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HEIGHT INEQUALITY FOR RATIONAL MAPS AND BOUNDS FOR PREPERIODIC POINTS

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Abstract. In this paper, we introduce the D-ratio of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, defined over $\overline{\mathbb{Q}}$, whose indeterminacy locus is contained in a hyperplane H on $\mathbb{P}^n$. The D-ratio $r(f;\overline{V})$ characterizes endomorphisms and provides a useful height inequality on $\mathbb{P}^n(\overline{\mathbb{Q}}) \setminus H$. We also provide a dynamical application: preperiodic points of dynamical systems of small D-ratio are of bounded height.

1. Introduction

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map, defined over $\overline{\mathbb{Q}}$, and suppose its indeterminacy locus $I(f)$ is contained in a hyperplane H on $\mathbb{P}^n$. The main purpose of this paper is to provide a way to find a number $r(f;\overline{V})$ which explains the height change by f.

Theorem A (Theorem 5.1)**.** *Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map defined over $\overline{\mathbb{Q}}$ such that indeterminacy only happens on a hyperplane H and let $r(f;\overline{V})$ be the D-ratio of f, associated with a resolution of indeterminacy $\overline{V}$ of f. Then there is a constant C such that the following inequality holds;*

$$\frac{r(f;\overline{V})}{\deg f} \cdot h\big(f(P)\big) + C > h(P) \quad \textit{for all } P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus H.$$

For an endomorphism, we have Northcott's theorem [11]: let $g : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{P}^n(\overline{\mathbb{Q}})$ be an endomorphism and let $h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to \mathbb{R}$ be the logarithmic absolute height function. Then there are two nonnegative constants C_1, C_2, only depending on g, such that

$$(1) \qquad \frac{1}{\deg g} h\big(g(P)\big) + C_1 > h(P) > \frac{1}{\deg g} h\big(g(P)\big) - C_2 \quad \text{for all } P \in \mathbb{P}^n(\overline{\mathbb{Q}}).$$

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It is one of the essential theorems in arithmetic dynamics. For example, the Call-Silverman canonical height function [1] for endomorphisms on projective spaces is well-defined because of Northcott's theorem. Note that we can find such C_2 in (1) for any rational maps on projective spaces [4, Theorem B.2.5(a)].

Unfortunately, the first inequality of Northcott's theorem only holds for endomorphisms [9, Theorem A]; we can find C_1 satisfying (1) on any dense subset of the projective space if and only if f is an endomorphism. If f is not an endomorphism but a rational map, we only expect weaker inequalities [10,12,13]. In this article, we will introduce the D-ratio $r(f;\overline{V})$ associated with f and a resolution of indeterminacy $\overline{V} = (V, \pi)$ of f which provides Theorem A.

The main idea of the D-ratio is to generalize the case of endomorphisms. The degree of an endomorphism $g = [g_0, \ldots, g_n] : \mathbb{P}^n \to \mathbb{P}^n$ is the total degree of homogeneous polynomials $g_0, \ldots, g_n$. In the Picard group $\mathrm{Pic}(\mathbb{P}^n)$, the degree of g is the coefficient of H in g^*H. By the functorial property of the Weil height machine [4, Theorem B.3.2], we can compare $h_H(P)$ and $h_H\big(g(P)\big)$ using the degree of ϕ.

$$h_H\big(g(P)\big) = h_{g^*H}(P) + O(1) = \deg g \cdot h_H(P) + O(1),$$

where h_H is the Weil height function associated with H (the logarithmic height function $h(P)$ and the Weil height function h_H are equivalent so that we may assume that they are the same function). It guarantees that the height function

$$\left(\frac{1}{\deg g} h_{g^*H} - h_H\right)(P) := \frac{1}{\deg g} h_H\big(g(P)\big) - h_H(P)$$

is bounded on $\mathbb{P}^n(\overline{\mathbb{Q}})$.

Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map. Then the above height function is not bounded below any more. To get a tool to examine the dynamical system defined by f, we want to find a constant $\delta > 1$ to build the height function

$$\delta \cdot \frac{1}{\deg f} h_H\big(f(P)\big) - h_H(P)$$

which is bounded below on a dense subset of $\mathbb{P}^n(\overline{\mathbb{Q}})$. Due to the failure of the functoriality of the Weil height machine for rational maps, we pass to a resolution of indeterminacy to work with morphisms: Hironaka [5] shows that there exists a nonsingular projective variety V with a composition of monoidal transformations $\pi = \rho_r \circ \cdots \circ \rho_1 : V \to \mathbb{P}^n$ such that $\phi = f \circ \pi$ extends to a morphism (in the case of endomorphisms, we may think π to be the identity map on $\mathbb{P}^n$).

$$\begin{tikzcd} V \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \mathbb{P}^n \arrow[r, dashed, "f"'] & \mathbb{P}^n \end{tikzcd} \qquad \begin{tikzcd} \mathbb{P}^n \arrow[d, "\mathrm{id}"'] \arrow[dr, "g"] & \\ \mathbb{P}^n \arrow[r, "g"'] & \mathbb{P}^n \end{tikzcd}$$

As we treat $h_H(P) = h_H\big(\mathrm{id}(P)\big)$ and $h_H\big(g(P)\big)$ in the endomorphism case, we will compare

$$h_H\big(\phi(Q)\big) = h_{\phi^*H}(Q) + O(1) \quad \text{and} \quad h_H\big(\pi(Q)\big) = h_{\pi^*H}(Q) + O(1).$$

Note that if $\pi(Q) = P$ and f is defined at P, then $\phi(Q) = f(P)$ so the comparison of these height functions provides the desired height relation [7].

To facilitate this comparison, we first define the $\mathbb{A}^n$-*effectiveness*, a new notion of effectivity for divisors on resolutions of indeterminacy of rational maps. The $\mathbb{A}^n$-effective divisors have better properties than effective divisors when we have a basis of the Picard group involving H and the exceptional divisors.

With the $\mathbb{A}^n$-effective divisors, we define the D-ratio $r(f;\overline{V})$ to be the number satisfying

$$r(f;\overline{V}) = \inf\left\{\delta \;\middle|\; \frac{\delta}{\deg f}\cdot\phi^*H - \pi^*H \text{ is } \mathbb{A}^n\text{-effective}\right\}$$

to provide Theorem A. Note that the D-ratio depends on the choice of a resolution of indeterminacy. However, we will show that it depends only on the "strong factorization class" of the resolution of indeterminacy. In particular, the D-ratio depends only on f in dimension 2 where the strong factorization always holds (Corollary 4.5).

The D-ratio is defined to satisfy some height inequalities, which are enough for dynamical implications. Theorem A only works on $\mathbb{A}^n = \mathbb{P}^n \setminus H$ so that we need a self map defined on $\mathbb{A}^n$ to make a dynamical system, forcing f to be polynomial maps. Theorem B shows that a polynomial map f has a dynamical property similar to an endomorphism if f has a small D-ratio.

Theorem B (Theorem 5.3). *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map defined over $\overline{\mathbb{Q}}$ such that $r(f;\overline{V}) < \deg f$ for a resolution of indeterminacy $\overline{V}$ of f. Then the set of preperiodic points of f is of bounded height.*

Note that the condition $r(f;\overline{V}) < \deg f$ is sharp: there is a polynomial map such that $r(f;\overline{V}) = \deg f$ and $\mathrm{Preper}_{\mathbb{A}^n}(f)$ is not of bounded height (Example 3). Still, we can find some dynamical information for such a rational map under the existence of a good counterpart, like the case of regular automorphisms. We refer [6, 8] to the reader for such cases.

We know that the height functions cannot be defined on entire $\mathbb{C}$. However, theorems are still valid for polynomial maps defined over $\mathbb{C}$ since preperiodic points are algebraic over the field generated by coefficients and we have arithmetic height defined on the algebraic closure of a finitely generated field over $\mathbb{Q}$.

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2. Preliminaries: resolution of indeterminacy

In this section, we see the basic theory of the resolution of indeterminacy, which will be used in later sections. For details, I refer [2, 3] to the reader for basic terminology and properties. We will let H be a fixed hyperplane of $\mathbb{P}^n$, let $\mathbb{A}^n = \mathbb{P}^n \setminus H$. For convenience, without loss of generality, we use the projective coordinate $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(\overline{\mathbb{Q}})$ and assume $H = \{P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \mid x_0 = 0\}$. Let

$$\mathrm{Rat}^n(H) := \{f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \mid I(f) \subset H\}$$

and let f be an element of $\mathrm{Rat}^n(H)$ defined over $\overline{\mathbb{Q}}$ unless stated otherwise. Also, we will consider the resolution of indeterminacy in Hironaka's paper [5]. All varieties are assumed to be irreducible unless stated otherwise.

Main idea of the resolution of indeterminacy is to blowup X several times until all possible limits of function values exist. We clarify the definition of blowups, in terms of the corresponding birational maps.

Definition 2.1. Let $\pi : \widetilde{X} \to X$ be a birational morphism. We say that π is a *monoidal transformation* if there is a smooth irreducible subvariety C of X, of codimension at least 2, such that $\pi^{-1}(C)$ is a smooth irreducible hypersurface in $\widetilde{X}$ and π is an isomorphism outside $\pi^{-1}(C)$. In such case, we say $\widetilde{X}$ *is a blowup of* X *along the center* C.

If the birational map $\pi : \widetilde{X} \to X$ is a composition of monoidal transformations, we say that $\widetilde{X}$ *is a successive blowup of* X. We say C *is the center of* $\widetilde{X}$ if C is the image of every center in each step, on X.

Theorem 2.2 (Hironaka). *Let $f : X \dashrightarrow Y$ be a rational map between proper varieties such that X is nonsingular. Then there is a finite sequence of proper varieties $X_0, \ldots, X_r$ such that*

(a) *$X_0 = X$.*
(b) *$\rho_i : X_i \to X_{i-1}$ is a monoidal transformation.*
(c) *If T_i is the center of the blowup $\rho_i : X_i \to X_{i-1}$, then $\rho_1 \circ \cdots \circ \rho_{i-1}(T_i) \subset I(f)$ on X.*
(d) *The rational map $f : X \dashrightarrow Y$ lifts to a morphism $\widetilde{f} : X_r \to Y$.*
(e) *Consider the composition of all monoidal transformations $\rho = \rho_1 \circ \cdots \circ \rho_r : X_r \to X$. Then $I(f)$ is precisely the set over which ρ fails to be injective.*

Proof. See [5, Question (E) and Main Theorem II]. □

For notational convenience, we will define the following:

Definition 2.3. Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map. We say that a pair (V, π) is *a resolution of indeterminacy of f* when V is a successive blowup of

$\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ such that $f \circ \pi : V \to \mathbb{P}^n$ extends to a morphism. We call this morphism $\phi := f \circ \pi$ the *resolved morphism of* f.

In Section 3, we find a basis of $\mathrm{Pic}(V)$ when (V, π) is a resolution of indeterminacy of a rational map f. In particular, we will use a special basis consisting of irreducible divisors. However, pullbacks of irreducible divisors may not be irreducible because of the exceptional part. So we find a basis of $\mathrm{Pic}(V)$ using a proper transformation of a divisor D, which is the closure of the inverse image of $D \setminus C$ where C is the center of V.

Proposition 2.4. *Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$: there are monoidal transformations $\rho_i : V_i \to V_{i-1}$ such that $V_r = V$ and $V_0 = \mathbb{P}^n$. Let F_i be the exceptional divisor of the blowup $\rho_i : V_i \to V_{i-1}$, let $\sigma_i = \rho_{i+1} \circ \cdots \circ \rho_r$ and let E_i be the proper transformation $\sigma_i^{\#} F_i$ of F_i by σ_i. Then* $\mathrm{Pic}(V)$ *is a free $\mathbb{Z}$-module of rank $(r+1)$ with a basis*

$$\mathfrak{B} = \{H_V = \pi^{\#} H, E_1, \ldots, E_r\}.$$

Proof. [3, Exer.II.7.9] shows that $\mathrm{Pic}(V_{i+1}) \simeq \mathrm{Pic}(V_i) \oplus \mathbb{Z}$ if $\rho_i : V_{i+1} \to V_i$ is a monoidal transformation. More precisely, $\mathrm{Pic}(V_{i+1}) = \{\rho^{\#} D + nE_{i+1} \mid D \in \mathrm{Pic}(V_i)\}$ where E_{i+1} is the exceptional divisor of ρ_i on V_{i+1}. $\square$

Lemma 2.5. *Let $\pi : V \to \mathbb{P}^n$ and $\rho : W \to V$ be compositions of monoidal transformations such that the centers of V and W, successive blowups of $\mathbb{P}^n$, are subsets of H, let $\{H_V, E_1, \ldots, E_r\}$ and $\{H_W, F_1, \ldots, F_s\}$ be the bases of* $\mathrm{Pic}(V)$ *and* $\mathrm{Pic}(W)$ *constructed from π and $\pi \circ \rho$ respectively, described in Proposition 2.4 and let*

$$\rho^* H_V = \rho^{\#} H_V + \sum_{j=1}^{s} m_{0j} F_j \quad \text{and} \quad \rho^* E_i = \rho^{\#} E_i + \sum_{j=1}^{s} m_{ij} F_j.$$

Then $m_{ij} \geq 0$ for all i, j. Furthermore, $\sum_{i=0}^{r} m_{ij} > 0$ for all $j = 1, \ldots, s$.

Proof. Fix $j \in \{1, \ldots, s\}$. If $\rho(F_j) \subset E_i$, then $m_{ij} > 0$. Otherwise, $m_{ij} = 0$. Furthermore, because of the assumption, $\rho(F_j)$ is over which π fails to be injective on V, it should be contained in one of $H_V, E_1, \ldots, E_r$. Note that m_{ij} are integers since we only blowup along smooth irreducible subvarieties. $\square$

3. $\mathbb{A}^n$-effective divisors

Recall that H is a fixed hyperplane of $\mathbb{P}^n$ with a uniformizer x_0, $\mathbb{A}^n = \mathbb{P}^n \setminus H$ and f is an element of $\mathrm{Rat}^n(H)$ defined over $\overline{\mathbb{Q}}$. The purpose of introducing the $\mathbb{A}^n$-effective divisor is to examine the corresponding Weil height function which is desired to be bounded below on $\mathbb{A}^n$. Since two Weil height functions are equivalent if associated divisors are linearly equivalent, we consider that the equality $D_1 = D_2$ in this paper means the linear equivalence for convenience though we call them divisors. Also, we use $\mathrm{Pic}_{\mathbb{R}}(V) = \mathrm{Pic}(V) \otimes \mathbb{R}$ instead of $\mathrm{Pic}(V)$ to describe the $\mathbb{A}^n$-effective cone and $r(f; \overline{V})$, which does not affect on applications of the Weil height machine. It follows that an effective divisor is considered to be a linear equivalent class in the effective cone of $\mathrm{Pic}_{\mathbb{R}}(V)$.

The precise definition of the D-ratio will be given in Section 4, but roughly the D-ratio is be the number having a following geometric meaning;

$$r(f;\overline{V}) := \inf\left\{\delta \;\middle|\; \frac{\delta}{\deg f}\cdot\phi^*H - \pi^*H \text{ is } \mathbb{A}^n\text{-effective in } \mathrm{Pic}_{\mathbb{R}}(V)\right\},$$

where $\overline{V} = (V,\pi)$ is a resolution of indeterminacy of f, $\phi = f\circ\pi$ is the resolved morphism. We may use 'effective' instead of the new term '$\mathbb{A}^n$-effective', but it is not easy to describe the effective cone of V even though $\mathrm{Pic}_{\mathbb{R}}(V)$ is a finite dimensional $\mathbb{R}$-vector space. Moreover, we cannot control the base loci of all effective divisors. So we will take the $\mathbb{A}^n$-effective cone $\mathrm{AE}(V)$ such that 1) $\mathrm{AE}(V)$ is a simple closed subset of the effective cone and 2) every element of $\mathrm{AE}(V)$ has the base locus outside $\mathbb{A}^n$.

Definition 3.1. Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$ such that the center of the successive blowup V lies in H and let

$$\mathrm{Pic}_{\mathbb{R}}(V) = \mathbb{R}H_V \oplus \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_r$$

with the basis $\mathfrak{B} = \{H_V, E_1, \ldots, E_r\}$ described in Proposition 2.4. We say that a divisor D on V is $\mathbb{A}^n$*-effective* if it is linearly equivalent to a nonnegative linear combination of $H_V, E_1, \ldots, E_r$. Moreover, we write $D_1 \succ D_2$ if $D_1 - D_2$ is $\mathbb{A}^n$-effective. We define *the* $\mathbb{A}^n$*-effective cone of* V to be

$$\mathrm{AE}(V) := \{D \mid D \text{ is an } \mathbb{A}^n\text{-effective divisor}\}.$$

Recall Proposition 2.4 provides a certain basis $\mathfrak{B}$ of $\mathrm{Pic}_{\mathbb{R}}(V)$. It implies that the representation of an element in $\mathrm{Pic}_{\mathbb{R}}(V)$ is unique and hence the $\mathbb{A}^n$-effectiveness is well-defined. The next proposition shows some useful properties of $\mathbb{A}^n$-effective divisors which will be important to define the D-ratio of a rational map in the next section.

Proposition 3.2. *Let V be a successive blowup of $\mathbb{P}^n$ with a birational morphism $\pi : V \to \mathbb{P}^n$, let $\mathfrak{B}$ be the basis defined on Proposition 2.4 and let $D, D_1, D_2, D_3 \in \mathrm{Pic}_{\mathbb{R}}(V)$.*

(a) (Effectiveness) *If D is $\mathbb{A}^n$-effective, then D is effective.*
(b) (Boundedness) *If D is $\mathbb{A}^n$-effective, then the Weil height function h_D associated with D is bounded below on the set $\pi^{-1}(\mathbb{A}^n)$:*

$$\pi^{-1}(\mathbb{A}^n) := V \setminus \left(H_V \cup \left(\bigcup_{i=1}^r E_i\right)\right).$$

(c) (Transitivity) *If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 \succ D_3$.*
(d) (Functoriality) *If $\rho : W \to V$ is a monoidal transformation and $D_1 \succ D_2$, then $\rho^*D_1 \succ \rho^*D_2$.*

Proof. (a) It is obvious since an $\mathbb{A}^n$-effective divisor is a nonnegative linear combination of effective divisors on V.

(b) Since D is $\mathbb{A}^n$-effective, it is effective. By the positivity of the Weil height machine [4, Theorem B.3.2(e)], we get that $h_D(P) > \mathcal{O}(1)$ for all $P \in V \setminus |D|$ where $|D|$ is the base locus of D. By the definition of the $\mathbb{A}^n$-effectiveness, $D = p_0 H_V + \sum_{i=1}^r p_i E_i$ for some nonnegative integers p_i's and hence $|D| \subset H_V \cup (\bigcup_{i=1}^r E_i) = \pi^{-1}(H)$.

(c) If $D_1 \succ D_2$ and $D_2 \succ D_3$, then $D_1 - D_2$ and $D_2 - D_3$ are in $\mathrm{AE}(V)$. Since $\mathrm{AE}(V)$ is closed under addition by definition, $D_1 - D_3 = (D_1 - D_2) + (D_2 - D_3) \in \mathrm{AE}(V)$.

(d) Let $\mathrm{Pic}_{\mathbb{R}}(V) = \mathbb{R}H_V \oplus \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_r$, and let W be a blowup of V with a monoidal transformation $\rho : W \to V$. Then $\mathrm{Pic}(W)$ is still an $\mathbb{R}$-vector space:

$$\mathrm{Pic}_{\mathbb{R}}(W) = \mathbb{R}H_V^{\#} \oplus \mathbb{R}E_1^{\#} \oplus \cdots \oplus \mathbb{R}E_r^{\#} \oplus \mathbb{R}F,$$

where $H_V^{\#} = \rho^{\#} H_V$, $E_i^{\#} = \rho^{\#} E_i$ and F is the exceptional divisor of W over V. Moreover, $\rho^* H_V = H_V^{\#} + m_0 F$ and $\rho^* E_i = E_i^{\#} + m_i F$ for some m_i, which are nonnegative integers by Lemma 2.5.

Therefore, for any $\mathbb{A}^n$-effective divisor $D = p_0 H_V + \sum_{i=1}^r p_i E_i \in \mathrm{Pic}_{\mathbb{R}}(V)$,

$$\rho^* D = p_0(\rho^* H_V) + \sum_{i=1}^r p_i(\rho^* E_i) = p_0 H_V^{\#} + \sum_{i=1}^r p_i E_i^{\#} + \left(\sum_{i=0}^r p_i m_i\right) F$$

is $\mathbb{A}^n$-effective on W because p_i's and m_i's are nonnegative integers. □

4. Maximal ratio of coefficient of divisors

In this section, we introduce the main idea of this paper - the D-ratio. In this section, we still fix the basis of the Picard group described in Proposition 2.4.

Definition 4.1. Let $f \in \mathrm{Rat}^n(H)$, let $\overline{V} = (V, \pi)$ be a resolution of indeterminacy of f and let ϕ be the resolved morphism of f on V:

$$\begin{tikzcd} V \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ \mathbb{P}^n \arrow[r, dashed, "f"'] & \mathbb{P}^n. \end{tikzcd}$$

Suppose that

$$\pi^* H = a_0 H_V + \sum_{i=1}^r a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=1}^r b_i E_i.$$

If b_i are nonzero for all i satisfying $a_i \neq 0$, we define *the D-ratio* to be

$$r(f; \overline{V}) := \deg f \cdot \max_{i=0,\ldots,r} \left(\frac{a_i}{b_i}\right).$$

If there is an i satisfying $a_i \neq 0$ and $b_i = 0$, we define

$$r(f;\overline{V}) := \infty.$$

Remark 1. We have some remarks for the definition of the D-ratio.

(a) We may define the D-ratio without degree. However, the author prefers the original one because of two reasons. 1) It shows how much f fails to be an endomorphism. f is an endomorphism if and only if $r(f;\overline{V}) = 1$. And $\mathrm{Preper}_{\mathbb{A}^n}(f)$ is of bounded height if $r(f;\overline{V}) < \deg f$. 2) We have better result for regular polynomial automorphisms in the original definition [8, Theorem 7.1(2)]: $r(f;\overline{V}) = r(f^{-1};\overline{V}) = \deg f \cdot \deg f^{-1}$.

(b) If we have a blowup along C which lies outside H, then we have an exceptional divisor that has trivial contribution to π^*H and ϕ^*H.

(c) Let $f \in \mathrm{Rat}^n(H)$ and let $\overline{V} = (V,\pi)$ be a resolution of indeterminacy of f. Then

$$r(f;\overline{V}) := \min\left\{\delta \;\middle|\; \frac{\delta}{\deg f}\cdot (f\circ\pi)^*H - \pi^*H \succ 0\right\}.$$

Note that $\mathrm{AE}(V)$ is closed by definition so that we can use the minimum.

In fact, the D-ratio $r(f,\overline{V})$ does not depend on the choice of $\overline{V}$, but only on the strong factorization class of $\overline{V}$. It is mainly because Remark 1 guarantees that we have the basis of $\mathrm{Pic}_{\mathbb{R}}(V)$ consisting of the proper transformation of H and all irreducible components of the exceptional divisor of π.

Definition 4.2. Let $\rho : X_1 \dashrightarrow X_2$ be an equivariant birational map. We say *ρ has a strong factorization* if there are birational endomorphisms $\pi_i : \widetilde{X} \to X_i$ which are compositions of monoidal transformations such that the following diagram commute:

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1 \arrow[rr, dashed, "\rho"'] & & X_2. \end{tikzcd}$$

Example 1. Any birational maps between smooth complete surfaces has a strong factorization [14, Chapter IV].

Lemma 4.3. *Let $\overline{V} = (V,\pi_V)$ and $\overline{W} = (W,\pi_W)$ be resolutions of indeterminacy of f with resolved morphisms $\phi_V = f\circ\pi_V$ and $\phi_W = f\circ\pi_W$, respectively. Suppose that $\tau = \pi_V^{-1}\circ\pi_W : W \dashrightarrow V$ has a strong factorization: there is a common successive blowup U of V and W where $\tau_V : U \to V$ and $\tau_W : U \to W$*

are compositions of monoidal transformations.

$$\begin{tikzcd} & U \arrow[dl, "\tau_W"'] \arrow[d, "\phi_U"] \arrow[dr, "\tau_V"] & \\ W \arrow[d, "\pi_W"'] \arrow[dr, "\phi_W"] & {} \arrow[d] & V \arrow[dl, "\phi_V"'] \arrow[d, "\pi_V"] \\ \mathbb{P}^n \arrow[r, dashed, "f"'] & \mathbb{P}^n & \mathbb{P}^n. \arrow[l, dashed, "f"] \end{tikzcd}$$

Then $r(f;\overline{V}) = r(f;\overline{W})$.

Proof. Suppose

$$\pi_V^* H = a_0 H_V + \sum_{i=1}^r a_i E_i \quad \text{and} \quad \phi_V^* H = b_0 H_V + \sum_{i=1}^r b_i E_i.$$

First, consider the case that W is a successive blowup of V. Suppose that $\rho : W \to V$ is a composition of monoidal transformations:

$$\begin{tikzcd} \mathbb{P}^n \arrow[d, dashed, "f"'] & V \arrow[l, "\pi_V"'] \arrow[dl, "\phi_V"] & W \arrow[l, "\rho"'] \arrow[dll, "\phi_W"] \\ \mathbb{P}^n. & & \end{tikzcd}$$

Since $\mathrm{Pic}_{\mathbb{R}}(V) = \mathbb{R}H_V \oplus \mathbb{R}E_1 \oplus \cdots \oplus \mathbb{R}E_r$, we get

$$\mathrm{Pic}_{\mathbb{R}}(W) = \mathbb{R}H_V^{\#} \oplus \mathbb{R}E_1^{\#} \oplus \cdots \oplus \mathbb{R}E_r^{\#} \oplus \mathbb{R}F_1 \oplus \cdots \oplus \mathbb{R}F_s,$$

where $H_V^{\#} = \rho^{\#} H_V$, $E_i^{\#} = \rho^{\#} E_i$ and F_j are the exceptional divisors of monoidal transformations in ρ. Moreover, we may assume that

$$\rho^* H_V = H_V^{\#} + \sum_{j=1}^s m_{0j} F_j \quad \text{and} \quad \rho^* E_i = E_i^{\#} + \sum_{j=1}^s m_{ij} F_j$$

for some integers m_{ij} which are nonnegative by Lemma 2.5. By assumption, $\phi_W = \phi_V \circ \rho$ and hence

$$\begin{aligned} \pi_W^* H = \rho^* \pi^* H &= \rho^* \left(a_0 H_V + \sum_{i=1}^r a_i E_i \right) \\ &= a_0 H_V^{\#} + \sum_{i=1}^r a_i E_i^{\#} + \sum_{j=1}^s \left(\sum_{i=0}^r a_i m_{ij} \right) F_j \end{aligned}$$

and

$$\phi_W^* H = \rho^* \phi_V^* H = \rho^* \left(b_0 H_V + \sum_{i=1}^r b_i E_i \right)$$

$$= b_0 H_V^{\#} + \sum_{i=1}^{r} b_i E_i^{\#} + \sum_{j=1}^{s} \left(\sum_{i=0}^{r} b_i m_{ij} \right) F_j.$$

Note that we assume $a_i > 0$ (Remark 1(b)). If $b_i = 0$ for some i, then $r(f, \overline{V}) = r(f, \overline{W}) = \infty$ by the definition of the D-ratio. So we may assume $b_i > 0$ for all i. Then also by the definition of the D-ratio, we obtain the following inequality:

$$r(f; \overline{V}) = \deg f \cdot \max_i \left(\frac{a_i}{b_i} \right) \geq \deg f \cdot \frac{a_i}{b_i} \quad \text{for all } i. \tag{2}$$

Because of Lemma 2.5 and the fact $b_i > 0$, we have

$$\sum_{i=0}^{r} b_i m_{ij} \geq \min_{0 \leq i \leq r} b_i \cdot \sum_{i=0}^{r} m_{ij} > 0 \quad \text{for all } j$$

and hence all coefficients of $\phi_W^* H$ are positive. Thus, we get

$$r(f; \overline{W}) = \deg f \cdot \max \left(\max_i \left(\frac{a_i}{b_i} \right), \max_j \left(\frac{\sum_{i=0}^{r} a_i m_{ij}}{\sum_{i=0}^{r} b_i m_{ij}} \right) \right).$$

Moreover, due to (2), we get the following which provides the desired result:

$$\max_j \left(\frac{\sum_{i=0}^{r} a_i m_{ij}}{\sum_{i=0}^{r} b_i m_{ij}} \right) \leq \max_j \left(\frac{\sum_{i=0}^{r} \frac{r(f,\overline{V})}{\deg f} b_i m_{ij}}{\sum_{i=0}^{r} b_i m_{ij}} \right) = \frac{r(f; \overline{V})}{\deg f} = \max_i \left(\frac{a_i}{b_i} \right).$$

Now let $\overline{V} = (V, \pi_V)$ and $\overline{W} = (W, \pi_W)$ be resolutions of indeterminacy of f allowing strong factorization: there is a common blowup U of V and W such that $\tau_V : U \to V$ and $\tau_W : U \to W$ are compositions of monoidal transformations. Then $\overline{U} = (U, \pi_U := \pi_V \circ \tau_V)$ is still a resolution of indeterminacy of f. Then the previous argument implies

$$r(f; \overline{V}) = r(f; \overline{U}) = r(f; \overline{W}). \qquad \square$$

If we can connect two resolutions of indeterminacy with a sequence of strong factorizations, then they will give the same D-ratios. We define such case as follows:

Definition 4.4. Let $\overline{V} = (V, \pi_V)$ and $\overline{W} = (W, \pi_W)$ be resolutions of indeterminacy of a rational map $f \in \mathrm{Rat}^n(H)$. We say $\overline{V}$ *and* $\overline{W}$ *are in the same strong factorization class* if there is a sequence $\overline{V} = \overline{V}_0, \overline{V}_1 = (V_1, \pi_1), \ldots, \overline{V}_k = \overline{W}$ of resolutions of indeterminacy of f such that $\pi_i^{-1} \circ \pi_{i+1}$ has a strong factorization for all i.

Corollary 4.5. *Let $\overline{V} = (V, \pi_V)$ and $\overline{W} = (W, \pi_W)$ be resolutions of indeterminacy of a rational map $f : \mathbb{P}^n(\overline{\mathbb{Q}}) \dashrightarrow \mathbb{P}^n(\overline{\mathbb{Q}})$, in the same strong factorization class. Then $r(f; \overline{V}) = r(f; \overline{W})$. In particular, if $n = 2$, then $\pi_V^{-1} \circ \pi_W$ is a birational map between smooth surfaces and hence it has a strong factorization so that $\overline{V} = (V, \pi_V)$ and $\overline{W} = (W, \pi_W)$ are in the same strong factorization class and hence $r(f; \overline{V}) = r(f; \overline{W})$.*

We will find some information of f by observing $r(f;\overline{V})$. For convenience, we will say f *is a polynomial map with respect to* H if $f(\mathbb{A}^n) \subset \mathbb{A}^n$. If f is a polynomial map with respect to H, then we evaluate $x_0 = 1$ to get $f = (f_1, \dots, f_n)$, where $f_i \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$.

Proposition 4.6. *Let $f, g \in \mathrm{Rat}^n(H)$ be rational maps defined over $\overline{\mathbb{Q}}$. Then*

(a) *$r(f;\overline{V}) = 1$ if f is an endomorphism on $\mathbb{P}^n$.*
(b) *If g is an endomorphism, then $r(g \circ f;\overline{V}) = r(f;\overline{V})$ for any resolution of indeterminacy $\overline{V}$ of f.*
(c) *Let f be a polynomial map with respect to H and let $\phi^* H = b_0 H^{\#} + \sum b_i E_i$. Then $b_0 = \deg f$.*
(d) *If f is a polynomial map with respect to H, then $r(f;\overline{V}) \geq 1$.*

Remark 2. In fact, (a) and (d) will be strengthened in Corollary 5.2: $r(f;\overline{V}) = 1$ if and only if f is an endomorphism, and $r(f;\overline{V}) \geq 1$ for any $f \in \mathrm{Rat}^n(H)$.

Proof of Proposition 4.6. (a) When f is an endomorphism, then $(\mathbb{P}^n, \mathrm{id})$ is a resolution of indeterminacy of f. Thus,

$$\mathrm{id}^* H = H \quad \text{and} \quad f^* H = \deg f \cdot H$$

and hence

$$r\big(f; (\mathbb{P}^n, \mathrm{id})\big) = \deg f \cdot \frac{1}{\deg f} = 1.$$

If $\overline{V} = (V, \pi)$ is an arbitrary resolution of indeterminacy of f, then V is a successive blowup of $\mathbb{P}^n$ so that

$$r\big(f; (\mathbb{P}^n, \mathrm{id})\big) = r(f;\overline{V})$$

because of Lemma 4.3. We will show that the equality only holds if f is an endomorphism in Corollary 5.2.

(b) Let (V, π) be a resolution of indeterminacy of f and suppose that

$$\pi^* H = a_0 H_V + \sum_{i=0}^{r} a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=0}^{r} b_i E_i.$$

We consider the following diagram:

$$\begin{tikzcd}
V \arrow[d, "\pi"'] \arrow[dr, "\phi"] & & \\
\mathbb{P}^n \arrow[r, dashed, "f"'] & \mathbb{P}^n \arrow[r, "g"] & \mathbb{P}^n.
\end{tikzcd}$$

We easily get $\phi^* g^* H = \deg g \cdot \left(b_0 H_V + \sum_{i=1}^{r} b_i E_i \right)$ and hence

$$\frac{r(g \circ f;\overline{V})}{\deg(g \circ f)} = \max_i \left(\frac{a_i}{\deg g \cdot b_i} \right) = \frac{1}{\deg g} \cdot \max_i \left(\frac{a_i}{b_i} \right) = \frac{r(f;\overline{V})}{\deg f \deg g}.$$

Furthermore, $\deg(g \circ f) = \deg f \cdot \deg g$ since g is an endomorphism. Therefore, we have the desired result.

(c) We may assume x_0 is a uniformizer at H, which generates the ideal of regular functions which becomes 0 on H. Then we have the following equality;

$$\phi^* H = \mathrm{ord}_{H_V}(x_0 \circ \phi) \cdot H_V + \sum_{i=1}^{r} \mathrm{ord}_{E_i}(x_0 \circ \phi) \cdot E_i.$$

Since f is a polynomial map, without loss of generality, we can say $H = (x_0 = 0)$ and $f(x_0, \ldots, x_n) = [x_0^d, \ldots, f_n(x_0, \ldots, x_n)]$. Moreover, since $\phi = f \circ \pi$ on $|H_V| \setminus \cup_{i=1}^r |E_i|$ and $\dim(|H_V| \cap \cup_{i=1}^r |E_i|) \leq n - 2 < n - 1 = \dim H_V$, we can claim that $\mathrm{ord}_{H_V}(x_0 \circ \phi) = \mathrm{ord}_{H_V}(x_0 \circ f)$. Thus, we obtain $b_0 = \mathrm{ord}_{H_V}(x_0 \circ f) = \mathrm{ord}_H(x_0^d) = d$.

(d) Let $\overline{V} = (V, \pi)$ be a resolution of indeterminacy of f with the resolved morphism $\phi = f \circ \pi$. We may assume that the underlying set of the center of blowup is $I(f)$ by Theorem 2.2. Suppose that

$$\pi^* H = a_0 H_V + \sum_{i=1}^{r} a_i E_i \quad \text{and} \quad \phi^* H = b_0 H_V + \sum_{i=1}^{r} b_i E_i.$$

We can easily check that $a_0 = 1$: because $\pi(E_i) \subset I(f)$ and $I(f)$ is a closed set of codimension at least 2, we have $\pi_* E_i = 0$. Thus,

$$\pi_* \pi^* H = \pi_{V*}\left(a_0 H_V + \sum_{i=1}^{r} a_i E_i\right) = \pi_* a_0 H_V = a_0 H.$$

On the other hand, choose another hyperplane H' which satisfies $I(f) \not\subset H'$. Since π is one-to-one outside of $\pi^{-1}(I(f))$, we get

$$\pi_* \pi^* H = \pi_* \pi^* H' = H' = H.$$

Therefore, $\pi_* H_V = H$ and $a_0 = 1$. Combine it with (c) to get

$$r(f; \overline{V}) = \deg f \cdot \max_i \left(\frac{a_i}{b_i}\right) \geq \deg f \cdot \frac{a_0}{b_0} = \deg f \cdot \frac{1}{\deg f} = 1. \qquad \square$$

5. Upper bounds of height for rational maps

In Introduction, we say that Northcott's Theorem does not work for rational maps and hence we need other tool to examine arithmetic relation between P and $f(P)$. In this section, we prove Theorem A, which is a tool to examine dynamical properties of $f \in \mathrm{Rat}^n(H)$.

Theorem 5.1. *Let $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map defined over $\overline{\mathbb{Q}}$ such that indeterminacy only happens on a hyperplane H and let $r(f; \overline{V})$ be the D-ratio of f, associated with a resolution of indeterminacy $\overline{V}$ of f. Then there is a constant C such that the following inequality holds.*

$$\frac{r(f; \overline{V})}{\deg f} \cdot h\big(f(P)\big) + C > h(P) \quad \text{for all } P \in \mathbb{P}^n(\overline{\mathbb{Q}}) \setminus H.$$

Proof. Let (V,π) be a resolution of indeterminacy of f with the resolved morphism $\phi = f\circ\pi_V$ and $r(f;\overline{V})$ be the D-ratio of f associated with V. Suppose that

$$\pi^*H = a_0H_V + \sum_{i=1}^{r} a_iE_i \quad \text{and} \quad \phi^*H = b_0H_V + \sum_{i=1}^{r} b_iE_i,$$

where $\mathfrak{B} = \{H_V, E_1, \ldots, E_r\}$ is the basis described in Proposition 2.4.

Let $D := \frac{r(f;\overline{V})}{\deg f}\phi^*H - \pi^*H$. By the definition of the D-ratio, D is $\mathbb{A}^n$-effective. So the height function associated with D,

$$h_D = \frac{r(f;\overline{V})}{\deg f} h_{\phi^*H} - h_{\pi^*H}$$

is bounded below on $\pi^{-1}(\mathbb{A}^n)$ by Proposition 3.2(b). Hence we have the following inequality:

$$\frac{r(f;\overline{V})}{\deg f} h\big(\phi(Q)\big) + C > h\big(\pi(Q)\big) \quad \text{for all } Q\in\pi^{-1}\big(\mathbb{A}^n\big).$$

Therefore, for $P\in\mathbb{A}^n(\overline{\mathbb{Q}})$, taking $Q=\pi^{-1}(P)$, we get the desired result. □

As an easy consequence of Theorem 5.1, we can strengthen Proposition 4.6 (a), (d) for any rational map $f\in \mathrm{Rat}^n(H)$.

Corollary 5.2. *Let $f \in \mathrm{Rat}^n(H)$ be a rational map defined over $\overline{\mathbb{Q}}$. Then $r(f;\overline{V}) \geq 1$. Moreover, $r(f;\overline{V}) = 1$ if only if f is an endomorphism on $\mathbb{P}^n$.*

Proof. By Theorem 5.1, we have

$$\frac{r(f;\overline{V})}{\deg f} h\big(f(P)\big) + C > h(P)$$

on $\mathbb{A}^n(\overline{\mathbb{Q}})$. In addition, we have

$$h(P) > \frac{1}{\deg f} h\big(f(P)\big) - C'$$

on $\mathbb{P}^n \setminus I(f)$ by the triangle inequality. (For details, see [4, Theorem B.2.5].) Therefore, we have $r(f;\overline{V}) \geq 1$ for any rational map $f\in\mathrm{Rat}^n(H)$ and for any resolution of indeterminacy $\overline{V}$ of f.

Moreover, by Proposition 4.6(a) and by Theorem 5.1, $r(f;\overline{V}) = 1$ only if f is an endomorphism. □

We apply Theorem 5.1 to study dynamical properties of a polynomial map. We can view a polynomial map $f:\mathbb{A}^n\to\mathbb{A}^n$ as an element $f\in\mathrm{Rat}^n(H)$ such that $f(\mathbb{A}^n)\subset\mathbb{A}^n$. Thus, we can find $r(f;\overline{V})$ and apply Theorem 5.1 at all forward images $f^m(P)\in\mathbb{A}^n$ of P. In fact, the inequality in Theorem 5.1 is only valid on $\mathbb{A}^n(\overline{\mathbb{Q}})$, forcing f to be a polynomial for dynamical application.

Theorem 5.3. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map, defined over $\overline{\mathbb{Q}}$, such that $r(f;\overline{V}) < \deg f$. Then*

$$\mathrm{Preper}_{\mathbb{A}^n}(f) = \big\{P \in \mathbb{A}^n(\overline{\mathbb{Q}}) \mid f^l(P) = f^m(P) \text{ for some } l \neq m\big\}$$

is a set of bounded height.

Proof. Let $u = \frac{r(f;\overline{V})}{\deg f} < 1$. By Theorem 5.1, we have

$$u \cdot h\big(f(P)\big) > h(P) - C \quad \text{for all } P \in \mathbb{A}^n(\overline{\mathbb{Q}}). \tag{5}$$

Then the iteration of (5) provides

$$u^l \cdot h\big(f^l(P)\big) > u^{l-1}\left[h\big(f^{l-1}(P)\big) - C\right] \cdots > h(P) - \left\{1 + u + \cdots + u^{l-1}\right\} C.$$

Hence, we have

$$u^l \cdot h\big(f^l(P)\big) > h(P) - \frac{C}{1-u}$$

for all $l > 0$. Moreover, if P is a preperiodic point of f, then $h\big(f^l(P)\big)$ is bounded and hence $\lim_{l\to\infty} u^l \cdot h\big(f^l(P)\big) = 0$. So we get

$$\frac{C}{1-u} \geq h(P) \quad \text{for all } P \in \mathrm{Preper}_{\mathbb{A}^n}(f).$$

□

Example 2. Let

$$f(x, y) = (x^3 + y, x + y^2).$$

Consider f as rational map on $\mathbb{P}^2$ by homogenization:

$$f(X, Y, Z) = [X^3 + YZ^2, XZ^2 + Y^2Z, Z^3].$$

Then after three blowing-ups, we get a resolution of indeterminacy of f:

$$\begin{aligned} f_1(x, z)[x_1, z_1] &= [x_1x^2 + z_1z, z_1xz + z_1, z_1z^2] \\ f_2(x, z_1)[x_2, z_2] &= [x_2x + x_2z_1^2, x_2z_1^2x + z_2, x_2z_1^3x] \quad \text{and} \\ f_3(x, z_2)[x_3, z_3] &= [x_3 + z_3z_2x^2, x_3z_3^2x^2 + z_3, x_3z_2^3x^3]. \end{aligned}$$

Note that we choose appropriate affine pieces for each step to make it simple. Now we can calculate two pullbacks

$$\pi^*H = H_V + E_1 + 2E_2 + 3E_3, \quad \phi^*H = 3H_V + 2E_1 + 4E_2 + 6E_3$$

and the D-ratio $r(f;\overline{V}) = 3/2 < 3$. Therefore, the preperiodic points of f on $\mathbb{A}^n(\overline{\mathbb{Q}})$ are of bounded height.

Example 3. The condition $r(f;\overline{V}) < \deg f$ in Theorem 5.3 is sharp: let

$$f(x, y) = (x, y^2).$$

Then after two blowing-ups along points, we get a resolution of indeterminacy of f. And we have

$$\pi^*H = H_V + E_1 + 2E_2, \quad \phi^*H = 2H_V + E_1 + 2E_2.$$

Thus, $r(f) = 2 = \deg f$. And it has infinitely many integral fixed points $(n, 0)$. Thus, $\mathrm{Preper}_{\mathbb{A}^n}(f)$ is not bounded.

Example 4. Let

$$f(x,y) = (y, x^2 + y).$$

Then after two blowing-ups, the indeterminacy of f is resolved. And, we have

$$\pi^* H = H_V + E_1 + 2E_2 \quad \text{and} \quad \phi^* H = 2H_v + E_1 + 2E_2$$

so that $r(f) = 2 = \deg f$. But we can check that $f^2(x,y) = (x^2+y, x^2+y^2+y)$ extends to a morphism. Thus, $r(f^2) = 1 < 2 = \deg f^2$. Therefore, preperiodic points of f on $\mathbb{A}^n(\overline{\mathbb{Q}})$ are of bounded height.

Corollary 5.4. *Let $f : \mathbb{A}^n \to \mathbb{A}^n$ be a polynomial map, defined over $\overline{\mathbb{Q}}$. If there is some number N satisfying $r(f^N; \overline{V}) < \deg(f^N)$, then* $\mathrm{Preper}_{\mathbb{A}^n}(f)$ *is a set of bounded height.*

Proof. It is clear since $\mathrm{Preper}_{\mathbb{A}^n}(f) = \mathrm{Preper}_{\mathbb{A}^n}(f^N)$. □

Example 5. Consider the Nagata map

$$f(x,y,z) = (x + (x^2 - yz)z,\ y + 2(x^2 - yz)x + (x^2 - yz)^2 z,\ z),$$

which is a polynomial map on $\mathbb{P}^3$. It has infinitely many rational fixed points: any point on the quadratic curve $x^2 = yz$ is a fixed point of f. So $\mathrm{Preper}_{\mathbb{A}^n}(f)$ is not of bounded height. Furthermore, the N-th iteration of f is still a polynomial map of degree 5;

$$f^N(x,y,z) = (x + N(x^2 - yz)z,\ y + 2N(x^2 - yz)x + N^2(x^2 - yz)^2 z,\ z).$$

So it follows from Corollary 5.4 that $r(f^N, \overline{V}_N) \geq \deg f^N = 5$ for any resolution of indeterminacy $\overline{V}_N$ of f^N.

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