}k^{-(n-1)}g'(b)^{n-1}b^m]s \\
&= r[b^l ab^m + k^{-1}b^l ag'(b)b^m - \cdots + (-1)^{n-1}k^{-(n-2)}b^l ag'(b)^{n-1}b^m \\
&\quad + k^{-1}b^l g'(b)ab^m - k^{-2}b^l g'(b)ag'(b)b^m \\
&\qquad - \cdots + (-1)^{n-1}k^{-(n-1)}b^l g'(b)ag'(b)^{n-1}b^m]s \\
&= (rb^l ab^m s) + r[k^{-1}b^l ag'(b)b^m - \cdots + (-1)^{n-1}k^{-(n-2)}b^l ag'(b)^{n-1}b^m \\
&\quad + k^{-1}b^l g'(b)ab^m - k^{-2}b^l g'(b)ag'(b)b^m \\
&\qquad - \cdots + (-1)^{n-1}k^{-(n-1)}b^l g'(b)ag'(b)^{n-1}b^m]s,
\end{aligned}$$

where $g'(b) = g + b^c h b^d$.

Next we observe the shapes of α and β more explicitly. Recall $\alpha = rb^l u$ and $\beta = u^{-1}b^m s$. Letting $u = k + g + b^p f b^q$ and $u^{-1} = k' + g' + b^{p'} f' b^{q'}$ as above, we obtain

$$\alpha = (r_1 g_1 + \cdots + r_n g_k)b^l u = tb^l(k + g + b^p f b^q) = t(kb^l + gb^l + b^{l+p} f b^q) \in Rb^{n_1}$$

and

$$\begin{aligned}
\beta &= u^{-1}b^m(h_1 s_1 + \cdots + h_p s_p) = (k' + g' + b^{p'} f' b^{q'})b^m t' \\
&= (k' b^m + b^m g' + b^{p'} f' b^{q'+m})t' \in b^{n_2}R,
\end{aligned}$$

where $n_1 = \min\{l, q\}$, $n_2 = \min\{m, p'\}$, $t = r_1 g_1 + \cdots + r_n g_k$, and $t' = h_1 s_1 + \cdots + h_p s_p$.

Here assume $rb^l ab^m s = 0$. Then $rb^l a \in Rb$ by the preceding argument, a contradiction because $rb^l a \neq 0$ and $b^m s \neq 0$. Thus $rb^l ab^m s \neq 0$.

If $u \in K$, then $\alpha a\beta = rb^l ab^m s$.

Suppose $u \notin K$, i.e., $g'(b) \neq 0$. Then, letting w be the sum of terms of least degree in $rb^l ab^m s$, w cannot occur in $\alpha a\beta - rb^l ab^m s$ by the existence of nonzero $g'(b)$. Consequently $\alpha a\beta \neq 0$ by the existence of the nonzero w, contrary to $\alpha a\beta \in \alpha R\beta = 0$.

Thus $\alpha = 0$ or $\beta = 0$, and therefore R is prime. □

Next we consider the structure of right or left regular elements in the ring R above.

Remark. Every element in R can be expressed by

$$k + f + gb \quad \text{or} \quad k' + f' + bg' \text{ with } f, f', g, g' \in R,$$

where every term of f (resp., f') does not end (resp., start) by b when nonzero.

(1) Suppose that α is not right regular in R. Then $\alpha \in Rb$ by the argument above. Hence, letting $\alpha = k+f+gb$, we get $k+f = 0$. The converse is obvious. Therefore $k + f \neq 0$ if and only if α is right regular.

(2) Suppose that β is not left regular in R. Then $\beta \in bR$ by the argument above. Hence, letting $\beta = k' + f' + bg'$, we get $k' + f' = 0$. The converse is obvious. Therefore $k' + f' \neq 0$ if and only if α is left regular.

(3) Every element in R can be also expressed by

$$k + f + f_1 b + bf_2 + g + bf_3 b \text{ with } f, f_i \in R \text{ and } g \in bK[b],$$

where every term of f does not start and does not end by b, every term of f_1 does not start by b, and every term of f_2 does not end by b.

Suppose that $\gamma = k + f + f_1 b + bf_2 + g + bf_3 b$ is regular in R. Then $k + f + bf_2 \neq 0$ by (1) since γ is right regular; and since γ is left regular, we moreover get $k + f \neq 0$ by (2). Therefore γ is regular if and only if $k + f \neq 0$.

3. Another kind of unit-IFP ring

We follow the construction and refer to the argument in [4, Example 14]. Let F be a field and $A = F\langle a, b, c\rangle$ (resp., $A_1 = F\langle a, b\rangle$) be the free algebra generated by the noncommuting indeterminates a, b, c (resp., a, b) over F. Next let B the subalgebra of A which consists of all polynomials with zero constant terms in A, and B_1 be the subalgebra of A_1 which consists of all polynomials with zero constant terms in A_1. Then $A = K + B$ and $A_1 = K + B_1$. Consider the ideal I of A generated by

$$cc, \ ac, \ \text{and } crc \text{ with } r \in B.$$

Set $R = A/I$, and identify a, b, c with their images in R for simplicity. Then R is not an IFP ring because $ac = 0$ but $abc \neq 0$. We will show that R is a unit-IFP ring.

Let C be the linear space, over F, of the monomials in B with exactly one c. Then $C^2 = 0$, $B = C + B_1 + I$, $A = K + C + B_1 + I$, and $R = K + C + B_1$; hence every element in R is expressed by

$$k + f_1 + f_2 \text{ with } k \in K, f_1 \in C, \text{ and } f_2 \in B_1.$$

Let $r = k + f_1 + f_2$ be a unit in R. Then $rs = 1$ for some $s = k' + g_1 + g_2 \in R$ with $k' \in K$, $g_1 \in C$, and $g_2 \in B_1$. This yields

$$1 = kk' + (k' f_1 + kg_1) + (k' f_2 + kg_2) + (f_1 g_1 + f_1 g_2 + f_2 g_1) + f_2 g_2$$

$$= 1 + (k'f_1 + kg_1) + (k'f_2 + kg_2) + (f_1g_2 + f_2g_1) + f_2g_2,$$

noting $kk' = 1$ and $f_1g_1 \in I$. This implies

$$(k'f_1 + kg_1) + (f_1g_2 + f_2g_1) + (k'f_2 + kg_2 + f_2g_2) = 0$$

with $(k'f_1 + kg_1) + (f_1g_2 + f_2g_1) \in C$ and $k'f_2 + kg_2 + f_2g_2 \in B_1$. So we have

$$(*) \qquad k'f_2 + kg_2 + f_2g_2 = 0 \text{ and } (k'f_1 + kg_1) + (f_1g_2 + f_2g_1) = 0.$$

From $k'f_2 + kg_2 = -f_2g_2$, we conclude $k'f_2 + kg_2 = 0$ and $f_2g_2 = 0$ by considering the degrees of both sides. So $f_2 = 0$ or $g_2 = 0$.

Suppose $f_2 = 0$. Then we get $kg_2 = 0$ from $k'f_2 + kg_2 = 0$, and $g_2 = 0$ follows because $k \neq 0$. Similarly $g_2 = 0$ implies $f_2 = 0$. Consequently we now have

$$r = k + f_1 \text{ and } s = k' + g_1 \text{ with } k'f_1 + kg_1 = 0,$$

noting $f_1g_1 = 0$. Therefore

$$U(R) = \{k + f \mid 0 \neq k \in K \text{ and } f \in C\}.$$

Next we observe the structure of zero-divisors in R. Let $\alpha\beta = 0$ for $0 \neq \alpha = h + a_1 + a_2$ and $0 \neq \beta = h' + b_1 + b_2$ in R, where $h, h' \in K$, $a_1, b_1 \in C$, and $a_2, b_2 \in B_1$. Then $hh' = 0$, so $h = 0$ or $h' = 0$.

Let $h = 0$, i.e., $\alpha = a_1 + a_2$. Then

$$0 = (h'a_1+a_1b_1+a_1b_2+a_2b_1)+(h'a_2+a_2b_2) = (h'a_1+a_1b_2+a_2b_1)+(h'a_2+a_2b_2),$$

noting $a_1b_1 = 0$. So $h'a_1 + a_1b_2 + a_2b_1 = 0$ and $h'a_2 + a_2b_2 = 0$. As above, $h'a_2 + a_2b_2 = 0$ implies that $a_2 = 0$ or $b_2 = 0$.

Suppose $a_2 = 0$. Then $h'a_1 + a_1b_2 + a_2b_1 = 0$ implies $h'a_1 + a_1b_2 = 0$. Since $b_2 \in B_1$, we get $h'a_1 = 0$ and $a_1b_2 = 0$. Here if $h' \neq 0$, then $a_1 = 0$ and so $\alpha = 0$, contrary to $\alpha \neq 0$. So $h' = 0$ and $\beta = b_1 + b_2$. Consequently $0 = \alpha\beta = a_1(b_1 + b_2) = a_1b_1 + a_1b_2 = a_1b_2$. If $b_2 \neq 0$, then $\alpha = a_1 = 0$ since $b_2 \in B_1$, contrary to $\alpha \neq 0$. So $b_2 = 0$ and $\beta = b_1$.

Suppose $b_2 = 0$. Then $h'a_2 + a_2b_2 = 0$ implies $h'a_2 = 0$. If $a_2 \neq 0$, then $h' = 0$, entailing $\alpha = a_1 + a_2$ and $\beta = b_1$. In this case, we have that $a_2 \in B_1a$ and $b_1 \in cB_1$ by help of the claim in [4, Example 14]. If $a_2 = 0$, then $a = a_1$, and $h'a_1 + a_1b_2 + a_2b_1 = 0$ implies $h'a_1 = 0$. So $h' = 0$ because $a_1 \neq 0$. Consequently $\alpha = a_1$ and $\beta = b_1$.

Therefore

$$\text{“}\alpha \in C \text{ and } \beta \in C\text{” or “}\alpha \in C + B_1a \text{ and } \beta \in cB_1\text{”}$$

when $\alpha\beta = 0$ for $0 \neq \alpha, \beta \in R$.

Consider $\alpha \in N(R)$. Then we have

$$\alpha \in C \cap C = C \text{ or } \alpha \in (C + B_1a) \cap cB_1 = cB_1$$

by the preceding argument. So $\alpha \in C$, and $N(R) \subseteq C$ follows. But $C^2 = 0$, and $C \subseteq N(R)$ follows. Thus $C = N(R)$, and hence

$$U(R) = \{k + f \mid 0 \neq k \in K \text{ and } f \in C\} = \{k + f \mid 0 \neq k \in K \text{ and } f \in N(R)\}$$

$= (K\backslash\{0\}) + C.$

Suppose that $\alpha\beta = 0$ for $\alpha, \beta \in R$. Then $\alpha, \beta \in C$ or $\alpha = a_1 + a_2'a, \beta = cb_2'$ with $a_1 \in C$ and $a_2', b_2' \in B_1$. So

$$\alpha U(R)\beta = \alpha((K\backslash\{0\}) + C)\beta = (K\backslash\{0\})\alpha\beta + \alpha C\beta = \alpha C\beta = 0$$

because $\alpha C\beta$ is either contained in $CCC = 0$ or $(a_1+a_2'a)C(cb_2') = 0$. Therefore R is a unit-IFP ring.

Acknowledgments. The authors thank the referee for very careful reading of the manuscript and many valuable suggestions that improved the paper by much. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIP; Ministry of Science, ICT & Future Planning) (No. 2017R1C1B5017863).

References

[1] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140.

[2] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[3] G. M. Bergman, *Modules over coproducts of rings*, Trans. Amer. Math. Soc. **200** (1974), 1–32.

[4] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[5] H. K. Kim, T. K. Kwak, Y. Lee, and Y. Seo, *Insertion of units at zero products*, J. Algebra Appl. **16** (2018), 1850043, 20 pages.

[6] H. K. Kim, T. K. Kwak, Y. Lee, and Y. Seo, *Corrigendum to "Insertion of units at zero products [J. Algebra Appl.* **16** *(2018), no. 11, 1850043, 20 pages]"*, J. Algebra Appl. **17** (2018), 1892002, 2 pages.

[7] Y. Lee, *Structure of unit-IFP rings*, submitted.

Sangwon Park
Department of Mathematics
Dong-A University
Busan 49315, Korea
Email address: swpark@donga.ac.kr

Sang Jo Yun
Department of Mathematics
Dong-A University
Busan 49315, Korea
Email address: sjyun@dau.ac.kr