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POLYNOMIALLY DEMICOMPACT OPERATORS AND SPECTRAL THEORY FOR OPERATOR MATRICES INVOLVING DEMICOMPACTNESS CLASSES

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Abstract. In the first part of this paper we show that, under some conditions, a polynomially demicompact operator can be demicompact. An example involving the Caputo fractional derivative of order α is provided. Furthermore, we give a refinement of the left and the right Weyl essential spectra of a closed linear operator involving the class of demicompact ones. In the second part of this work we provide some sufficient conditions on the inputs of a closable block operator matrix, with domain consisting of vectors which satisfy certain conditions, to ensure the demicompactness of its closure. Moreover, we apply the obtained results to determine the essential spectra of this operator.

1. Introduction

Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$). We denote by $\mathcal{K}(X,Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, we use the following notations: $\alpha(T)$ is the dimension of the kernel $\mathcal{N}(T)$ and $\beta(T)$ is the codimension of the range $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y),$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $T \in \Phi_\pm(X,Y)$, the index is defined as $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in $\Phi_{+T}, \Phi_{-T}, \Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$,

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$\Phi_{\pm}(X,Y)$ or $\Phi(X,Y)$, respectively. If $X=Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_{\pm}(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_{\pm}(X)$, respectively. If $T\in\mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. Let $T\in\mathcal{C}(X)$. For $x\in\mathcal{D}(T)$, the graph norm $\|\cdot\|_T$ of x is defined by $\|x\|_T=\|x\|+\|Tx\|$. It follows from the closedness of T that $X_T:=(\mathcal{D}(T),\|\cdot\|_T)$ is a Banach space. Clearly, for every $x\in\mathcal{D}(T)$ we have $\|Tx\|\leq\|x\|_T$, so that $T\in\mathcal{L}(X_T,X)$. A linear operator B is said to be T-defined if $\mathcal{D}(T)\subseteq\mathcal{D}(B)$. If the restriction $\widehat{B}$ of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded.

Remark 1.1. Notice that if $T\in\mathcal{C}(X)$ and B is T-bounded, then we get the obvious relations:

$$\begin{cases}\alpha(\widehat{T})=\alpha(T),\beta(\widehat{T})=\beta(T),\mathcal{R}(\widehat{T})=\mathcal{R}(T),\\ \alpha(\widehat{T}+\widehat{B})=\alpha(T+B),\beta(\widehat{T}+\widehat{B})=\beta(T+B),\text{ and }\mathcal{R}(\widehat{T}+\widehat{B})=\mathcal{R}(T+B).\end{cases}$$

Hence, $T\in\Phi(X)$, (resp. $\Phi_+(X),\Phi_-(X)$) if, and only if, $\widehat{T}\in\Phi(X_T,X)$, (resp. $\Phi_+(X_T,X)$, $\Phi_-(X_T,X)$).

Definition 1.1. Let $T\in\mathcal{L}(X,Y)$, where X and Y are two Banach spaces.

(i) T is said to have a left Fredholm inverse if there exist $T_l\in\mathcal{L}(Y,X)$ and $K\in\mathcal{K}(X)$ such that $T_lT=I_X-K$. The operator T_l is called left Fredholm inverse of T.

(ii) T is said to have a right Fredholm inverse if there exists $T_r\in\mathcal{L}(Y,X)$ such that $I_Y-TT_r\in\mathcal{K}(Y)$. The operator T_r is called right Fredholm inverse of T.

(iii) T is said to have a Fredholm inverse if there exists a map which is both a left and a right Fredholm inverse of T.

Definition 1.2. Let $T\in\mathcal{C}(X)$, where X is a Banach space. T is said to have a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse) if $\widehat{T}$ has a left Fredholm inverse (resp. right Fredholm inverse, Fredholm inverse).

The sets of left and right Fredholm inverses are respectively defined by:

$$\Phi_l(X):=\{T\in\mathcal{C}(X)\text{ such that }T\text{ has a left Fredholm inverse}\},\text{ and}$$
$$\Phi_r(X):=\{T\in\mathcal{C}(X)\text{ such that }T\text{ has a right Fredholm inverse}\}.$$

A complex number λ is in $\Phi_{lT}(X)$, $\Phi_{rT}(X)$ or $\Phi_T(X)$ if $\lambda-T$ is in $\Phi_l(X)$, $\Phi_r(X)$ or $\Phi(X)$, respectively.

Definition 1.3. Let X and Y be two Banach spaces and let $F\in\mathcal{L}(X,Y)$. The operator F is called:

(a) Fredholm perturbation if $T+F\in\Phi(X,Y)$ whenever $T\in\Phi(X,Y)$.
(b) Upper semi-Fredholm perturbation if $T+F\in\Phi_+(X,Y)$ whenever $T\in\Phi_+(X,Y)$.
(c) Lower semi-Fredholm perturbation if $T+F\in\Phi_-(X,Y)$ whenever $T\in\Phi_-(X,Y)$.

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.

The concept of demicompactness appeared in the literature since 1966 in order to discuss fixed points. It was introduced by W. V. Petryshyn [15] as follows:

Definition 1.4. An operator $T : \mathcal{D}(T) \subseteq X \to X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n$ converges in X, there exists a convergent subsequence of $(x_n)_n$.

The family of demicompact operators on X is denoted by $\mathcal{DC}(X)$. It is clear that the sum, the product of demicompact operators and the product of a complex number by a demicompact operator are not necessarily demicompact. W. V. Petryshyn [15] and W. Y. Akashi [1] used the class of demicompact operators to obtain some results on Fredholm perturbation. In 2014, B. Krichen [10], gave a generalization of this notion by introducing the class of relative demicompact linear operator with respect to a given linear operator. Recently, W. Chaker, A. Jeribi and B. Krichen [5, 6] continued this study to investigate the essential spectra of densely defined linear operators. Moreover, they established the relationship between demicompact operators and upper semi-Fredholm and Fredholm ones.

The study of spectral theory of block operator matrices has been around for many years. From the most important works on this subject, we quote [8,18]. Let us consider the following operator matrix L_0 acting on the Banach space product $X \times Y$

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In general, the operators occurring in L_0 are unbounded and L_0 need not to be a closed nor a closable operator, even if its entries are closed. However, under some conditions L_0 is closable and its closure L can be determined.

In the theory of unbounded block operator matrices, the Frobenius-Schur factorization is of vital importance in the study of the spectrum and the various spectral theory. In [3], the authors showed that under some hypothesis, the operator L_0 is closable and they described its closure. In [13], N. Moalla, M. Damak and A. Jeribi have described the essential spectra of the closure of L_0. But to determine the essential spectra of L, they must absolutely know the essential spectrum of the entry A. In [4], the authors have generalized these results and have described the essential spectrum of the closure of L_0, which is supposed satisfying some hypothesis. In fact, they have assumed that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and the intersection of the domains of the operators B and D is sufficiently large. Moreover, they have assumed that the domain of L_0 is defined by an additional relation of the form $\Gamma_X x = \Gamma_Y y$ between the two components of its elements. In the same work, they have investigated the essential spectra of L by only knowing the essential spectrum of $A_1 := A|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}$.

In this paper, we will study some demicompactness properties of the closure of the operator matrix L_0 under some hypotheses introduced in [4] and then we will investigate its essential spectrum. More precisely, we are concerned with the following essential spectra:

$$\begin{aligned}
\sigma_{e1}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T},\\
\sigma_{e_2}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_-(X)\} := \mathbb{C}\backslash\Phi_{-T},\\
\sigma_{e_4}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_T,\\
\sigma_{e_5}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma(T+K),\\
\sigma_{e_7}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K),\\
\sigma_{e_8}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{\delta}(T+K),\\
\sigma_{ewl}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{l}(T+K),\\
\sigma_{ewr}(T) &:= \bigcap_{K\in\mathcal{K}(X)} \sigma_{r}(T+K),
\end{aligned}$$

where,

$$\begin{aligned}
\sigma_{ap}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in D(T);\|x\|=1} \|(\lambda - T)x\| = 0\},\\
\sigma_{\delta}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is not surjective}\},\\
\sigma_{l}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_l(X)\} := \mathbb{C}\backslash\Phi_{lT},\\
\sigma_{r}(T) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \notin \Phi_r(X)\} := \mathbb{C}\backslash\Phi_{rT}.
\end{aligned}$$

This paper is organized in the following way. In Section 2, we recall some definitions and results needed in the rest of the paper. In Section 3, we show that under some conditions, a polynomially demicompact operator is demicompact and we give an example involving the Caputo fractional derivative of order α. In Section 4, we give a fine description of the left and the right Weyl essential spectra. In Section 5, we prove that under some conditions, μL is demicompact and we give a necessary condition for which $I - L$ is an upper semi-Fredholm operator on a Banach space. In Section 6, we investigate the essential spectra of the matrix operator L.

2. Preliminary results

We start this section by recalling some Fredholm results related with demicompact operators.

Theorem 2.1 ([5, 6]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. Then, $T \in \mathcal{DC}(X)$ if, and only if, $I - T \in \Phi_+(X)$.*

Theorem 2.2 ([5]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. If μT is demicompact for each $\mu \in [0,1]$, then $I - T$ is a Fredholm operator of index zero.*

Theorem 2.3 ([5]). *Let X be a Banach space and let $T : D(T) \subseteq X \to X$ be a closed linear operator. If T is a 1-set-contraction, then μT is demicompact for each $\mu \in [0,1)$.*

Now let us recall the following results:

Theorem 2.4 ([14,17]). *Let X, Y and Z be Banach spaces, $A \in \mathcal{L}(Y,Z)$ and $B \in \mathcal{L}(X,Y)$.*

(i) *If $AB \in \Phi_+(X,Z)$, then $B \in \Phi_+(X,Y)$.*
(ii) *If $AB \in \Phi_-(X,Z)$, then $A \in \Phi_-(Y,Z)$.*
(iii) *If $X = Y = Z$, $AB \in \Phi(X)$ and $BA \in \Phi(X)$, then $A \in \Phi(X)$ and $B \in \Phi(X)$.*
(iv) *If $A \in \Phi_+(Y,Z)$ and $B \in \Phi_+(X,Y)$, then $AB \in \Phi_+(X,Z)$.*
(v) *If $A \in \Phi(Y,Z)$ and $B \in \Phi(X,Y)$, then $AB \in \Phi(X,Z)$ and $i(A+B) = i(A) + i(B)$.*

Proposition 2.1 ([17]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. Then, $\lambda \notin \sigma_{e_5}(T)$ if, and only if, $(\lambda - T) \in \Phi(X)$ and $i(\lambda - T) = 0$.*

Theorem 2.5 ([8]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. Then,*
(i) *$\lambda \notin \sigma_{e_7}(T)$ if, and only if, $\lambda - T \in \Phi_+(X)$ and $i(\lambda - T) \leq 0$.*
(ii) *$\lambda \notin \sigma_{e_8}(T)$ if, and only if, $\lambda - T \in \Phi_-(X)$ and $i(\lambda - T) \geq 0$.*

Theorem 2.6 ([2,9]). *Let X be a Banach space and let $T \in \mathcal{C}(X)$. Then,*
(i) *$\lambda \notin \sigma_{ewl}(T)$ if, and only if, $\lambda - T \in \Phi_l(X)$ and $i(\lambda - T) \leq 0$.*
(ii) *$\lambda \notin \sigma_{ewr}(T)$ if, and only if, $\lambda - T \in \Phi_r(X)$ and $i(\lambda - T) \geq 0$.*
(iii) *$\sigma_{e5}(T) = \sigma_{ewl}(T) \cup \sigma_{ewr}(T)$.*

In the rest of this section, we will give some results which guarantee the demicompactness of the sum and the product of two operators.

Proposition 2.2. *Let X be a Banach space and let $A \in \mathcal{C}(X) \cap \mathcal{DC}(X)$ and $B \in \mathcal{L}(X)$. If the operator $I - A$ has a left (resp. right) Fredholm inverse A_l (resp. A_r) such that BA_l (resp. A_rB) $\in \mathcal{DC}(X)$, then $A + B \in \mathcal{DC}(X)$.*

Proof. Since A_l is a left Fredholm inverse of $I - A$, then

$$A_l(I - A) = I - K, \text{ where } K \in \mathcal{K}(X).$$

Then, the operator $I - A - B$ can be written as follows

$$I - A - B = (I - BA_l)(I - A) - BK. \tag{2.1}$$

Now, let $(x_n)_n$ be a bounded sequence in X satisfying $(I - A - B)x_n$ converges to an element of X. It follows from Eq. (2.1) together with the compactness of BK and the demicompactness of BA_l that $(I - A)x_n$ has a convergent subsequence. Using the demicompactness of A, we infer that $(x_n)_n$ admits a convergent subsequence and this shows the demicompactness of $A + B$. □

Theorem 2.7. *Let X be a Banach space and let A and B be two bounded operators. If A has a left Fredholm inverse A_l such that $A_l + B \in \mathcal{F}_+(X)$, then $AB \in \mathcal{DC}(X)$.*

Proof. Let A_l be a left Fredholm inverse of A, then

$$A_l A = I - K,$$

where $K \in \mathcal{K}(X)$. It follows that

$$A_l(I - AB) = A_l - (I - K)B = A_l - B - KB.$$

Using the fact that $KB \in \Phi_+(X)$ together with the fact that $A_l - B \in \mathcal{F}_+(X)$, we infer that $A_l(I-AB) \in \Phi_+(X)$. Now, the result follows from both Theorems 2.4(i) and 2.1. □

3. Polynomially demicompact operators

In this section, we will generalize the following result proved in [11] for the class of polynomially demicompact operators on X. In fact, the authors proved that a polynomially compact operator T, element of $\mathcal{P}(X) := \{T \in \mathcal{L}(X)$ such that there exists a nonzero complex polynomial $P(z) = \sum_{r=0}^{p} a_r z^r$ satisfying $P(1) \neq 0$, $P(1) - a_0 \neq 0$, and $P(T) \in \mathcal{K}(X)\}$, is demicompact. In order to state our results, we need to introduce the following set, denoted by $\mathcal{PDC}(X)$, which is defined by:

$$\mathcal{PDC}(X) = \bigcup_{P \in \mathbb{C}[z]\setminus\{0\}, P(1)\neq 0} \mathcal{H}_P,$$

where

$$\mathcal{H}_P := \left\{ T \in \mathcal{L}(X) \text{ such that } \frac{1}{P(1)} P(T) \in \mathcal{DC}(X) \right\}.$$

We note that $\mathcal{PDC}(X)$ contains the set $\mathcal{P}(X)$.

Theorem 3.1. *Let X be a Banach space. Then, $T \in \mathcal{PDC}(X)$ if, and only if, T is demicompact.*

Proof. We first establish the following relation that we are using in the proof. Since $I - T$ commutes with I, Newton's binomial formula allows us to write the following relation

$$T^j = I + \sum_{i=1}^{j} (-1)^i C_j^i (I - T)^i.$$

By making some simple calculations, we may write

$$P(T) = P(1)I + \sum_{j=1}^{p} a_j \left(\sum_{i=1}^{j} (-1)^i C_j^i (I - T)^i \right).$$

Since $P(1) \neq 0$, we have

$$I - \frac{1}{P(1)}P(T) = \frac{1}{P(1)}\sum_{j=1}^{p} a_j \left(\sum_{i=1}^{j} (-1)^i C_j^i (I-T)^i \right). \tag{3.1}$$

Now, let $(x_n)_n$ be a bounded sequence in X satisfying $(I-T)x_n \to x_0$. Using the continuity of T together with the relation (3.1), we infer that there exists x such that

$$\left(I - \frac{1}{P(1)}P(T) \right) x_n \to x.$$

By demicompactness of $\frac{1}{P(1)}P(T)$, we conclude that $(x_n)_n$ admits a convergent subsequence. The converse can be checked by taking $P(z) = z$. □

Theorem 3.2. *Let X be a Banach space and let $T \in \mathcal{L}(X)$. Suppose that there exists a complex polynomial P such that $P(0) = 0$. Then, $I - P(T)$ is demicompact if, and only if, $I - T$ is demicompact.*

Proof. Assume that $I - P(T)$ is a demicompact operator, it follows from Theorem 2.1 that $P(T) \in \Phi_+(X)$. Now, take $x \in \mathcal{N}(T)$, then $Tx = 0$ which implies that for all $j \geq 1$, $T^j = 0$. Hence,

$$P(T)x = \sum_{j=0}^{m} a_j T^j x = P(0)x + \sum_{j=1}^{m} a_j T^j x = 0,$$

where $P(z) = \sum_{j=0}^{m} a_j z^j$. Then, $\mathcal{N}(T) \subset \mathcal{N}(P(T))$ is obvious and this shows that $\alpha(T) < \infty$. Next, since $\mathcal{R}(P(T))$ is closed, we deduce from Theorem 3.12 in [17] that there exists $k > 0$ such that $\forall y \in X$, $\|y\| \leq k\|P(T)y\|$. In particular,

$$\begin{aligned} \|x\| &\leq k\|P(T)x\| \\ &\leq k\sum_{j=1}^{m} |a_j| \|T\|^{j-1} \|Tx\|. \end{aligned}$$

The use of Theorem 3.12 in [17] shows that $\mathcal{R}(T)$ is closed. Therefore, $T \in \Phi_+(X)$ and we conclude by Theorem 2.1 in [6] that $I - T$ is demicompact. Conversely, the result can be obtained by taking $P(z) = z$. □

Remark 3.1. Using Theorem 3.1 in [5] and Theorem 2.1 in [6], we deduce that if P is a complex polynomial such that $P(0) = 0$ and $T \in \mathcal{L}(X)$, then $P(T) \in \Phi_+(X)$ if, and only if, $T \in \Phi_+(X)$.

Theorem 3.3 ([12])**.** *If $x(t) \in C^1[0,T]$ for $T > 0$, then*

$${}_C D_{0,t}^{(\alpha_2)} {}_C D_{0,t}^{(\alpha_1)} x(t) = {}_C D_{0,t}^{(\alpha_1)} {}_C D_{0,t}^{(\alpha_2)} x(t) = {}_C D_{0,t}^{(\alpha_1+\alpha_2)} x(t), \ t \in [0,T],$$

where α_1 and $\alpha_2 \in \mathbb{R}_+$ and $\alpha_1 + \alpha_2 \leq 1$.

Theorem 3.4 ([12]). *If $x(t) \in C^m[0,T]$, $m \in \mathbb{N}$ for $T > 0$, then*

$$ {}_C D_{0,t}^{(\alpha)} x(t) = {}_C D_{0,t}^{(\alpha_n)} \cdots {}_C D_{0,t}^{(\alpha_2)} {}_C D_{0,t}^{(\alpha_1)} x(t);\ t \in [0,T], $$

where $\alpha = \sum_{i=1}^{n} \alpha_i$; $\alpha_i \in (0,1]$, $m-1 \leq \alpha < m$ and there exists $i_k < n$ such that $\sum_{j=1}^{i_k} \alpha_j = k$, and $k = 1, 2, \ldots, m-1$.

Example 3.1. Let C_ω be the space of continuous ω-periodic functions $x : \mathbb{R} \to \mathbb{R}$ and C'_ω the space of continuously differentiable ω-periodic functions $x : \mathbb{R} \to \mathbb{R}$. C_ω equipped with the maximum norm $\|\cdot\|_\infty$ and C'_ω with the norm given by $\|u\|_\infty^1 = \max\{\|u\|_\infty, \|u'\|_\infty\}$ for $u \in \mathcal{C}'$ are Banach spaces. Let us consider the following differential equation:

$$ x'(t) = a(t)x'(t-h_1) + b(t)x(t-h_2) + f(t). $$

Here, a and b are continuous ω-periodic functions such that $|a(t)| < k$, $(k < \infty)$, where $k < \frac{1}{\omega}$ if $\omega > 2$ or $k < \frac{1}{2}$ if $\omega \leq 2$; $f \in \mathcal{C}_\omega$ is a given function and $x \in \mathcal{C}'_\omega$ is an unknown function. This equation can be rewritten in the operator from:

$$ Gx - Ax = f, $$

where $G : \mathcal{C}'_\omega \to \mathcal{C}_\omega$ is given by the formula:

$$ (Gx)(t) = x'(t), $$

and the operator $A : \mathcal{C}'_\omega \to \mathcal{C}_\omega$ by the formula:

$$ (Ax)(t) = a(t)x'(t-h_1) + b(t)x(t-h_2). $$

Let us consider the polynomial $P(z) = \frac{1}{\mu^n} z^n$ and the operator $T = \mu_C D^{(\frac{1}{n})}$; $n \in \mathbb{N}\backslash\{0\}$, where ${}_C D^{(\frac{1}{n})}$ is the Caputo derivative of fractional order $\frac{1}{n}$ and $\mu \in \mathbb{C}\backslash\{0\}$. Applying Theorem 3.4, we get

$$ P(T) = \frac{1}{\mu^n} T^n(x) = \frac{1}{\mu^n} [\mu_C D^{(\frac{1}{n})}]^n x(t) = x'(t). $$

Clearly, $P(T)$ is a bounded linear operator with $\|P(T)\| = 1$ and therefore, $P(T)$ is 1-set-contractive. It follows from the use of Theorem 2.3 that $Q(T) = \alpha^n P(T)$ is demicompact, for all $\alpha \in]0,1[$. Remark that $Q(1) \neq 0$, we get from Theorem 3.1

$$ \mu_C D^{(\frac{1}{n})} \in \mathcal{DC}(X)\ \forall \mu \in \mathbb{C}\backslash\{0\}. $$

Then,

$$ \mu_C D^{(\frac{1}{n})} \in \mathcal{DC}(X)\ \forall \mu \in [0,1]. $$

Using Theorem 2.2, we infer that

$$ I -_C D^{(\frac{1}{n})} \in \Phi(X) \text{ and } i(I -_C D^{(\frac{1}{n})}) = 0. $$

4. Characterization of left and right Weyl essential spectra

The aim of this section is to give a refinement of the left and the right Weyl essential spectra. For this, let X be a Banach space and $T \in \mathcal{C}(X)$. Let us consider the following sets $\Lambda_X, \Upsilon_T(X)$, and $\Psi_T(X)$, respectively, defined by:

$$\Lambda_X = \{J \in \mathcal{L}(X) \text{ such that } \mu J \text{ is demicompact for all } \mu \in [0,1]\}.$$

$$\Upsilon_T(X) = \{K \in \mathcal{L}(X) \text{ such that for all } \lambda \in \rho(T+K), \\ -(\lambda - T - K)^{-1}K \in \Lambda_X\}.$$

$$\Psi_T(X) = \{K \text{ is } T\text{-bounded such that for all } \lambda \in \rho(T+K), \\ -K(\lambda - T - K)^{-1} \in \Lambda_X\}.$$

We also denote for $T \in \mathcal{C}(X)$, the following sets:

$$\sigma_e^l(T) = \bigcap_{K \in \Upsilon_T(X)} \sigma_l(T+K) \text{ and } \sigma_e^r(T) = \bigcap_{K \in \Psi_T(X)} \sigma_r(T+K).$$

Theorem 4.1. *Let X be a Banach space and let $T \in \mathcal{C}(X)$. Then,*

$$\sigma_{ewl}(T) = \sigma_e^l(T) \text{ and } \sigma_{ewr}(T) = \sigma_e^r(T).$$

Proof. Let us notice that for $T \in \mathcal{C}(X)$, K is a T-bounded operator such that $\lambda \in \rho(T+K)$, then, according to closed graph theorem (Lemma 2.1 in [16]), $K(\lambda - T - K)^{-1}$ is a closed linear operator defined on X and then bounded. We first prove that $\sigma_{ewl}(T) \subset \sigma_e^l(T)$ (resp. $\sigma_{ewr}(T) \subset \sigma_e^r(T)$). Indeed, for $\lambda \notin \sigma_e^l(T)$ (resp. $\lambda \notin \sigma_e^r(T)$), there exists $K \in \Upsilon_T(X)$ (resp. $K \in \Psi_T(X)$) such that $\lambda \notin \sigma_l(T+K)$ (resp. $\lambda \notin \sigma_r(T+K)$). Hence,

$$\lambda - T - K \in \Phi_l(X) \text{ and } i(\lambda - T - K) = 0,$$

$$(\text{resp. } \lambda - T - K \in \Phi_r(X) \text{ and } i(\lambda - T - K) = 0).$$

Which implies that

$$\lambda - T - K \in \Phi_l(X) \text{ and } i(\lambda - T - K) \leq 0,$$

$$(\text{resp. } \lambda - T - K \in \Phi_r(X) \text{ and } i(\lambda - T - K) \geq 0).$$

Next, since $K \in \Upsilon_T(X)$, (resp. $K \in \Psi_T(X)$) then $-(\lambda - T - K)^{-1}K \in \Lambda_X$, (resp. $-K(\lambda - T - K)^{-1} \in \Lambda_X$), whenever $\lambda \in \rho(T+K)$. Thus, applying Theorem 2.2, one has

$$I + (\lambda - T - K)^{-1}K \in \Phi(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] = 0,$$

which implies that

$$I + (\lambda - T - K)^{-1}K \in \Phi_l(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] \leq 0$$

(resp.

$$I + (\lambda - T - K)^{-1}K \in \Phi_r(X) \text{ and } i[I + (\lambda - T - K)^{-1}K] \geq 0).$$

Using the equality

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}K]$$

(resp.

$$\lambda - T = [I + K(\lambda - T - K)^{-1}](\lambda - T - K)),$$

we deduce from Theorem 2.5 in [7] that

$$\lambda - T \in \Phi_l(X) \text{ and } i(\lambda - T) \leq 0$$

(resp.

$$\lambda - T \in \Phi_r(X) \text{ and } i(\lambda - T) \geq 0)).$$

We conclude from Theorem 2.6 that $\lambda \notin \sigma_{ewl}(T)$ (resp. $\lambda \notin \sigma_{ewr}(T)$). The inverse inclusion follows from the fact that $\mathcal{K}(X) \subset \Upsilon_T(X)$, (resp. $\mathcal{K}(X) \subset \Psi_T(X)$). □

Corollary 4.1. *Let X be a Banach space, $T \in \mathcal{C}(X)$ and let $\Gamma(X)$ be a subset of X containing $\mathcal{K}(X)$.*

(i) *If $\Gamma(X) \subset \Upsilon_T(X)$, then* $\sigma_{ewl}(T) = \bigcap_{K \in \Gamma(X)} \sigma_l(T + K)$.

(ii) *If $\Gamma(X) \subset \Psi(X)$, then* $\sigma_{ewr}(T) = \bigcap_{K \in \Gamma(X)} \sigma_r(T + K)$.

Proof. Since $\mathcal{K}(X) \subset \Gamma(X) \subset \Upsilon_T(X)$ (resp. $\mathcal{K}(X) \subset \Gamma(X) \subset \Psi(X)$), we obtain

$$\bigcap_{K \in \Upsilon_T(X)} \sigma_l(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_l(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_l(T + K) := \sigma_{ewl}(T)$$

(resp.

$$\bigcap_{K \in \Psi_T(X)} \sigma_r(T + K) \subset \bigcap_{K \in \Gamma(X)} \sigma_r(T + K) \subset \bigcap_{K \in \mathcal{K}(X)} \sigma_r(T + K) := \sigma_{ewr}(T)).$$

The use of Theorem 4.1 allows us to conclude that

$$\sigma_{ewl}(T) = \bigcap_{K \in \Gamma(X)} \sigma_l(T + K),$$

and

$$\sigma_{ewr}(T) = \bigcap_{K \in \Gamma(X)} \sigma_r(T + K).$$

Hence, we get the desired result. □

5. Demicompactness results for operator matrices

Let X, Y and Z be three Banach spaces. In this paper, we consider the linear operators Γ_X from X into Z and Γ_Y from Y into Z. In this section, we are concerned with some new results which can be used to determinate the essential spectra of the matrix operator L, the closure of L_0, on the space $X \times Y$. Let us consider an operator which is formally defined by a matrix

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.1}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space Y and the intertwining operators B and C are defined on the domains $\mathcal{D}(B)$ and $\mathcal{D}(C)$, respectively, and act between these spaces. Then the operator L_0 is defined on the domain $[\mathcal{D}(A)\cap\mathcal{D}(C)]\times[\mathcal{D}(D)\cap\mathcal{D}(B)]$ and

$$\mathcal{D}(L_0)=\left\{\begin{pmatrix} x \\ y \end{pmatrix} \text{ such that } x\in\mathcal{D}(A),\ y\in\mathcal{D}(D)\cap\mathcal{D}(B) \text{ and } \Gamma_X x=\Gamma_Y y\right\}.$$

In the following, it is always assumed that the entries of this matrix satisfy the following conditions, introduced in [4].

- $(H1)$ The operator A is densely defined and closable.
  It follows that $\mathcal{D}(\overline{A})$, the domain of closure $\overline{A}$ of A, coincides with the Banach space X_A which is contained in X.
- $(H2)$ $\mathcal{D}(A)\subset\mathcal{D}(\Gamma_X)\subset X_A$ and Γ_X is bounded as a mapping from X_A into Z.
  The extension of Γ_X by continuity to $X_A=\mathcal{D}(A)$ is denoted by $\overline{\Gamma}_X$, which is a bounded operator from X_A into Z.
- $(H3)$ The lineal $\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)$ is dense in X and the resolvent set of restriction $A_1:=A_{|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}}$ is not empty, i.e., $\rho(A_1)\neq\emptyset$.
- $(H4)$ $\mathcal{D}(A)\subset\mathcal{D}(C)\subset X_A$ and C is closable as an operator from X_A into Y.

Remark 5.1. It follows from $(H3)$ that A_1 is a closed operator, whence $\mathcal{D}(A_1)$ is a closed subspace of X_A. The closed graph theorem and $(H4)$ imply that for $\lambda\in\rho(A_1)$ the operator $C_\lambda:=C(A_1-\lambda I)^{-1}$ from X into Y is bounded.

Remark 5.2. Under the assumptions $(H1)$-$(H3)$, we infer that for each $\lambda\in\rho(A_1)$,

(i) the following decomposition holds:
$$\mathcal{D}(A)=\mathcal{D}(A_1)\oplus\mathcal{N}(A-\lambda I).$$
(ii) $\Gamma_\lambda:=\Gamma_{X|_{\mathcal{N}(A-\lambda I)}}$ is injective and $\mathcal{R}(\Gamma_\lambda)=\Gamma_X(\mathcal{N}(A-\lambda I))=\Gamma_X(\mathcal{D}(A))$ $:=Z_1$ does not depend on λ.
(iii) the inverse K_λ of the operator Γ_λ is
$$K_\lambda:=(\Gamma_{X|_{\mathcal{N}(A-\lambda I)}})^{-1}:Z_1\to\mathcal{N}(A-\lambda I)\subset X.$$

Concerning the operators K_λ, D, Γ_Y and B we impose the following conditions:

- $(H5)$ For some (hence for all) $\lambda\in\rho(A_1)$, the operator K_λ is bounded as a mapping from Z into X.
- $(H6)$ The operator D is densely defined and closed with $\rho(D)\neq\emptyset$.
- $(H7)$ $\mathcal{D}(\Gamma_Y)\supset\mathcal{D}(D)\cap\mathcal{D}(B)$, the set
  $$Y_1=\{y \text{ such that } y\in\mathcal{D}(D)\cap\mathcal{D}(B) \text{ and } \Gamma_Y y\in Z_1\}$$
  is dense in Y and the restriction of Γ_Y to this set is bounded as an operator from Y into Z.

($H8$) For some (hence for all) $\lambda \in \rho(A_1)$, the operator $(A_1 - \lambda I)^{-1}B$ is closable and its closure $\overline{(A_1 - \lambda I)^{-1}B}$ is bounded.

We recall the following results which describe the operator L_0.

Remark 5.3. We will denote by

(i) $\overline{\Gamma}_X$ the extension of Γ_X by continuity to $\mathcal{D}(\overline{A}) := X_A$. It is a bounded operator from X_A into Z.

(ii) $\overline{\Gamma}_Y^{\circ}$ the extension of $\Gamma_Y|_{Y_1}$.

(iii) $\overline{K}_\lambda$ the extension of K_λ to the closure $\overline{Z}_1$ of Z_1 with respect to the norm of Z. Without loss of generality we assume that $\overline{Z}_1 = Z$. Clearly, the operator $\overline{K}_\lambda$ is also bounded as a mapping from $\overline{Z_1}$ to X_A.

We consider in the space Y, for $\lambda \in \rho(A_1)$, the operator

$$M_\lambda := D + CK_\lambda\Gamma_Y - C_\lambda B.$$

The operator M_λ is defined on the set Y_1, which is dense in Y according to ($H7$). Here we observe that Γ_Y is bounded on this domain by assumption ($H7$), that K_λ is bounded by assumption ($H8$). From ($H4$) and ($H8$), we infer that if M_λ is closable as an operator in Y for some $\lambda \in \rho(A_1)$, then it is closable for all $\lambda \in \rho(A_1)$. We emphasize also that the domain of M_λ does not depend on λ.

Theorem 5.1 ([4]). *Let conditions ($H1$)-($H8$) be satisfied. Then, the operator L_0 is closable in $X \times Y$ if, and only if, the operator $M_\lambda := D + CK_\lambda\Gamma_Y - C_\lambda B$ is closable for some $\lambda \in \rho(A_1)$, and equivalently, for all $\lambda \in \rho(A_1)$. Moreover, the closure L of L_0 is given by*:

$$L = \lambda I + \begin{pmatrix} I & 0 \\ C_\lambda & I \end{pmatrix} \begin{pmatrix} A_1 - \lambda I & 0 \\ 0 & \overline{M}_\lambda - \lambda I \end{pmatrix} \begin{pmatrix} I & G_\lambda \\ 0 & I \end{pmatrix}, \tag{5.2}$$

where $G_\lambda := -\overline{K}_\lambda\overline{\Gamma}_Y^{\circ} + \overline{(A_1 - \lambda I)^{-1}B}$.

Remark 5.4. For each $\lambda \in \mathbb{C}$, we have:

$$\begin{aligned} \lambda - L &= \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} \lambda - A_1 & 0 \\ 0 & \lambda - \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (\lambda - \mu)N_\mu \\ &:= UV(\lambda)W - (\lambda - \mu)N_\mu, \end{aligned} \tag{5.3}$$

where

$$N_\mu = \begin{pmatrix} 0 & G_\mu \\ C_\mu & C_\mu G_\mu \end{pmatrix}.$$

Proposition 5.1. *Let L_0 assumptions ($H1$)-($H8$) hold and suppose that there is $\mu \neq 0$ such that $\frac{1}{\mu} \in \rho(A_1)$. If the operator $\mu\overline{M}_{\frac{1}{\mu}}$ is demicompact, then μL is a demicompact operator.*

Proof. Let $\binom{x_n}{y_n}_n \in \mathcal{D}(L)$ be a bounded sequence such that

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} := (I - \mu L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Recalling the factorization (5.2) for $\frac{1}{\mu} \in \rho(A_1)$, one has

$$L = \frac{1}{\mu} I - \begin{pmatrix} I & 0 \\ C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} - A_1 & 0 \\ 0 & \frac{1}{\mu} - \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix}.$$

Then,

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} I - \mu A_1 & 0 \\ 0 & I - \mu \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

means exactly,

$$\begin{pmatrix} I & 0 \\ -C_{\frac{1}{\mu}} & I \end{pmatrix} \begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I - \mu A_1 & 0 \\ 0 & I - \mu \overline{M}_{\frac{1}{\mu}} \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\mu}} \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Hence, we get the following system:

$$\begin{cases} (I - \mu A_1)^{-1} x_n' = x_n + G_{\frac{1}{\mu}} y_n, \\ -C_{\frac{1}{\mu}} x_n' + y_n' = (I - \mu \overline{M}_{\frac{1}{\mu}}) y_n. \end{cases} \tag{5.4}$$

Since the operator $C_{\frac{1}{\mu}}$ is bounded, then we infer from the second equation of the system (5.4) that $(I - \mu \overline{M}_{\frac{1}{\mu}}) y_n$ is convergent. Combining this result together with the demicompactness of $\mu \overline{M}_{\frac{1}{\mu}}$, we show that $(y_n)_n$ has a convergent subsequence. Hence, the first equation of the system (5.4) allows us to conclude that $(x_n)_n$ has a convergent subsequence, which proves the demicompactness of μL and this shows our claim. □

For more generalization, we give the following result.

Theorem 5.2. *Let assumptions* $(H1)$-$(H8)$ *be satisfied and suppose that for every* $\mu \in \rho(A_1)$*, there is* $\lambda \in \mathbb{C}\backslash\{0\}$ *such that* λA_1 *is demicompact. Then, if* C_μ *is compact and* $\lambda \overline{M}_\mu$ *is demicompact, then* $\lambda L \in \mathcal{DC}(X \times Y)$.

Proof. Let $\mu \in \rho(A_1)$. By assumption, there is a complex nonzero number λ verifying $\lambda A_1 \in \mathcal{DC}(X)$. Take a bounded sequence $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)_n \in \mathcal{D}(L)$ such that

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} := (I - \lambda L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Let $\mu \in \rho(A_1)$ be such that there is a complex nonzero number λ verifying $\lambda A_1 \in \mathcal{DC}(X)$. Thanks to Remark 5.4, one has

$$\frac{1}{\lambda} - L = \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} - A_1 & 0 \\ 0 & \frac{1}{\lambda} - \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (\frac{1}{\lambda} - \mu) N_\mu.$$

Thus,

$$I - \lambda L = \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} I - \lambda A_1 & 0 \\ 0 & I - \lambda \overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (1 - \lambda\mu) N_\mu.$$

Therefore,

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} I-\lambda A_1 & 0 \\ 0 & I-\lambda\overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} - (1-\lambda\mu)N_\mu \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{5.5}$$

Observe that (5.5) is equivalent to

$$\begin{pmatrix} I & 0 \\ -C_\mu & I \end{pmatrix} \begin{pmatrix} x_n' \\ y_n' \end{pmatrix} + (1-\lambda\mu) \begin{pmatrix} I & 0 \\ -C_\mu & I \end{pmatrix} N_\mu \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} I-\lambda A_1 & 0 \\ 0 & I-\lambda\overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Moreover, by making some simple calculations, we may show that

$$\begin{pmatrix} x_n' \\ -C_\mu x_n' + y_n' \end{pmatrix} + \begin{pmatrix} (I-\lambda\mu)G_\mu y_n \\ (I-\lambda\mu)C_\mu x_n \end{pmatrix} = \begin{pmatrix} (I-\lambda A_1)x_n + (I-\lambda A_1)G_\mu y_n \\ (I-\lambda\overline{M}_\mu)y_n \end{pmatrix},$$

in equivalent way,

$$\begin{cases} x_n' - \lambda(\mu - A_1)G_\mu y_n = (I-\lambda A_1)x_n, \\ -C_\mu x_n' + y_n' + (I-\lambda\mu)C_\mu x_n = (I-\lambda\overline{M}_\mu)y_n. \end{cases} \tag{5.6}$$

We deduce from the fact that G_μ is compact and $(y_n)_n$ is bounded, that $\lambda(\mu - A_1)G_\mu y_n$ has a convergent subsequence. Hence, from the first equation of system (5.6), we infer that $(I-\lambda A_1)x_n$ has a convergent subsequence. Using the demicompactness of λA_1, we deduce that there exists a convergent subsequence of $(x_n)_n$. Now, since C_μ is bounded, we conclude from the second equation of system (5.6) that $(I-\lambda\overline{M}_\mu)y_n$ has a convergent subsequence. This together with the fact that $\lambda\overline{M}_\mu$ is demicompact allows us to conclude that $(y_n)_n$ has a convergent subsequence. Therefore, there exists a subsequence of $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)_n$ which converges on $\mathcal{D}(L)$. Thus, λL is demicompact. □

Theorem 5.3. *Let assumptions* $(H1)$-$(H8)$ *be satisfied and let* $\mu \in \rho(A_1)$ *such that* $(1-\mu)N_\mu \in \mathcal{F}_+(X\times Y)$. *If the operators* A_1 *and* $\overline{M}_\mu$ *are demicompact, then* $I-L$ *is an upper semi-Fredholm operator.*

Proof. According to the factorization (5.3), we have:

$$\begin{aligned} I-L &= \begin{pmatrix} I & 0 \\ C_\mu & I \end{pmatrix} \begin{pmatrix} I-A_1 & 0 \\ 0 & I-\overline{M}_\mu \end{pmatrix} \begin{pmatrix} I & G_\mu \\ 0 & I \end{pmatrix} - (1-\mu)N_\mu \\ &:= UV(1)W - (1-\mu)N_\mu. \end{aligned}$$

Since A_1 and $\overline{M}_\mu$ are demicompact, it follows from Theorem 2.1, the operators $I-A_1$ and $I-\overline{M}_\mu$ are upper semi-Fredholm, hence $V(1) \in \Phi_+(X\times Y)$. Taking into account that the operators U and W are invertible, we deduce that $UV(1)W$ is an upper semi-Fredholm operator. Now, owing to the fact that $(1-\mu)N_\mu \in \mathcal{F}_+(X\times Y)$, we conclude that $I-L$ is an upper semi-Fredholm operator. □

Before giving the next results, we may state the following notations: Assume that L_0 satisfies $(H1)$-$(H8)$ and suppose that $[1,+\infty[\subset \rho(A_1(\lambda))$. According to the Frobenius-Schur factorization, one has

$$\lambda L = I - \begin{pmatrix} I & 0 \\ C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix},$$

where,

$$\begin{aligned}
&A_1(\lambda) = \lambda A_{|_{\mathcal{D}(A)\cap\mathcal{N}(\Gamma_X)}},\\
&C_1(\lambda) = \lambda C(A_1(\lambda) - I)^{-1},\\
&K_1(\lambda) = (\Gamma_{X|_{\mathcal{N}(\lambda A - I)}})^{-1},\\
&\Gamma_Y(\lambda) = \Gamma_{X|_{\mathcal{N}(\lambda A - I)}},\\
&\overline{\Gamma}_Y^{\circ}(\lambda) \text{ the extension of } \Gamma_Y(\lambda)_{|_{Y_1}},\\
&M_1(\lambda) = \lambda(D + CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B), \text{ and}\\
&G_1(\lambda) = -\overline{K}_1(\lambda)\overline{\Gamma}_Y^{\circ}(\lambda) + \overline{\lambda(A_1(\lambda) - I)^{-1}B}.
\end{aligned}$$

Theorem 5.4. *Let assumptions $(H1)$-$(H8)$ hold and let $\lambda \in \mathbb{C}$ such that $CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B$ is bounded on Y and $[1,+\infty[\subset \rho(A_1(\lambda))$. If λD is demicompact and $I-\lambda D$ has a left (resp. right) Fredholm inverse D_l (resp. D_r) such that $\lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)D_l$ (resp. $\lambda D_r(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)$) is demicompact, then $\lambda L \in \mathcal{DC}(X \times Y)$.*

Proof. Let $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)_n$ be a bounded sequence in $\mathcal{D}(L)$ which verifies

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} := (I - \lambda L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

It follows from Eq. (5.2) that

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

thus,

$$\begin{pmatrix} I & 0 \\ -C_1(\lambda) & I \end{pmatrix} \begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I - A_1(\lambda) & 0 \\ 0 & I - \overline{M}_1(\lambda) \end{pmatrix} \begin{pmatrix} I & G_1(\lambda) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

which allows us to get the following system

$$\left\{ \begin{array}{l} x_n' = (I - A_1(\lambda))x_n + (I - A_1(\lambda)G_1(\lambda)y_n, \\ -C_1(\lambda)x_n' + y_n' = (I - \overline{M}_1(\lambda))y_n. \end{array} \right. \tag{5.7}$$

Since both λD and $\lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)D_l$ (resp. $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$) are demicompact, we infer from Proposition 2.2 that the operator $\lambda D + \lambda(CK_1(\lambda)\Gamma_Y(\lambda) - C_1(\lambda)B)$ is such too. Now, it is easy to show that if a closable operator is demicompact, then its closure is also demicompact. Consequently, we get the demicompactness of $\overline{M}_1(\lambda)$. Moreover, it should be observed that the second equation of the system (5.7) implies the convergence

of $\big((I-\overline{M}_1(\lambda))y_n\big)_n$, hence $(y_n)_n$ has a convergent subsequence. Next, the first equation of the system (5.7) implies that $(x_n)_n$ has a convergent subsequence. Therefore, there exists a convergent subsequence of $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)_n$ which converges in $\mathcal{D}(L)$. Hence, the demicompactness of λL is proved. □

The following corollary gives a sufficient condition to guarantee the demicompactness of L, the closure of the closable matrix operator L_0.

Corollary 5.1. *Let assumptions $(H1)$-$(H8)$ hold. Suppose that $CK_1\Gamma_Y - C_1B$ is bounded on Y and $[1,+\infty[\subset \rho(A_1)$. If D is demicompact and $I-D$ has a left (resp. right) Fredholm inverse D_l (resp. D_r) such that $(CK_1\Gamma_Y - C_1B)D_l$ (resp. $D_r(CK_1\Gamma_Y - C_1B)$) is demicompact, then $L \in \mathcal{DC}(X\times Y)$.*

Proof. The proof is a direct application of Theorem 5.4 for $\lambda = 1$. □

6. Essential spectra of matrix operators by means of demicompactness classes

We start this section by giving some notations that we will need in the proof. Assume that L_0 satisfies $(H1)$-$(H8)$ and let $\alpha \in \mathbb{C}\backslash\{0\}$. Suppose that $[1,+\infty[\subset \rho(A_1(\frac{1}{\alpha}))$. Applying Remark 5.4 on the operator $\frac{1}{\alpha}L$ and for the case $\lambda = 1$, one has

$$\begin{aligned} I-\frac{1}{\alpha}L = &\begin{pmatrix} I & 0 \\ C_\mu(\frac{1}{\alpha}) & I \end{pmatrix}\begin{pmatrix} I-A_1(\frac{1}{\alpha}) & 0 \\ 0 & I-\overline{M}_\mu(\frac{1}{\alpha}) \end{pmatrix}\begin{pmatrix} I & G_\mu(\frac{1}{\alpha}) \\ 0 & I \end{pmatrix} \\ &- (1-\mu)N_\mu(\frac{1}{\alpha}) \\ := &\, UV(1)W - (1-\mu)N_\mu(\frac{1}{\alpha}), \end{aligned} \tag{6.1}$$

where

$$N_\mu(\frac{1}{\alpha}) := \begin{pmatrix} 0 & G_\mu(\frac{1}{\alpha}) \\ C_\mu(\frac{1}{\alpha}) & C_\mu(\frac{1}{\alpha})G_\mu(\frac{1}{\alpha}) \end{pmatrix}.$$

Theorem 6.1. *Let assumptions $(H1)$-$(H8)$ hold and suppose that $[1,+\infty[\subset \rho(A_1(\frac{1}{\alpha}))$, then we have*:

(i) *If for all $\alpha \in \mathbb{C}\backslash\{0\}$, the operator $\frac{1}{\alpha}D$ is demicompact and*

$$\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)$$

is bounded on Y and if $I-\frac{1}{\alpha}D$ has a left (resp. right) Fredholm inverse D_l (resp. D_r) such that $\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)D_l$ (resp. $\frac{1}{\alpha}D_r(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha}) - C_1(\frac{1}{\alpha})B)$) is demicompact and $(1-\mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X\times Y)$, then

$$\sigma_{e_1}(L)\backslash\{0\} = \sigma_{e_1}(A_1)\backslash\{0\} \cup \sigma_{e_1}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}.$$

(ii) *If for all $\lambda \in [0,1]$ and $\alpha \in \mathbb{C}\backslash\{0\}$ the operator $\frac{\lambda}{\alpha}D$ is demicompact and*

$$\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$$

is bounded on Y *and if* $I-\frac{\lambda}{\alpha}D$ *has a left (resp. right) Fredholm inverse* D_l *(resp.* D_r*) such that* $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_l$ *(resp.* $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$*) is demicompact and* $(1-\mu)N_\mu(\frac{\lambda}{\alpha}) \in \mathcal{F}(X\times Y)$*, then*

$$\sigma_{e_i}(L)\backslash\{0\} = \sigma_{e_i}(A_1)\backslash\{0\} \cup \sigma_{e_i}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}, \text{ where } i \in \{4,5\},$$

and

$$\sigma_{e_i}(L)\backslash\{0\} \subseteq \sigma_{e_i}(A_1)\backslash\{0\} \cup \sigma_{e_i}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}, \text{ where } i \in \{7,8\}.$$

Proof. (i) Let $\alpha \in \mathbb{C}\backslash\{0\}$ be such that $\alpha \notin \sigma_{e_1}(L)$. Then,

$$\alpha - L = \alpha(I - \frac{1}{\alpha}L) \in \Phi_+(X\times Y). \tag{6.2}$$

Clearly, αI is an upper semi Fredholm operator. We get then the following equivalence

$$\alpha - L \in \Phi_+(X\times Y) \Longleftrightarrow (I - \frac{1}{\alpha}L) \in \Phi_+(X\times Y).$$

Since both $\frac{1}{\alpha}D$ and $\frac{1}{\alpha}(CK_1(\frac{1}{\alpha})\Gamma_Y(\frac{1}{\alpha})-C_1(\frac{1}{\alpha})B)D_l$ (resp. $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha})$ $-C_1(\frac{\lambda}{\alpha})B)$) are demicompact, it follow from Theorem 5.4 that the operator $\frac{1}{\alpha}L$ is such too. Hence, thanks to Theorem 2.1, the operator $I-\frac{1}{\alpha}L \in \Phi_+(X\times Y)$. Using the fact that $(1-\mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X\times Y)$, we infer that $I-\frac{1}{\alpha}L \in \Phi_+(X\times Y)$ if, and only if, the operator $UV(1)W$ is such too. Now, observe that U and W are invertible and have bounded inverses, hence $I-\frac{1}{\alpha}L \in \Phi_+(X\times Y)$ if, and only if, $V(1)$ has this property, if and only if, both $I-A_1(\frac{1}{\alpha})$ and $I-\overline{M}_\mu(\frac{1}{\alpha})$ are demicompact operators. Equivalently, the operators $\alpha-A_1$ and $\alpha-\alpha\overline{M}_\mu(\frac{1}{\alpha})$ are demicompact. Thus,

$$\sigma_{e_1}(L)\backslash\{0\} = \sigma_{e_1}(A_1)\backslash\{0\} \cup \sigma_{e_1}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}.$$

(ii) We claim that

$$\sigma_{e_4}(L)\backslash\{0\} = \sigma_{e_4}(A_1)\backslash\{0\} \cup \sigma_{e_4}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}.$$

For this purpose, take $\alpha \in \mathbb{C}\backslash\{0\}$. Since αI is a Fredholm operator, then $\alpha-L \in \Phi(X\times Y)$ if, and only if, the operator $(I-\frac{1}{\alpha}L) \in \Phi(X\times Y)$. Next, using the demicompactness of both $\frac{\lambda}{\alpha}D$ and $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_l$ (resp. $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha})-C_1(\frac{\lambda}{\alpha})B)$) for all $\lambda \in [0,1]$, we deduce from Theorem 5.4 that the operator $\frac{\lambda}{\alpha}L$ is demicompact. Hence, according to Theorem 2.2, we have $I-\frac{1}{\alpha}L \in \Phi(X\times Y)$. Using Eq. (6.1) and the fact that $(1-\mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X\times Y)$, we infer that $I-\frac{1}{\alpha}L$ is a Fredholm operator if, and only if, the operator $UV(1)W$ is such too. Now, observe that U and W are invertible and have bounded inverses, hence $I-\frac{1}{\alpha}L \in \Phi(X\times Y)$ if, and only if, $V(1)$ has this property if, and only if, $\alpha-A_1$ and $\alpha-\alpha\overline{M}_\mu(\frac{1}{\alpha})$ are Fredholm operators. Thus, the desired result follows.

Now, we prove the same equality for the Schechter's essential spectrum. To this end, we take $\alpha \in \mathbb{C}\backslash\{0\}$. It is easy to see that $\alpha - L \in \Phi(X \times Y)$ and $i(\alpha - L) = 0$ if, and only if, the operator $(I - \frac{1}{\alpha}L) \in \Phi(X \times Y)$ and $i(I - \frac{1}{\alpha}L) = 0$. By demicompactness of both $\frac{\lambda}{\alpha}D$ and $\frac{\lambda}{\alpha}(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)D_l$ (resp. $\frac{\lambda}{\alpha}D_r(CK_1(\frac{\lambda}{\alpha})\Gamma_Y(\frac{\lambda}{\alpha}) - C_1(\frac{\lambda}{\alpha})B)$) for all $\lambda \in [0, 1]$, we get from Theorem 5.4 that the operator $\frac{\lambda}{\alpha}L$ is demicompact. Hence, according to Theorem 2.2, the operator $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$ and $i(I - \frac{1}{\alpha}L) = 0$. Using Eq. (6.1) and the fact that $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X \times Y)$, we infer that $I - \frac{1}{\alpha}L$ is a Fredholm operator with index zero if, and only if, the operator $UV(1)W$ is such too. Note that U and W are invertible and have bounded inverses, then $I - \frac{1}{\alpha}L$ is Fredholm with index zero if, and only if, $V(1)$ has this property, if and only if, $I - A_1(\frac{1}{\alpha})$ and $I - \overline{M}_\mu(\frac{1}{\alpha})$ are Fredholm operators with indexes zero. Therefore, $\alpha - A_1$ and $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$ have also this property. Hence $\alpha \notin \sigma_{e_5}(A_1)\backslash\{0\} \cap \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}$. Thus,

$$\sigma_{e_5}(A_1)\backslash\{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\} \subseteq \sigma_{e_5}(L)\backslash\{0\}. \tag{6.3}$$

Conversely, let $0 \neq \alpha \notin \sigma_{e_5}(A_1) \cap \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))$, then $\alpha - A_1$ and $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$ are Fredholm operators with indexes zero. Equivalently, $I - A_1(\frac{1}{\alpha})$ and $I - \overline{M}_\mu(\frac{1}{\alpha})$ are such too. The boundedness of the operators U and W and their inverses and the fact that $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}(X \times Y)$ imply that $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$ and $i(I - \frac{1}{\alpha}L) = 0$. Therefore, $\alpha - L \in \Phi(X \times Y)$ and $i(\alpha - L) = 0$, hence $\alpha \notin \sigma_{e_5}(L)\backslash\{0\}$. This immediately shows that

$$\sigma_{e_5}(L)\backslash\{0\} \subseteq \sigma_{e_5}(A_1)\backslash\{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}. \tag{6.4}$$

Now, the use of Eqs. (6.3) and (6.4) makes us to conclude that

$$\sigma_{e_5}(L)\backslash\{0\} = \sigma_{e_5}(A_1)\backslash\{0\} \cup \sigma_{e_5}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}.$$

We give now the proof for $i = 7$. Note that the case $i = 8$ can be checked similarly. Let $\alpha \in \mathbb{C}\backslash\{0\}$, reasoning in the same way as the case $i = 5$, we prove that $I - \frac{1}{\alpha}L \in \Phi(X \times Y)$ and $i(I - \frac{1}{\alpha}L) = 0$. This implies that $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$ and $i(I - \frac{1}{\alpha}L) \leq 0$. If $\alpha \notin \sigma_{e_7}(A_1) \cap \sigma_{e_7}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))$, then $\alpha - A_1$ and $\alpha - \alpha\overline{M}_\mu(\frac{1}{\alpha})$ are upper semi Fredholm operators with negative indexes. It remains to get the same properties for the operators $I - A_1(\frac{1}{\alpha})$ and $I - \overline{M}_\mu(\frac{1}{\alpha})$. Since U and W are invertible and have bounded inverses and using the fact that $(1 - \mu)N_\mu(\frac{1}{\alpha}) \in \mathcal{F}_+(X \times Y)$, we infer that $I - \frac{1}{\alpha}L \in \Phi_+(X \times Y)$ and $i(I - \frac{1}{\alpha}L) \leq 0$. Therefore, $\alpha - L \in \Phi_+(X \times X)$ and $i(\alpha - L) \leq 0$. Now, by applying Lemma 2.5, we conclude that $\alpha \notin \sigma_{e_7}(L)\backslash\{0\}$ and then,

$$\sigma_{e_7}(L)\backslash\{0\} \subseteq \sigma_{e_7}(A_1)\backslash\{0\} \cup \sigma_{e_7}(\alpha\overline{M}_\mu(\frac{1}{\alpha}))\backslash\{0\}.$$

Hence, the theorem is proved. □

References

[1] W. Y. Akashi, *On the perturbation theory for Fredholm operators*, Osaka J. Math. **21** (1984), no. 3, 603–612.

[2] A. Ammar, B. Boukettaya, and A. Jeribi, *Stability of the S-left and S-right essential spectra of a linear operator*, Acta Math. Sci. Ser. B (Engl. Ed.) **34** (2014), no. 6, 1922–1934.

[3] F. V. Atkinson, H. Langer, R. Mennicken, and A. A. Shkalikov, *The essential spectrum of some matrix operators*, Math. Nachr. **167** (1994), 5–20.

[4] A. Bátkai, P. Binding, A. Dijksma, R. Hryniv, and H. Langer, *Spectral problems for operator matrices*, Math. Nachr. **278** (2005), no. 12-13, 1408–1429.

[5] W. Chaker, A. Jeribi, and B. Krichen, *Demicompact linear operators, essential spectrum and some perturbation results*, Math. Nachr. **288** (2015), no. 13, 1476–1486.

[6] ———, *Some Fredholm theory results around relative demicompactness concept*, Preprint, 2018.

[7] M. Gonzalez and V. M. Onieva, *On Atkinson operators in locally convex spaces*, Math. Z. **190** (1985), no. 4, 505–517.

[8] A. Jeribi, *Spectral Theory and Applications of Linear Operators and Block Operator Matrices*, Springer, Cham, 2015.

[9] ———, *Linear Operators and Their Essential Pseudospectra*, Apple Academic Press, Oakville, ON, 2018.

[10] B. Krichen, *Relative essential spectra involving relative demicompact unbounded linear operators*, Acta Math. Sci. Ser. B (Engl. Ed.) **34** (2014), no. 2, 546–556.

[11] K. Latrach and A. Jeribi, *Some results on Fredholm operators, essential spectra, and application*, J. Math. Anal. Appl. **225** (1998), no. 2, 461–485.

[12] C. Li and W. Deng, *Remarks on fractional derivatives*, Appl. Math. Comput. **187** (2007), no. 2, 777–784.

[13] N. Moalla, M. Damak, and A. Jeribi, *Essential spectra of some matrix operators and application to two-group transport operators with general boundary conditions*, J. Math. Anal. Appl. **323** (2006), no. 2, 1071–1090.

[14] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Operator Theory: Advances and Applications, **139**, Birkhäuser Verlag, Basel, 2003.

[15] W. V. Petryshyn, *Remarks on condensing and k-set-contractive mappings*, J. Math. Anal. Appl. **39** (1972), 717–741.

[16] M. Schechter, *On the essential spectrum of an arbitrary operator. I*, J. Math. Anal. Appl. **13** (1966), 205–215.

[17] ———, *Principles of Functional Analysis*, Academic Press, New York, 1971.

[18] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008.

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