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NOTES ON MINIMAL UNIT KILLING VECTOR FIELDS

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Abstract. We will find a necessary and sufficient condition for unit Killing vector fields to be minimal and provide an application of the obtained result.

1. Introduction

Let (M,g) be an n-dimensional Riemannian manifold and (T_1M,g^s) be its unit tangent sphere bundle equipped with the Sasaki metric g^s. We denote by $\chi(M)$ (resp. $\chi^1(M)$) the set of all smooth vector fields (resp. all smooth unit vector fields) on M. Every $V \in \chi^1(M)$ determines a mapping from (M,g) to (T_1M,g^s), embedding M into T_1M. This mapping is isometry only when V is parallel. If the manifold M is compact and orientable, we can define the *energy* of V as the energy of the corresponding map and the *volume* of V as the volume of the immersion, which is regarded as the functionals on the space $\chi^1(M)$. A unit vector field which is critical for the energy functional (resp. the volume functional) is called a *harmonic* vector field (resp. a *minimal* vector field). The notion of harmonic vector fields and minimal vector field can be extended to unit vector fields which on possibly non-compact or non-orientable manifolds in the canonical ways.

Minimal unit vector fields have been worked by many authors – see, for example, [2, 7, 10–13]. In [8], it is shown that minimal vector fields correspond to minimal submanifolds of the unit tangent sphere bundle. The authors have studied Riemannian manifolds (M,g) whose unit tangent sphere bundle (T_1M,g^s) admits harmonic characteristic vector field [3–5].

We begin by establishing the requisite notations. Let (M,g) be an n-dimensional Riemannian manifold. Let ∇ be the Levi-Civita connection of g. We adopt the sign convention for the curvature

$$R(X,Y,Z,W) = -g((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z,W) \tag{1}$$

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for $X, Y, Z, W \in \chi(M)$. Now, we assume that there exists a unit Killing vector field V on (M, g) under consideration. Then, we set

$$\begin{aligned} L_V &= \mathrm{Id} + (\nabla V)^t \circ \nabla V, \quad f(V) = \sqrt{\det(L_V)}, \\ K_V &= f(V) L_V^{-1} \circ (\nabla V)^t, \quad \omega_V(X) = \mathrm{tr}(Z \to (\nabla_Z K_V)(X)). \end{aligned} \tag{2}$$

For compact orientable (M, g), the volume functional $F(V)$ is given by

$$F(V) = \int_M \sqrt{\det(L_V)} dv \tag{3}$$

for $V \in \chi^1(M)$, where dv is the volume density on M defined by g.

Now, we shall write the critical point condition of the functional F on $\chi^1(M)$ ([8], Proposition 4).

Theorem 1.1. *A unit vector field $V \in \chi^1(M)$ is a critical point of F if and only if the 1-form ω_V annihilates $\mathcal{H}^v$, or equivalently, if and only if the vector field X_V, given by $\omega_V(X) = g(X_V, X)$, is in the distribution $\mathcal{V}$ determined by V.*

In this paper, we shall prove the following:

Theorem 1.2. *Let V be a unit Killing vector field. Then*

$$\omega_V(X) = f \tilde{\rho}_V(X) - ((L_V^{-1} \circ \nabla V)X) f \tag{4}$$

holds, where $\tilde{\rho}_V(X)$ is defined to be

$$\sum_{j=1}^{n} (R((L_V^{-1} \circ \nabla V)(X), (L_V^{-1} \circ \nabla V)(E_j), V, E_j) - R(L_V^{-1}(X), L_V^{-1}(E_j), V, E_j)).$$

Consequently, V is minimal if and only if $f \tilde{\rho}_V(X) = ((L_V^{-1} \circ \nabla V)X) f$ holds for any vector field $X \in \mathcal{H}^v = V^{\perp}$, by taking account of Theorem 1.1.

In Section 2, we give a proof of Theorem 1.2, which is a modification of the one of Theorem 2 in our previous article ([6]) in which we pointed out a gap in the proof of the result ([8], Theorem 14) and corrected it to Theorem 1.2. In Section 3, we give a proof to the fact that the characteristic vector field on a K-contact manifold is minimal as an application of Theorem 1.2.

2. Proof of Theorem 1.2

In order to prove Theorem 1.2, we shall prepare some fundamental formulas. Let (M, g) be an n-dimensional Riemannian manifold and assume that there exists a unit Killing vector field V. In the sequel, we fix it and set $f = f(V)$. Since V is a Killing vector field on (M, g), the rank of V must be even, say, $2m$ (since ∇V is regarded as a skew-symmetric linear endomorphism on the tangent space at each point of M). We further normalize the choice of local orthonormal frame field $\{E_j\}_{j=1}^n$ so that $\nabla V(E_i) = -\lambda_i E_{i^*}$ and $\nabla V(E_{i^*}) = \lambda_i E_i$ for $1 \le i \le m$ and $\nabla V(E_\alpha) = 0$ for $2m+1 \le \alpha \le n$, where $i^* = m + i$ $(1 \le i \le m)$.

Thus we can take our frame field to be $\{E_1, \ldots, E_m, E_{1^*}, \ldots, E_{m^*}, E_n = V\}$, which will be called a local adapted orthonormal frame field.

Now, by the definition of the operator L_V in (2), we have

$$\begin{aligned} L_V(E_i) &= (1+\lambda_i^2)E_i, \\ L_V(E_{i^*}) &= (1+\lambda_i^2)E_{i^*}, \\ L_V(E_\alpha) &= E_\alpha \end{aligned} \tag{5}$$

for $1 \leq i \leq m$, $2m+1 \leq \alpha \leq n$. Similarly, we have

$$\begin{aligned} (L_V^{-1} \circ \nabla V)(E_i) &= -\frac{\lambda_i}{1+\lambda_i^2} E_{i^*}, \\ (L_V^{-1} \circ \nabla V)(E_{i^*}) &= \frac{\lambda_i}{1+\lambda_i^2} E_i, \\ (L_V^{-1} \circ \nabla V)(E_\alpha) &= 0 \end{aligned} \tag{6}$$

for $1 \leq i \leq m$, $2m+1 \leq \alpha \leq n$. Further, by the definition of the operator K_V in (2), we get

$$\begin{aligned} K_V(E_i) &= -\frac{f\lambda_i}{1+\lambda_i^2} E_{i^*}, \\ K_V(E_{i^*}) &= \frac{f\lambda_i}{1+\lambda_i^2} E_i, \\ K_V(E_\alpha) &= 0 \end{aligned} \tag{7}$$

for $1 \leq i \leq m$, $2m+1 \leq \alpha \leq n$. By the definition of the function $f = f(V)$, we get also $f = f(V) = \Pi_{j=1}^m (1+\lambda_j^2)$, and hence

$$\begin{aligned} E_i(f) &= 2f \sum_{j=1}^m \frac{\lambda_j}{1+\lambda_j^2} E_i(\lambda_j), \\ E_{i^*}(f) &= 2f \sum_{j=1}^m \frac{\lambda_j}{1+\lambda_j^2} E_{i^*}(\lambda_j). \end{aligned} \tag{8}$$

Let $G_{ij}^k = g(\nabla_{E_i} E_j, E_k)$, $(1 \leq i, j, k \leq n)$ describe the components of covariant differentiation with respect to a local adapted orthonormal frame field $\{E_j\}_{j=1}^n$. We then have

$$(\nabla V)_i^j = G_{in}^j, \quad G_{ij}^k = -G_{ik}^j \tag{9}$$

and further, $G_{in}^j = -G_{jn}^i$ since V is a unit Killing vector field. With respect to a local adapted orthonormal frame field $\{E_j\}_{j=1}^n$, the components of the curvature tensor are given by

$$\begin{aligned} R_{jikr} &= g(R(E_j, E_i)E_k, E_r) \\ &= E_i(G_{jk}^r) - E_j(G_{ik}^r) + \sum_{l=1}^n \{G_{jk}^l G_{il}^r - G_{ik}^l G_{jl}^r - G_{ij}^l G_{lk}^r + G_{ji}^l G_{lk}^r\}. \end{aligned}$$

In particular, for a unit Killing vector field $V = E_n$, we have

$$
\begin{aligned}
R_{jikn} = & -E_i((\nabla V)^k_j) + E_j((\nabla V)^k_i) \\
& + \sum_{l=1}^{n-1}\{-G^k_{il}(\nabla V)^l_j + G^k_{jl}(\nabla V)^l_i + G^l_{ij}(\nabla V)^k_l - G^l_{ji}(\nabla V)^k_l\}.
\end{aligned} \tag{10}
$$

From the definition of the 1-form ω_V, taking account of (5), (7) and (8), we have

$$
\begin{aligned}
\frac{1}{f}\omega_V(E_i) = & \frac{2\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}E_{i^*}(\lambda_j) + \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2}E_{i^*}(\lambda_i) \\
& + \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{n-1}G^j_{ji^*} + \sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}(G^{j^*}_{ji} - G^j_{j^*i}),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{1}{f}\omega_V(E_{i^*}) = & -\frac{2\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}E_i(\lambda_j) - \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2}E_i(\lambda_i) \\
& - \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{n-1}G^j_{ji} - \sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}(G^j_{j^*i^*} - G^{j^*}_{ji^*}),
\end{aligned} \tag{12}
$$

$$
\frac{1}{f}\omega_V(E_\alpha) = -\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}(G^j_{j^*\alpha} - G^{j^*}_{j\alpha}). \tag{13}
$$

On the other hand, from the definition of $\tilde{\rho}_V$ in Theorem 1.2, taking account of (6), (9) and (10), we have

$$
\begin{aligned}
\tilde{\rho}_V(E_i) = & \frac{1}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{1}{1+\lambda_j^2}(R(E_i,E_j,E_j,V)+R(E_i,E_{j^*},E_{j^*},V)) \\
& + \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}R(E_{j^*},E_j,E_{i^*},V) \\
& + \frac{1}{1+\lambda_i^2}\sum_{\beta=2m+1}^{n}R(E_i,E_\beta,E_\beta,V),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\tilde{\rho}_V(E_{i^*}) = & \frac{1}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{1}{1+\lambda_j^2}(R(E_{i^*},E_j,E_j,V) + R(E_{i^*},E_{j^*},E_{j^*},V)) \\
& - \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}R(E_{j^*},E_j,E_i,V) \\
& + \frac{1}{1+\lambda_i^2}\sum_{\beta=2m+1}^{n}R(E_{i^*},E_\beta,E_\beta,V),
\end{aligned} \tag{15}
$$

$$\tilde{\rho}_V(E_\alpha) = \sum_{j=1}^{m} \frac{1}{1+\lambda_j^2}(R(E_\alpha, E_j, E_j, V) + R(E_\alpha, E_{j^*}, E_{j^*}, V)) + \sum_{\beta=2m+1}^{n} R(E_\alpha, E_\beta, E_\beta, V). \tag{16}$$

Using (14) and applying (10), we obtain

$$\begin{aligned}
&\tilde{\rho}_V(E_i)\\
&= \frac{1}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{1}{1+\lambda_j^2}\{\lambda_i G_{ji^*}^{j} - \lambda_j G_{ij^*}^{j} + \lambda_j G_{ji}^{j^*} - \lambda_j G_{ij}^{j^*}\\
&\qquad + E_{j^*}(\lambda_j g_{ij}) + \lambda_i G_{j^*i^*}^{j^*} + \lambda_j G_{ij}^{j^*} - \lambda_j G_{j^*i}^{j} + \lambda_j G_{ij^*}^{j}\}\\
&\quad + \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}\{-E_{j^*}(\lambda_i g_{ij}) - \lambda_j G_{jj}^{i^*} - \lambda_j G_{j^*j^*}^{i^*} - \lambda_i G_{jj^*}^{i} + \lambda_i G_{j^*j}^{i}\}\\
&\quad + \frac{1}{1+\lambda_i^2}\sum_{\beta=2m+1}^{n}\lambda_i G_{\beta i^*}^{\beta}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\tilde{\rho}_V(E_i) &= \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{1}{1+\lambda_j^2}\{(1+\lambda_j^2)G_{ji^*}^{j} + (1+\lambda_j^2)G_{j^*i^*}^{j^*}\}\\
&\quad + \frac{1}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}\{(1+\lambda_i^2)G_{ji}^{j^*} - (1+\lambda_i^2)G_{j^*i}^{j}\}\\
&\quad + \frac{\lambda_i}{1+\lambda_i^2}\sum_{\beta=2m+1}^{n} G_{\beta i^*}^{\beta} + \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2}E_{i^*}(\lambda_i).
\end{aligned}$$

This yields

$$\begin{aligned}
\tilde{\rho}_V(E_i) &= \frac{\lambda_i}{1+\lambda_i^2}\{\sum_{j=1}^{m} G_{ji^*}^{j} + \sum_{j=1}^{m} G_{j^*i^*}^{j^*} + \sum_{\beta=2m+1}^{n} G_{\beta i^*}^{\beta}\}\\
&\quad + \sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}(G_{ji}^{j^*} - G_{j^*i}^{j}) + \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2}E_{i^*}(\lambda_i) \qquad (17)\\
&= \frac{\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{n-1} G_{ji^*}^{j} + \sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}(G_{ji}^{j^*} - G_{j^*i}^{j}) + \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2}E_{i^*}(\lambda_i).
\end{aligned}$$

Similarly, from (15) and (10), we obtain

$$\begin{aligned}
&\tilde{\rho}_V(E_{i^*})\\
&= \frac{1}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{1}{1+\lambda_j^2}\{-E_j(\lambda_i g_{ij}) - \lambda_i G_{ji}^{j} - \lambda_j G_{i^*j^*}^{j} + \lambda_j G_{ji^*}^{j^*}
\end{aligned}$$

$$- \lambda_j G^{j^*}_{i^*j} - \lambda_i G^{j^*}_{j^*i} + \lambda_j G^{j^*}_{i^*j} - \lambda_j G^{j}_{j^*i^*} + \lambda_j G^{j}_{i^*j^*}\}$$

$$- \frac{\lambda_i}{1+\lambda_i^2} \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} \{-E_j(\lambda_j g_{ij}) - \lambda_j G^{i}_{jj} - \lambda_j G^{i}_{j^*j^*} + \lambda_i G^{i^*}_{jj^*} - \lambda_i G^{i^*}_{j^*j}\}$$

$$- \frac{1}{1+\lambda_i^2} \sum_{\beta=2m+1}^{n} \lambda_i G^{\beta}_{\beta i}.$$

This simplifies to become

$$\begin{aligned}
\tilde{\rho}_V(E_{i^*}) = & -\frac{\lambda_i}{1+\lambda_i^2} \sum_{j=1}^{m} \frac{1}{1+\lambda_j^2} \{(1+\lambda_j^2) G^{j}_{ji} + (1+\lambda_j^2) G^{j^*}_{j^*i}\} \\
& + \frac{1}{1+\lambda_i^2} \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} \{(1+\lambda_i^2) G^{j^*}_{ji^*} - (1+\lambda_i^2) G^{j}_{j^*i^*}\} \qquad (18) \\
& - \frac{\lambda_i}{1+\lambda_i^2} \sum_{\beta=2m+1}^{n} G^{\beta}_{\beta i} - \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2} E_i(\lambda_i) \\
= & -\frac{\lambda_i}{1+\lambda_i^2} \{\sum_{j=1}^{m} G^{j}_{ji} + \sum_{j=1}^{m} G^{j^*}_{j^*i} + \sum_{\beta=2m+1}^{n} G^{\beta}_{\beta i}\} \\
& + \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} (G^{j^*}_{ji^*} - G^{j}_{j^*i^*}) - \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2} E_i(\lambda_i) \\
= & -\frac{\lambda_i}{1+\lambda_i^2} \sum_{j=1}^{n-1} G^{j}_{ji} - \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} (G^{j}_{j^*i^*} - G^{j^*}_{ji^*}) - \frac{1-\lambda_i^2}{(1+\lambda_i^2)^2} E_i(\lambda_i).
\end{aligned}$$

From (16), we also have

$$\begin{aligned}
\tilde{\rho}_V(E_\alpha) &= \sum_{j=1}^{m} \frac{1}{1+\lambda_j^2} \{-\lambda_j G^{j}_{\alpha j^*} + \lambda_j G^{j^*}_{j\alpha} - \lambda_j G^{j^*}_{\alpha j} + \lambda_j G^{j^*}_{\alpha j} - \lambda_j G^{j}_{j^*\alpha} + \lambda_j G^{j}_{\alpha j^*}\} \\
&= -\sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} (G^{j}_{j^*\alpha} - G^{j^*}_{j\alpha}). \qquad (19)
\end{aligned}$$

From Equations (11) and (17), we get

$$\frac{1}{f} \omega_V(E_i) - \tilde{\rho}_V(E_i) = \frac{2\lambda_i}{1+\lambda_i^2} \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} E_{i^*}(\lambda_j). \qquad (20)$$

Similarly, from (12) and (18), (13) and (19), we have the following equalities:

$$\frac{1}{f} \omega_V(E_{i^*}) - \tilde{\rho}_V(E_{i^*}) = -\frac{2\lambda_i}{1+\lambda_i^2} \sum_{j=1}^{m} \frac{\lambda_j}{1+\lambda_j^2} E_i(\lambda_j), \qquad (21)$$

$$\frac{1}{f} \omega_V(E_\alpha) - \tilde{\rho}_V(E_\alpha) = 0. \qquad (22)$$

From (6) and (8), we also get

$$\begin{aligned} ((L_V^{-1} \circ \nabla V)E_i)f &= -f\frac{2\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}E_{i^*}(\lambda_j), \\ ((L_V^{-1} \circ \nabla V)E_{i^*})f &= f\frac{2\lambda_i}{1+\lambda_i^2}\sum_{j=1}^{m}\frac{\lambda_j}{1+\lambda_j^2}E_i(\lambda_j), \\ ((L_V^{-1} \circ \nabla V)E_\alpha)f &= 0. \end{aligned} \tag{23}$$

From (20)∼(23), Theorem 1.2 follows. This completes the proof of Theorem 1.2.

3. Characteristic vector field on K-contact manifolds

In [9], J. C. González-Dávila and L. Vanhecke proved the following:

Theorem 3.1. *The characteristic vector field ξ on a K-contact manifold is minimal.*

In this section, we shall give another proof to the above theorem based on the discussion in the previous sections. Let (M, ϕ, ξ, η, g) be an $n(= 2m + 1)$-dimensional K-contact manifold. Then, by the definition, (M, ϕ, ξ, η, g) is a contact metric manifold such that the characteristic vector field ξ is a unit Killing vector field on M. It is known that the following equalities hold on M ([1], [9]).

$$\nabla_X \xi = -\phi X, \tag{24}$$

$$R(X, \xi, \xi, Y) = \eta(X)\eta(Y) - g(X, Y), \tag{25}$$

$$(\nabla_X \phi)Y = R(X, \xi)Y, \tag{26}$$

where we define $R(X, Y)Z \in \chi(M)$ by $g(R(X, Y)Z, W) = R(X, Y, Z, W)$ for $X, Y, Z, W \in \chi(M)$. From (25), we have immediately

$$\rho(\xi, X) = 2m\eta(X) = 2mg(\xi, X) \tag{27}$$

for any $X \in \chi(M)$, where ρ is the Ricci tensor of (M, ϕ, ξ, η, g). We here recall the definition of the $*$-Ricci tensor ρ^* of (M, ϕ, ξ, η, g) given by

$$\rho^*(X, Y) = \frac{1}{2}\ \mathrm{tr}\ (Z \longmapsto -R(X, \phi Y)\phi Z) \tag{28}$$

for $X, Y, Z \in \chi(M)$ ([1], p. 7). Now, we set $V = \xi$ and $f \equiv f(V)$. Then, from (24), we have

$$\nabla V = -\phi, \text{ and hence, } (\nabla V)^t = \phi. \tag{29}$$

Thus, from (2) and (29), we have also

$$L_V = 2I - \eta \otimes \xi. \tag{30}$$

Now, let $\{E_j\}_{j=1}^{2m+1}$ be a local adapted orthonormal frame field given by

$$\{E_i, E_{i^*} = \phi E_i, E_{2m+1} = V\}, \tag{31}$$

where $i^* = i + m$, $1 \le i \le m$. Thus, from (29) and (31), we see that $\lambda_i = 1$, $(1 \le i \le m)$, and

$$(\nabla V)E_i = -E_{i^*} = -\phi E_i,$$
$$(\nabla V)E_{i^*} = -\phi E_{i^*} = E_i, \tag{32}$$
$$(\nabla V)E_n = (\nabla V)E_{2m+1} = (\nabla V)V = 0.$$

Thus, from (2), (30), (31) and (32), we have

$$L_V(E_i) = 2E_i, \;\; L_V(E_{i^*}) = 2E_{i^*}, \tag{33}$$
$$L_V(E_n) = L_V(E_{2m+1}) = L_V(V) = V,$$

and hence,

$$L_V^{-1}(E_i) = \frac{1}{2}E_i, \;\; L_V^{-1}(E_{i^*}) = \frac{1}{2}E_{i^*},$$
$$L_V^{-1}(E_n) = L_V^{-1}(E_{2m+1}) = L_V^{-1}(V) = V.$$

Thus, from (2), (30), (32) and (33), we have

$$f = f(V) = 2^m, \tag{34}$$
$$K_V = 2^m L_V^{-1}(\nabla V)^t = 2^m L_V^{-1} \circ \phi, \tag{35}$$

and hence

$$K_V(E_i) = 2^{m-1}E_{i^*}, \;\; K_V(E_{i^*}) = -2^{m-1}E_i,$$
$$K_V(E_n) = K_V(E_{2m+1}) = K_V(V) = 0.$$

From (24), (31), (32) and (33), we have also

$$(L_V^{-1} \circ \nabla V)(E_i) = -\frac{1}{2}E_{i^*},$$
$$(L_V^{-1} \circ \nabla V)(E_{i^*}) = \frac{1}{2}E_i, \tag{36}$$
$$(L_V^{-1} \circ \nabla V)(E_{2m+1}) = (L_V^{-1} \circ \nabla V)(V) = 0$$

for $1 \le i \le m$.

The following equality plays an important role in the proof of Theorem 3.1.

Lemma 3.2. *On a K-contact manifold (M, ϕ, ξ, η, g), the equality*

$$\rho^*(V, X) = 0$$

holds for any $X \in \chi(M)$.

Proof. We set

$$\phi_{ij} = g(\phi E_i, E_j), \;\; \eta_k = \eta(E_k) = g(\xi, E_k),$$
$$\nabla_i \phi_{jk} = g((\nabla_{E_i}\phi)E_j, E_k), \;\; \rho^*_{ij} = \rho^*(E_i, E_j)$$

and

$$R_{ijkl} = R(E_i, E_j, E_k, E_l)$$

for $1 \leq i, j, k, l \leq n$. Then from (28), we get

$$\rho^*_{ij} = \frac{1}{2} \sum_{a,b,k=1}^{n} R_{iakb} \phi_{ja} \phi_{kb}. \tag{37}$$

Further, from (26), we get also

$$\nabla_i \phi_{jk} = \sum_{a=1}^{n} R_{iajk} \eta_a. \tag{38}$$

Transvecting ϕ_{jk} with (38) and taking account of (37), we have

$$\sum_{a,j,k=1}^{n} \phi_{jk} R_{iajk} \eta_a = \sum_{j,k=1}^{n} (\nabla_i \phi_{jk}) \phi_{jk} = \nabla_i (|\phi|^2) = 0$$

and hence

$$\begin{aligned} 0 &= - \sum_{j=1}^{n} R(\phi^2 E_i, V, E_j, \phi E_j) \\ &= \sum_{j=1}^{n} R(V, \phi E_{i^*}, E_j, \phi E_j) \\ &= 2\rho^*(V, E_{i^*}) \end{aligned} \tag{39}$$

for any i $(1 \leq i \leq n)$. Thus, we have required equality. □

Now, from the definition of the 1-form $\tilde{\rho}_V$ and (1), (27), (32), (33), (36), taking account of Lemma 3.2, we have

$$\begin{aligned} \tilde{\rho}_V(E_i) = & \sum_{j=1}^{m} R((L_V^{-1} \circ \nabla V)(E_i), (L_V^{-1} \circ \nabla V)(E_j), V, E_j) \\ & - \sum_{j=1}^{m} R(L_V^{-1}(E_i), L_V^{-1}(E_j), V, E_j) \\ & + \sum_{j=1}^{m} R((L_V^{-1} \circ \nabla V)(E_i), (L_V^{-1} \circ \nabla V)(E_{j^*}), V, E_{j^*}) \\ & - \sum_{j=1}^{m} R(L_V^{-1}(E_i), L_V^{-1}(E_{j^*}), V, E_{j^*}) \\ = & \frac{1}{4} \sum_{j=1}^{m} R(E_{i^*}, E_{j^*}, V, E_j) - \frac{1}{4} \sum_{j=1}^{m} R(E_i, E_j, V, E_j) \\ & - \frac{1}{4} \sum_{j=1}^{m} R(E_{i^*}, E_j, V, E_{j^*}) - \frac{1}{4} \sum_{j=1}^{m} R(E_i, E_{j^*}, V, E_{j^*}) \end{aligned} \tag{40}$$

$$
\begin{aligned}
&= -\frac{1}{4}\rho(V,E_i) - \frac{1}{4}\sum_{j=1}^{m} R(E_{i^*},V,E_j,E_{j^*})\\
&= -\frac{1}{4}\rho(V,E_i) + \frac{1}{8}\sum_{j=1}^{2m} R(V,E_{i^*},E_j,E_{j^*})\\
&= -\frac{1}{4}\rho(V,E_i) + \frac{1}{4}\rho^*(V,E_i)\\
&= 0
\end{aligned}
$$

for any i $(1 \leq i \leq m)$. Similarly, we have

$$
\begin{aligned}
\tilde{\rho}_V(E_{i^*}) = &\sum_{j=1}^{m} R((L_V^{-1}\circ\nabla V)(E_{i^*}),(L_V^{-1}\circ\nabla V)(E_j),V,E_j)\\
&-\sum_{j=1}^{m} R(L_V^{-1}(E_{i^*}),L_V^{-1}(E_j),V,E_j)\\
&+\sum_{j=1}^{m} R((L_V^{-1}\circ\nabla V)(E_{i^*}),(L_V^{-1}\circ\nabla V)(E_{j^*}),V,E_{j^*})\\
&-\sum_{j=1}^{m} R(L_V^{-1}(E_{i^*}),L_V^{-1}(E_{j^*}),V,E_{j^*})\\
= &-\frac{1}{4}\sum_{j=1}^{m} R(E_i,E_{j^*},V,E_j) - \frac{1}{4}\sum_{j=1}^{m} R(E_{i^*},E_j,V,E_j)\\
&+\frac{1}{4}\sum_{j=1}^{m} R(E_i,E_j,V,E_{j^*}) - \frac{1}{4}\sum_{j=1}^{m} R(E_{i^*},E_{j^*},V,E_{j^*})\\
= &-\frac{1}{4}\rho(V,E_{i^*}) - \frac{1}{4}\sum_{j=1}^{m} R(E_i,V,E_{j^*},E_j)\\
= &-\frac{1}{4}\rho(V,E_{i^*}) + \frac{1}{8}\sum_{j=1}^{2m} R(V,\phi E_{i^*},E_j,\phi E_j)\\
= &-\frac{1}{4}\rho(V,E_{i^*}) + \frac{1}{4}\rho^*(V,E_{i^*})\\
= &\ 0
\end{aligned}
\tag{41}
$$

for any i $(1 \leq i \leq m)$. Thus, from (40) and (41), it follows that $\tilde{\rho}_V(X) = 0$ for any $X \in \mathcal{H}^v$. On the other hand, from (34), $f = f(V)$ is constant and hence $((L_V^{-1}\circ\nabla V)X)f = 0$ for any $X \in \mathcal{H}^v$. Therefore, from Theorem 1.2, it follows that $V = \xi$ is minimal. This completes the proof of Theorem 3.1.

Corollary 3.3. *The characteristic vector field of a Sasakian manifold is minimal.*

We note that Gil-Medrano et al. showed the same results as a Corollary of Theorem 14 in [8]. Here, we have to note the following.

Remark. Let (M, ϕ, ξ, η, g) be a $(2m+1)$-dimensional K-contact manifold and $\tilde{\rho}'_V$ be the 1-form defined in [8] (Theorem 14). Then, from the calculations in section 3, we may easily check that $\frac{1}{f}\omega_V(V) = -m$ and $\tilde{\rho}'_V(V) = m$, and hence, $\frac{1}{f}\omega_V \neq \tilde{\rho}'_V$ on M. This shows that the assertion of Theorem 14 in [8] is not correct.

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, **203**, Birkhäuser Boston, Inc., Boston, MA, 2002.

[2] V. Borrelli and O. Gil-Medrano, *Area-minimizing vector fields on round 2-spheres*, J. Reine Angew. Math. **640** (2010), 85–99.

[3] S. H. Chun, H. K. Pak, J. H. Park, and K. Sekigawa, *A remark on H-contact unit tangent sphere bundles*, J. Korean Math. Soc. **48** (2011), no. 2, 329–340.

[4] S. H. Chun, J. H. Park, and K. Sekigawa, *H-contact unit tangent sphere bundles of four-dimensional Riemannian manifolds*, J. Aust. Math. Soc. **91** (2011), no. 2, 243–256.

[5] ———, *H-contact unit tangent sphere bundles of Einstein manifolds*, Q. J. Math. **62** (2011), no. 1, 59–69.

[6] ———, *Correction to "Minimal unit vector fields"*, arXiv:1121.1841.

[7] A. Fawaz, *Total curvature and volume of foliations on the sphere S^2*, Cent. Eur. J. Math. **7** (2009), no. 4, 660–669.

[8] O. Gil-Medrano and E. Llinares-Fuster, *Minimal unit vector fields*, Tohoku Math. J. (2) **54** (2002), no. 1, 71–84.

[9] J. C. González-Dávila and L. Vanhecke, *Examples of minimal unit vector fields*, Ann. Global Anal. Geom. **18** (2000), no. 3-4, 385–404.

[10] A. Hurtado, *Instability of Hopf vector fields on Lorentzian Berger spheres*, Israel J. Math. **177** (2010), 103–124.

[11] D. Perrone, *Stability of the Reeb vector field of H-contact manifolds*, Math. Z. **263** (2009), no. 1, 125–147.

[12] ———, *Minimality, harmonicity and CR geometry for Reeb vector fields*, Internat. J. Math. **21** (2010), no. 9, 1189–1218.

[13] S. Yi, *Left-invariant minimal unit vector fields on the semi-direct product $\mathbb{R}^n \times_p \mathbb{R}$*, Bull. Korean Math. Soc. **47** (2010), no. 5, 951–960.

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