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ON SUBMANIFOLDS OF ALMOST PSEUDO SYMMETRIC MANIFOLDS

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Abstract. The object of the present paper is to study a decomposable $(APS)_n$ and hypersurfaces of $(APS)_n$. Also, the existence of a decomposable $(APS)_n$ is ensured by a proper example.

1. Introduction

In 1967, Sen and Chaki [6] studied certain curvature restrictions on a certain kind of conformally flat manifold of class one and they obtained the following expressions of the covariant derivative of curvature tensor R:

$$\begin{aligned}(\nabla_X R)(Y,Z,V,W) =& 2A(X)R(Y,Z,V,W) + A(Y)R(X,Z,V,W)\\ &+ A(Z)R(Y,X,V,W) + A(V)R(Y,Z,X,W)\\ &+ A(W)R(Y,Z,V,X),\end{aligned}$$

where A is a nowhere vanishing 1-form. Later in 1987, Chaki [1] called the manifold whose curvature tensor satisfies the above relation as a pseudo symmetric manifold. If $A = 0$, then the manifold reduces to a symmetric manifold in the sense of Cartan. A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be almost pseudo symmetric if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z,V,W) =& [A(X) + B(X)]R(Y,Z,V,W)\\ &+ A(Y)R(X,Z,V,W) + A(Z)R(Y,X,V,W)\\ &+ A(V)R(Y,Z,X,W) + A(W)R(Y,Z,V,X),\end{aligned} \tag{1.1}$$

where A and B are nowhere vanishing 1-forms. An n-dimensional almost pseudo symmetric manifold is denoted by $(APS)_n$. In [3], $(APS)_n$ was

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introduced by De and Gazi, and they showed the physical significance of such a manifold in the general theory of relativity. Moreover, they proved its existence by several examples. If $A = B$ in (1.1), then the manifold reduces to a pseudo symmetric manifold. The purpose of this paper is to investigate some properties of a decomposable $(APS)_n$ and hypersurfaces of $(APS)_n$. Section 2 is concerned with preliminaries. Section 3 deals with a study of hypersurfaces of $(APS)_n$. In the last section, we study a decomposable $(APS)_n$ and provide a proper example of a decomposable $(APS)_n$.

2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; y^\alpha\}$ and $(\bar{M}^{n-1}, \bar{g})$ a hypersurface of (M^n, g) covered by a system of coordinate neighborhoods $\{V; x^i\}$. Let $y^\alpha = y^\alpha(x^i)$ be the parametric representation of the hypersurface $\bar{M}^{n-1}$ in M^n, where Greek indices take the values $1, 2, ..., n$ and Latin indices take the values $1, 2, ..., n-1$. Then we have

$$\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Here we adopt the Einstein convention, that is, when an index variable appears once in an upper and once in a lower position in a term, it implies summation of that term over all the values of the index. Let N^α be a local unit normal to $(\bar{M}^{n-1}, \bar{g})$. Then we have the relations

$$g_{\alpha\beta} N^\alpha \frac{\partial y^\beta}{\partial x^j} = 0, g_{\alpha\beta} N^\alpha N^\beta = 1, g^{\alpha\beta} = \bar{g}^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + N^\alpha N^\beta.$$

The structure equations of Gauss, Codazzi for a hypersurface $(\bar{M}^{n-1}, \bar{g})$ of (M^n, g) can be respectively written as

$$\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \bar{\omega}_{il}\bar{\omega}_{jk} - \bar{\omega}_{ik}\bar{\omega}_{jl}, \tag{2.1}$$

$$R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta = \bar{\omega}_{jk;i} - \bar{\omega}_{ik;j}, \tag{2.2}$$

where $\bar{R}_{ijkl}$ and $R_{\alpha\beta\gamma\delta}$ are the curvature tensors of $(\bar{M}^{n-1}, \bar{g})$ and (M^n, g) respectively, and $\bar{\omega}_{ij}$ is the second fundamental form of $(\bar{M}^{n-1}, \bar{g})$.

The hypersurface $(\bar{M}^{n-1}, \bar{g})$ is said to be a totally umbilic hypersurface of (M^n, g) [2] if its second fundamental form $\bar{\omega}_{ij}$ satisfies

$$\bar{\omega}_{ij} = H\bar{g}_{ij}, (\frac{\partial y^\alpha}{\partial x^i})_{;j} = \bar{g}_{ij} H N^\alpha, \tag{2.3}$$

where H denotes the mean curvature of $(\bar{M}^{n-1}, \bar{g})$ defined by $H = \frac{1}{n-1}\bar{g}^{ij}\bar{\omega}_{ij}$, and semicolon ";" indicates covariant differentiation. In particular, if H=0, then the totally umbilic hypersurface $(\bar{M}^{n-1}, \bar{g})$ is called a totally geodesic hypersurface of (M^n, g) [2]. The equations of Weingarten, Gauss and Codazzi for a totally umbilic hypersurface $(\bar{M}^{n-1}, \bar{g})$ of (M^n, g) are respectively obtained as

$$N^{\alpha}_{;i} = -H\frac{\partial y^{\alpha}}{\partial x^i}, \tag{2.4}$$

$$\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}\frac{\partial y^{\delta}}{\partial x^l} + H^2(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}), \tag{2.5}$$

$$R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}N^{\delta} = H_{;i}\bar{g}_{jk} - H_{;j}\bar{g}_{ik}. \tag{2.6}$$

3. Hypersurfaces of $(APS)_n$

In this section we deal with some hypersurfaces of $(APS)_n$. At first, we can state the following Lemma which we need for the proofs of main results in this section, and for the sake of completeness, we have provided the proof of this one which was already appeared in [5].

Lemma 3.1 ([5])**.** Let $(\bar{M}^{n-1}, \bar{g})$ be a totally umbilic hypersurface of (M^n, g). Then we have

$$\begin{aligned}\bar{R}_{ijkl;p} =& R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}\frac{\partial y^{\delta}}{\partial x^l}\frac{\partial y^{\mu}}{\partial x^p} + HH_{;i}(\bar{g}_{lp}\bar{g}_{jk} - \bar{g}_{kp}\bar{g}_{jl})\\ &+ HH_{;j}(\bar{g}_{il}\bar{g}_{kp} - \bar{g}_{ik}\bar{g}_{lp}) + HH_{;k}(\bar{g}_{jp}\bar{g}_{li} - \bar{g}_{ip}\bar{g}_{lj})\\ &+ HH_{;l}(\bar{g}_{ip}\bar{g}_{kj} - \bar{g}_{jp}\bar{g}_{ki}) + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}),\end{aligned} \tag{3.1}$$

$$\begin{aligned}R_{\alpha\beta\gamma\delta;\mu}&\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}N^{\delta}\frac{\partial y^{\mu}}{\partial x^p} + H(R_{\alpha\beta\gamma\delta}N^{\alpha}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}N^{\delta}\bar{g}_{ip}\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^i}N^{\beta}\frac{\partial y^{\gamma}}{\partial x^k}N^{\delta}\bar{g}_{jp} + R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}N^{\gamma}N^{\delta}\bar{g}_{kp}\\ &- R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j}\frac{\partial y^{\gamma}}{\partial x^k}\frac{\partial y^{\delta}}{\partial x^p}) = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}.\end{aligned} \tag{3.2}$$

Proof. Differentiating (2.5) covariantly, we have

$$\begin{aligned}\bar{R}_{ijkl;p} =& R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\mu}{\partial x^p}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta}(\frac{\partial y^\alpha}{\partial x^i})_{;p}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}(\frac{\partial y^\beta}{\partial x^j})_{;p}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}(\frac{\partial y^\gamma}{\partial x^k})_{;p}\frac{\partial y^\delta}{\partial x^l}\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}(\frac{\partial y^\delta}{\partial x^l})_{;p} + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).\end{aligned}$$

By virtue of (2.3) and the last relation, we obtain

$$\begin{aligned}\bar{R}_{ijkl;p} =& R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\mu}{\partial x^p}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + \bar{g}_{ip}HR_{\alpha\beta\gamma\delta}N^\alpha\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\\ &+ \bar{g}_{jp}HR_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + \bar{g}_{kp}HR_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}N^\gamma\frac{\partial y^\delta}{\partial x^l}\\ &+ \bar{g}_{lp}HR_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).\end{aligned}$$

It follows from (2.6) that the last relation reduces to

$$\begin{aligned}\bar{R}_{ijkl;p} =& R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p} + HH_{;i}(\bar{g}_{lp}\bar{g}_{jk} - \bar{g}_{kp}\bar{g}_{jl})\\ &+ HH_{;j}(\bar{g}_{il}\bar{g}_{kp} - \bar{g}_{ik}\bar{g}_{lp}) + HH_{;k}(\bar{g}_{jp}\bar{g}_{li} - \bar{g}_{ip}\bar{g}_{lj})\\ &+ HH_{;l}(\bar{g}_{ip}\bar{g}_{kj} - \bar{g}_{jp}\bar{g}_{ki}) + 2HH_{;p}(\bar{g}_{il}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}).\end{aligned}$$

On the other hand, differentiating (2.6) covariantly, we get

$$\begin{aligned}&R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\mu}{\partial x^p}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta + R_{\alpha\beta\gamma\delta}(\frac{\partial y^\alpha}{\partial x^i})_{;p}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}(\frac{\partial y^\beta}{\partial x^j})_{;p}\frac{\partial y^\gamma}{\partial x^k}N^\delta + R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}(\frac{\partial y^\gamma}{\partial x^k})_{;p}N^\delta\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta_{;p} = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}.\end{aligned}$$

Taking account of (2.3), (2.4) and the last relation, we have

$$\begin{aligned}&R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} + H(R_{\alpha\beta\gamma\delta}N^\alpha\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\bar{g}_{ip}\\ &+ R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta\frac{\partial y^\gamma}{\partial x^k}N^\delta\bar{g}_{jp} + R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}N^\gamma N^\delta\bar{g}_{kp}\\ &- R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^p}) = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}.\end{aligned}$$

This completes the proof. □

Theorem 3.2. Let (M^n, g) be an $(APS)_n$. If $(\bar{M}^{n-1}, \bar{g})$ is a totally geodesic hypersurface of (M^n, g), then the manifold $(\bar{M}^{n-1}, \bar{g})$ is an $(APS)_{n-1}$.

Proof. By virtue of $H = 0$, we have from (3.1)

$$\bar{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}.$$

Since (M^n, g) is an $(APS)_n$, the next relation yields from (1.1)

$$\begin{aligned}\bar{R}_{ijkl;p} &= (A_\mu + B_\mu)R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}\\ &\quad + A_\alpha R_{\mu\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p} + A_\beta R_{\alpha\mu\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}\\ &\quad + A_\gamma R_{\alpha\beta\mu\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p} + A_\delta R_{\alpha\beta\gamma\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}.\end{aligned}$$

Because of (2.5) and $H = 0$, the last relation reduces to

$$\bar{R}_{ijkl;p} = (A_p + B_p)\bar{R}_{ijkl} + A_i\bar{R}_{pjkl} + A_j\bar{R}_{ipkl} + A_k\bar{R}_{ijpl} + A_l\bar{R}_{ijkp},$$

showing that the manifold is an $(APS)_{n-1}$. The proof is completed. □

Note that $A^\sharp$ is a vector field associated with the 1-form A in (1.1), that is, $g(A^\sharp, V) = A(V)$. Now we can state the following.

Theorem 3.3. Let (M^n, g) be an $(APS)_n$. If $(\bar{M}^{n-1}, \bar{g})$ is a totally geodesic hypersurface of (M^n, g), then we have $g(A^\sharp, N) = 0$.

Proof. Taking account of $H = 0$, we obtain from (3.2)

$$R_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} = 0.$$

Since (M^n, g) is an $(APS)_n$, the next relation yields from (1.1)

$$\begin{aligned}&(A_\mu + B_\mu)R_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} + A_\alpha R_{\mu\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p}\\ &\quad + A_\beta R_{\alpha\mu\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} + A_\gamma R_{\alpha\beta\mu\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p}\\ &\quad + A_\delta R_{\alpha\beta\gamma\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p} = 0.\end{aligned}$$

Because of (2.6) and $H = 0$, the last relation reduces to

$$A_\delta N^\delta R_{\alpha\beta\gamma\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\mu}{\partial x^p} = 0.$$

It follows from (2.5) and $H = 0$ that

$$A_\delta N^\delta \bar{R}_{ijkp} = 0,$$

which implies that either $A_\delta N^\delta = 0$ or $\bar{R}_{ijkp} = 0$. According to Theorem 3.2 and the definition of $(APS)_{n-1}$, $(\bar{M}^{n-1}, \bar{g})$ should be non-flat. Therefore we have $A_\delta N^\delta = 0$ and this completes the proof. □

4. Decomposable $(APS)_n$

Let (M^n, g) be a Riemannian product manifold $(M^p \times M^{n-p}, \widehat{g} + \widetilde{g})$. In local coordinates, we adopt the Latin indices (resp. the Greek indices) for tensor components which are constructed on $(M^p, \widehat{g})$ (resp. $(M^{n-p}, \widetilde{g})$). Therefore, the Latin indices take the values from $1, ..., p$ whereas the Greek indices run over the range $p + 1, ..., n$. Now we can state the following.

Theorem 4.1. Let a Riemannian manifold

$$(M^n,\, g) = (M^p \times M^{n-p},\, \widehat{g} + \widetilde{g}),\ p \geq 2$$

be an $(APS)_n$, $n \geq 4$. Then either one decomposition manifold $(M^p, \widehat{g})$ is flat or the other decomposition manifold $(M^{n-p}, \widetilde{g})$ is locally symmetric.

Proof. Since any tensor components of R and its covariant derivatives with both Latin and Greek indices together should be zero, we have from (1.1) and $R_{ijkl;\alpha} = 0$

$$0 = (A_\alpha + B_\alpha) R_{ijkl}. \tag{4.1}$$

Similarly, from (1.1) and $R_{\alpha jkl;i} = 0$ it follows that

$$0 = A_\alpha R_{ijkl}. \tag{4.2}$$

Taking account of (4.1) and (4.2), we get either

$$A_\alpha = B_\alpha = 0 \tag{4.3}$$

or

$$R_{ijkl} = 0, \tag{4.4}$$

From (1.1) and (4.3) it follows that

$$R_{\alpha\beta\gamma\delta;\mu} = 0.$$

Therefore we show that one decomposition manifold $(M^p, \widehat{g})$ is flat or the other decomposition manifold $(M^{n-p}, \widetilde{g})$ is locally symmetric. This completes the proof. □

Example 4.2. Let (R_+^{n+4}, g) $(n \geq 2)$ be a Riemannian manifold given by

$$R_+^{n+4} = \{(x^1, x^2, x^3, x^4, ..., x^{n+4}) | x^4 > 0\}$$

and

$$g = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 + (dx^5)^2 + ... + (dx^{n+4})^2.$$

This kind of metric was appeared in [4]. In the metric described as above, the only nonvanishing components for the Christoffel symbols Γ_{ij}^k, the curvature tensors R_{ijkl} and their covariant derivatives $R_{ijkl;p}$ are

$$\Gamma_{14}^1 = \Gamma_{24}^2 = \Gamma_{34}^3 = \frac{2}{3x^4},$$
$$\Gamma_{11}^4 = \Gamma_{22}^4 = \Gamma_{33}^4 = \frac{-2}{3}(x^4)^{\frac{1}{3}},$$
$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}},$$
$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}},$$
$$R_{1221;4} = R_{1331;4} = R_{2332;4} = \frac{-24}{27}(x^4)^{-\frac{1}{3}},$$
$$R_{1441;4} = R_{2442;4} = R_{3443;4} = \frac{12}{27}(x^4)^{-\frac{5}{3}}.$$

Let us define the associated 1-forms A and B of (1.1) on (R_+^{n+4}, g) as follows:

$$A_1 = \frac{1}{x^4}, A_2 = \frac{2}{x^4}, A_3 = \frac{3}{x^4}, A_4 = ... = A_{n+4} = 0,$$
$$B_1 = \frac{-3}{x^4}, B_2 = \frac{-6}{x^4}, B_3 = \frac{-9}{x^4}, B_4 = \frac{-2}{x^4}, B_5 = \cdots = B_{n+4} = 0.$$

It is easy to see that (1.1) holds on (R_+^{n+4}, g). For instance,

$$\begin{aligned} R_{1221;4} &= \frac{-24}{27}(x^4)^{-\frac{1}{3}} \\ &= (A_4 + B_4)R_{1221} + A_1 R_{4221} + A_2 R_{1421} + A_2 R_{1241} + A_1 R_{1224}, \end{aligned}$$

$$\begin{aligned} R_{1441;4} &= \frac{12}{27}(x^4)^{-\frac{5}{3}} \\ &= (A_4 + B_4)R_{1441} + A_1 R_{4441} + A_4 R_{1441} + A_4 R_{1441} + A_1 R_{1444}. \end{aligned}$$

Hence the Riemannian manifold $(R_+^{n+4}, g) = (R_+^4 \times R^n, \widetilde{g} + \widehat{g})$ is a decomposable $(APS)_{n+4}$ with a flat decomposition manifold $(R^n, \widehat{g})$.

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