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# A QUESTION ON $*$-REGULAR RINGS

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Abstract. A $*$-ring $R$ is called $*$-regular if every principal one-sided ideal of $R$ is generated by a projection. In this note, several characterizations of $*$-regular rings are provided. In particular, it is shown that a matrix ring $M_n(R)$ is $*$-regular if and only if $R$ is regular and $1+x_1^*x_1+\cdots+x_{n-1}^*x_{n-1}$ is a unit for all $x_i$ of $R$; which answers a question raised in the literature recently.

## 1. Introduction

Regular rings were invented by von Neumann in order to coordinatize certain lattices of projections. Recall that a ring $R$ is *regular* if for every $a \in R$ the principal right ideal $aR$ is generated by an idempotent, or equivalently there exists $b \in R$ such that $a = aba$; and in addition, if $b$ can be chosen as a unit, then $R$ is said to be *unit regular*. It is well known that a regular ring $R$ is unit regular if and only if $R$ possesses the stable range one (see [4]). Regularity of rings plays an important role in ring theory and module theory, one may refer to [4, 8] for more general theory.

A ring $R$ is a $*$-*ring* (or *ring with involution*) if there exists a map $* : R \to R$ such that for all $x,\ y \in R$

$$(x + y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad \text{and } (x^*)^* = x.$$

An involution $*$ of $R$ is *proper* if $x^*x = 0$ implies $x = 0$ for all $x \in R$. Recall that an element $p$ of a $*$-ring $R$ is a *projection* if $p^2 = p = p^*$. Due to [1, Proposition 3], a $*$-ring $R$ is called $*$-*regular* if for every $a$ in $R$ there exists a projection $p \in R$ such that $aR = pR$, or equivalently $R$ is a regular ring and the involution is proper. Clearly the property of being $*$-regularity is left-right symmetric. Recently, the authors studied properties of $*$-regular rings in [3], and proved that if $R$ is unit regular, then a matrix ring $M_n(R)$ is $*$-regular if and only if $R$ is regular and $1 + x_1^*x_1 + \cdots + x_{n-1}^*x_{n-1}$ is a unit for all $x_i$ of $R$; but it is a question that whether the words '$R$ is unit regular' can be removed.

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In this note, we answer the above question, and prove that a matrix ring $M_n(R)$ is $*$-regular if and only if $R$ is regular and $1+x_1^*x_1+\cdots+x_{n-1}^*x_{n-1}$ is a unit for all $x_i$ of $R$. Some basic properties of $*$-regular rings are also considered.

Rings considered are associative with unity. The set of all idempotents, all projections and all units of a ring $R$ are denoted by $Id(R)$, $P(R)$ and $U(R)$, respectively. The symbol $M_n(R)$ stands for the $n\times n$ matrix ring over $R$. For a $*$-ring $R$, $M_n(R)$ has a natural involution inherited from $R$: if $A=(a_{ij})\in M_n(R)$, $A^*$ equals $(a_{ji}^*)$. Henceforth we consider $M_n(R)$ as a $*$-ring with respect to this natural involution.

## 2. Main results

We begin with the following result.

**Proposition 1.** *Let $R$ be a $*$-ring. The following are equivalent*:
(1) *$R$ is $*$-regular.*
(2) *For every $a\in R$, $Ra=Ra^*a$.*
(3) *$R$ is regular and $Re^*e=Re$ for every idempotent $e$ of $R$.*

*Proof.* (1) $\Rightarrow$ (2) follows from [3, Lemma 2.1].

(2) $\Rightarrow$ (3). It suffices to show that $R$ is regular. Let $a\in R$. By hypothesis, there exists $r\in R$ such that $a=r^*a^*a$. Then we have $ar=r^*a^*ar=(ar)^*ar$. It follows that $ar=(ar)^*$ and $ara=(ar)^*a=r^*a^*a=a$, as required.

(3) $\Rightarrow$ (1). Let $x\in R$ with $x^*x=0$. Since $R$ is regular, $x=xyx$ for some $y\in R$. Write $e=xy$. It is clear that $e\in Id(R)$ and $e^*e=y^*x^*xy=0$. So $Re=Re^*e=0$. Thus $e=0$ and $x=ex=0$, which implies that the involution of $R$ is proper. Therefore, $R$ is a $*$-regular ring. $\square$

Recall that an element $a$ of a $*$-ring $R$ is called *Moore-Penrose invertible* [7] if there exists $b\in R$ such that $a=aba, b=bab, (ab)^*=ab$ and $(ba)^*=ba$, where $b$ is called the *Moore-Penrose inverse* of $a$ and denoted by $b=a^\dagger$. The following result is known in literature, we give the proof for a convenience.

**Proposition 2.** *Let $R$ be a $*$-ring. Then $R$ is $*$-regular if and only if every element of $R$ is Moore-Penrose invertible.*

*Proof.* Suppose that $R$ is $*$-regular. Given $a\in R$. Then $Ra=Rp$ for some $p\in P(R)$. So there is an element $r\in R$ such that $p=ra$ and $a=ap=ara$. Similarly, $aR=qR$ for some $q\in P(R)$, which implies that there exists $s\in R$ satisfying $q=as$ and $a=qa=asa$. Let $b=ras$. Then $aba=(ara)sa=asa=a$ and $bab=r(asa)ras=r(ara)s=ras=b$. Further, $(ab)^*=(aras)^*=(as)^*=as=(ara)s=ab$ and $(ba)^*=(rasa)^*=(ra)^*=ra=r(asa)=ba$. This proves that $a$ is Moore-Penrose invertible and $b=a^\dagger$.

Conversely, let $a\in R$. Then there exists $b\in R$ such that $a=aba$ and $(ab)^*=ab$. So $a=(ab)^*a=b^*a^*a$. It follows that $Ra\subseteq Ra^*a$. Clearly, $Ra^*a\subseteq Ra$. In view of Proposition 1, the result follows. $\square$

Due to [3], a $*$-ring $R$ is said to have the *$k$-GN property* if $1 + x_1^* x_1 + \cdots + x_k^* x_k \in U(R)$ for all $x_1, \ldots, x_k$ in $R$; if $k = 1$, then $R$ is known as a $*$-ring which possesses the *Gelfand–Naimark property* [6] (written *GN property*).

**Lemma 3.** (1) [3, Proposition 3.7] *If $R$ has the $k$-GN property, then $R$ has the $l$-GN property for any integer $1 \le l \le k$.*

(2) [3, Lemma 2.5] *If $R$ is a regular $*$-ring with the GN property, then the involution of $R$ is proper.*

In [3, Theorem 3.8], it was shown that for a $*$-ring $R$ and an integer $n \ge 2$, $M_n(R)$ is $*$-regular and unit regular if and only if $R$ is unit regular and $1 + x_1^* x_1 + \cdots + x_{n-1}^* x_{n-1} \in U(R)$ for all $x_i \in R$. The property of being unit regularity is Morita invariant ([4, Corollary 4.7]), so it is a question that whether the words 'unit regular' be weakened as 'regular'. We give an affirmative answer.

**Theorem 4.** *Let $R$ be a $*$-ring and an integer $n \ge 2$. The following are equivalent*:

(1) *$M_n(R)$ is $*$-regular.*

(2) *$R$ is regular with the $(n-1)$-GN property.*

(3) *$R$ is regular and $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$ implies $x_i = 0$ for all $x_i \in R$.*

*Proof.* (1) $\Rightarrow$ (2). Write $S = M_n(R)$. Since $S$ is $*$-regular, it is regular. By [4, Theorem 1.7], $R$ is regular. Take any $x_1, x_2, \ldots, x_{n-1} \in R$. Let $E = \begin{pmatrix} 1 & 0 \\ \alpha & O_{n-1} \end{pmatrix}$ be a $2 \times 2$ block matrix with $\alpha = (x_1, x_2, \ldots, x_{n-1})^T$ and $O_{n-1}$ the $(n-1) \times (n-1)$ zero matrix. Clearly, $E^2 = E \in S$. In view of Proposition 1, $SE^*E = SE$. Note that $E^*E = \begin{pmatrix} 1 + \sum_{i=1}^{n-1} x_i^* x_i & 0 \\ 0 & O_{n-1} \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix} \in SE$. So there exists a $2 \times 2$ block matrix $Y = \begin{pmatrix} y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} \in S$ such that $YE^*E = \begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix}$, which yields $y_1(1 + \sum_{i=1}^{n-1} x_i^* x_i) = 1$. It follows that $(1 + \sum_{i=1}^{n-1} x_i^* x_i) y_1^* = [y_1(1 + \sum_{i=1}^{n-1} x_i^* x_i)]^* = 1$. So $1 + \sum_{i=1}^{n-1} x_i^* x_i \in U(R)$, and therefore, $R$ has the $(n-1)$-GN property.

(2) $\Rightarrow$ (3). Assume (2) holds. In view of Lemma 3, $R$ has the GN property. Then $R$ is a $*$-regular ring since it is regular. By Proposition 2, every element of $R$ is Moore-Penrose invertible. Let $x_1, x_2, \ldots, x_n \in R$ with $x_1^* x_1 + x_2^* x_2 + \cdots + x_n^* x_n = 0$. Multiplying the above equation by $x_n^\dagger$ on the right and by $(x_n^\dagger)^*$ on the left, we obtain

$$\begin{aligned} 0 &= (x_n^\dagger)^* x_1^* x_1 x_n^\dagger + (x_n^\dagger)^* x_2^* x_2 x_n^\dagger + \cdots + (x_n^\dagger)^* x_n^* x_n x_n^\dagger \\ &= (x_1 x_n^\dagger)^* x_1 x_n^\dagger + (x_2 x_n^\dagger)^* x_2 x_n^\dagger + \cdots + (x_n x_n^\dagger)^* x_n x_n^\dagger. \end{aligned}$$

Set $p = x_n x_n^\dagger$. Clearly, $p^* = p$ and $p^2 = p$. Since $R$ has the $(n-1)$-GN property, $1 + (x_1 x_n^\dagger)^* x_1 x_n^\dagger + (x_2 x_n^\dagger)^* x_2 x_n^\dagger + \cdots + (x_{n-1} x_n^\dagger)^* x_{n-1} x_n^\dagger = 1 - (x_n x_n^\dagger)^* x_n x_n^\dagger = 1 - p \in Id(R) \cap U(R) = \{1\}$. Hence $p = 0$ and $x_n = x_n x_n^\dagger x_n = p x_n = 0$.

So $x_1^*x_1 + x_2^*x_2 + \cdots + x_{n-1}^*x_{n-1} = 0$. By Lemma 3, for each positive integer $l \leq n-1$, $R$ possesses the $l$-GN property. Repeating the above procedure, we will get $x_{n-1} = x_{n-2} = \cdots = x_2 = x_1 = 0$.

(3) $\Rightarrow$ (1) follows from [5, Theorem 1]. □

**Corollary 5.** *Let $R$ be a $*$-ring. Then $M_2(R)$ is $*$-regular if and only if $R$ is a regular ring with GN property.*

**Corollary 6.** *Let $R$ be a $*$-ring. If $M_n(R)$ is $*$-regular, then $M_m(R)$ is $*$-regular for any positive integer $m \leq n$.*

By Theorem 4 and Corollary 6, we have the following result.

**Corollary 7** ([3, Proposition 2.9])**.** *Let $R$ be a $*$-ring. If $M_n(R)$ is $*$-regular, then $m \cdot 1 \in U(R)$ for any positive integer $m \leq n$.*

For a ring $R$, we use $1_R$ to denote the identity endomorphism of $R$.

*Remark* 8. (1) The property of being $*$-regularity relies on the choice of the involution. Let $\mathbb{C}$ be the field of complex numbers and $* = 1_{\mathbb{C}}$. Then $\mathbb{C}$ is regular. For any $n \geq 2$, $M_n(\mathbb{C})$ is not $*$-regular by Theorem 4 (indeed, $M_2(\mathbb{C})$ is not $*$-regular as $1 + i^*i = 0 \notin U(R)$). Nevertheless, if the involution $*$ of $\mathbb{C}$ is defined by $x \mapsto \bar{x}$ where $\bar{x}$ is the conjugation of $x$, then $M_n(\mathbb{C})$ is $*$-regular.

(2) Let $\mathbb{R}$ be the field of real numbers. Set $* = 1_{\mathbb{R}}$. Then $\mathbb{R}$ is regular with $k$-GN property for each $k \geq 1$. So $M_n(\mathbb{R})$ is $*$-regular for every integer $n \geq 2$.

(3) Let $R = \mathbb{Z}_3$ be the ring of integers modulo 3 and $* = 1_R$. Clearly, $R$ is regular with the GN property. So $M_2(R)$ is $*$-regular. But $M_3(R)$ is not $*$-regular since $R$ does not possess the 2-GN property.

It is well known that $C^*$-algebras possess the GN property (see also [6]). So we have the following result immediately.

**Example 9.** If $R$ is a regular $C^*$-algebra, then $M_2(R)$ is $*$-regular.

Let $I$ be an ideal of a ring $R$. Recall that $I$ is called *regular* provided that for each $x \in I$, there exists $y \in I$ such that $x = xyx$ (see [4, Definition, p. 2]).

**Lemma 10** ([4, Lemma 1.3])**.** *Let $I$ be an ideal of a ring $R$. Then $R$ is regular if and only if $I$ and the factor ring $R/I$ are both regular.*

Let $R$ be a $*$-ring. An ideal $I$ of $R$ is called $*$-*invariant* if $I^* \subseteq I$. In this way, $I$ is a $*$-ring (possibly without the identity of $R$) and the involution of $R$ can be extended to the factor ring $R/I$ which is still denoted by $*$. We call an $*$-invariant ideal $I$ is $*$-*regular* if $I$ is regular and the involution of $I$ is proper.

**Lemma 11.** *Let $I$ be an $*$-invariant ideal of $R$. Then $R$ is $*$-regular if and only if $I$ and $R/I$ are both $*$-regular.*

*Proof.* Assume that $R$ is $*$-regular. In view of Lemma 10, $I$ and $R/I$ are regular. Let $x \in I$ with $x^*x = 0$. As $I \subseteq R$ and the involution of $R$ is proper, $x = 0$. So $I$ is $*$-regular. To show that the involution of $R/I$ is proper. Take

$\bar{x} = x + I \in R/I$. If $\bar{x}^*\bar{x} = 0$, then $x^*x \in I$. By Proposition 2, we may write $p = xx^\dagger$. Then $p \in P(R)$ and $x = xx^\dagger x = px = p^*x = (x^\dagger)^*x^*x \in I$ as $I$ is an ideal, whence $\bar{x} = 0 \in R/I$, which implies that $R/I$ is $*$-regular.

Conversely, $R$ is regular by Lemma 10. It suffices to show that the involution of $R$ is proper. Let $x^*x = 0$ with $x \in R$. Then $\bar{x}^*\bar{x} = x^*x + I = 0 \in R/I$. Since $R/I$ is $*$-regular, $\bar{x} = 0$. So $x \in I$. Notice that the involution of $I$ is proper. It follows from $x^*x = 0$ that $x = 0$. As desired. $\square$

For an ideal $I$ of a ring $R$, it is well known that $M_n(I)$ is an ideal of $M_n(R)$ and $M_n(R/I) \cong M_n(R)/M_n(I)$. So we may treat $M_n(R/I)$ and $M_n(R)/M_n(I)$ as the same.

**Proposition 12.** *Let $I$ be an $*$-invariant ideal of a $*$-ring $R$. Then $M_n(R)$ is $*$-regular if and only if both of the following hold*:

(1) *$I$ is regular and $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ implies $x_i = 0$ for all $x_i \in I$.*

(2) *$R/I$ is regular and $y_1^*y_1 + y_2^*y_2 + \cdots + y_n^*y_n \in I$ implies $y_i \in I$ for all $y_i \in R$.*

*Proof.* We will use the following facts freely: (i) $M_n(R)$ is regular if and only if $R$ is regular (by [4, Corollary 4.7]); (ii) $M_n(R)$ is regular if and only if $I$ and $R/I$ are regular if and only if $M_n(I)$ and $M_n(R/I)$ are regular (by Lemma 10).

Suppose that $M_n(R)$ is $*$-regular. Clearly, both $I$ and $R/I$ are regular. Let $x_1^*x_1 + x_2^*x_2 + \cdots + x_n^*x_n = 0$ with $x_i \in I \subseteq R$. By Theorem 4, we have $x_i = 0$ for all $i$. So (1) follows. Since $M_n(R)$ is $*$-regular, $M_n(R/I) \cong M_n(R)/M_n(I)$ is $*$-regular by Lemma 11. Now, given $y_1^*y_1 + y_2^*y_2 + \cdots + y_n^*y_n \in I$. Then $\bar{y_1}^*\bar{y_1} + \bar{y_2}^*\bar{y_2} + \cdots + \bar{y_n}^*\bar{y_n} = 0 \in R/I$. So one has $\bar{y_i} = 0 \in R/I$ by applying Theorem 4 again. Thus $y_i \in I$ for $i = 1, 2, \ldots, n$, and (2) follows.

Conversely, it is enough to verify both $M_n(I)$ and $M_n(R/I)$ are $*$-regular. Clearly, $M_n(I)$ and $M_n(R/I)$ are regular. Let $X = (x_{ij}) \in M_n(I)$ and $X^*X = O$. Then $x_{1j}^*x_{1j} + x_{2j}^*x_{2j} + \cdots + x_{nj}^*x_{nj} = 0$ for $j = 1, \ldots, n$. By (1), $x_{ij} = 0$ for all $i$ and $j$. So $X = O$, which implies that $M_n(I)$ is $*$-regular. Next we show that the involution of $M_n(R/I)$ is proper. Let $Y = (y_{ij}) \in M_n(R)$ be such that $(\bar{Y})^*\bar{Y} = O \in M_n(R/I)$. Then we obtain $Y^*Y \in M_n(I)$. It follows that $y_{1j}^*y_{1j} + y_{2j}^*y_{2j} + \cdots + y_{nj}^*y_{nj} \in I$ for all $j$. By (2), $y_{ij} \in I$ for $i, j = 1, \ldots, n$. Hence $Y \in M_n(I)$, whence $\bar{Y} = O \in M_n(R/I)$. So the involution of $M_n(R/I)$ is proper, and therefore, $M_n(R/I)$ is $*$-regular. As desired. $\square$

Recall that an element $e \in Id(R)$ is *left* (resp., *right*) *semicentral* in $R$ if $re = ere$ (resp., $er = ere$) for all $r \in R$ (see [2]); $e$ is central if and only if $e$ is left and right semicentral. We now provide an application of $*$-regular rings.

**Proposition 13.** *If $R$ is $*$-regular ring, then every left* (*right*) *semicentral idempotent of $R$ is central.*

*Proof.* Without loss of generality, we suppose that $e \in Id(R)$ is left semicentral. Since $R$ is $*$-regular, there exists $p \in P(R)$ such that $Re = Rp$. It follows that

$e = ep$ and $p = pe$. As $e$ is left semicentral, $p = pe = epe = e^2 = e \in P(R)$. For any $r \in R$, we have $er = e^*r = (r^*e)^* = (er^*e)^* = e^*re^* = ere$, which implies that $e$ is right semicentral. So $e$ is central in $R$. □

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