Honam Mathematical J. **40** (2018), No. 3, pp. 391–400
https://doi.org/10.5831/HMJ.2018.40.3.391

ESTIMATE FOR INITIAL MACLAURIN COEFFICIENTS OF GENERAL SUBCLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER INVOLVING SUBORDINATION

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Abstract. The object of this paper to construct a new class

$$\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\Psi)$$

of bi-univalent functions of complex order defined in the open unit disc. The second and the third coefficients of the Taylor-Maclaurin series for functions in the new subclass are determined. Several special consequences of the results are also indicated.

1. Introduction

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1}$$

which are analytic in the open unit disc $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized under the conditions

$$f(0) = 0, \;\; f'(0) = 1.$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ. With a view to recalling the principle of subordination between analytic functions, let the functions f and g be analytic in Δ. Given functions $f, g \in A$, f is subordinate to g if there exists a Schwarz function $w \in \Lambda$, where

$$\Lambda = \{w : w(0) = 0, \;\; |w(z)| < 1, \;\; z \in \Delta\},$$

Received October 27, 2017. Revised July 27, 2018. Accepted July 31, 2018.
2010 Mathematics Subject Classification. 30C45.
Key words and phrases. bi-starlike functions, Coefficient bounds, subordination, differential operator.
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such that

$$f(z) = g(w(z)) \qquad (z \in \Delta).$$

We denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).$$

In particular, if the function g is univalent in Δ, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta).$$

According to the Koebe-One Quarter Theorem [11], it ensures that the image of Δ under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus every univalent function $f \in A$ has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z$ and $f\left(f^{-1}(w)\right) = w \ \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$\text{(2)} \quad g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots .$$

A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions in Δ given by (1). For a brief history and interesting examples in the class Σ, see [24] (see also [6], [7], [15], [18]). Furthermore, many recent papers have been devoted to the problem of finding non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) (see, for example, [5], [13], [16] and [23]). There are, however, few papers that discuss the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([4], [12], [21]). The problem to find the coefficient bounds on

$$|a_n| \qquad (n \in \mathbb{N} \backslash \{1, 2\}\ ;\ \mathbb{N} = \{1, 2, 3, ...\})$$

for functions $f \in \Sigma$ is still an open problem.

The study of operators plays an important role in the Geometric Function Theory and its related fields. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better (see, for example [2], [3], [8], [14] and [17]). Recently, Amourah and Darus [10] introduced the following differential operator as given below:

$$\boldsymbol{A}^0_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = f(z),$$

$$\boldsymbol{A}^1_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = \left[1 - \frac{(\lambda-\alpha)\beta}{\lambda+\mu}\right]f(z) + \frac{(\lambda-\alpha)\beta}{\lambda+\mu}zf'(z) + \frac{\delta}{\lambda+\mu}z^2f''(z),$$

$$\vdots$$

$$\vdots$$

$$\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = \boldsymbol{A}_{\mu,\lambda,\delta}(\alpha,\beta)\left(\boldsymbol{A}^{m-1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right),$$

or equivalently

$$\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z) = z + \sum_{n=2}^{\infty}\left[1 + \frac{(n-1)\left[(\lambda-\alpha)\beta + n\delta\right]}{\lambda+\mu}\right]^m a_n z^n,$$

where

$$(f \in A,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \alpha,\delta \geq 0,\ \beta,\lambda,\mu > 0,\ \alpha \neq \lambda;\ z,w \in \Delta).$$

It should be remarked that the operator $\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)$ is a generalization of many other linear operators studied by earlier researchers. Namely:

- for $\delta = 0, \alpha = 0, \beta = 1$, the operator $\boldsymbol{A}^m_{\mu,\lambda,0}(0,1) \equiv \boldsymbol{I}^m_{\mu,\lambda}$ has been studied Swamy (see [25]),
- for $\delta = 0, \mu = 1-\lambda$, the operator $\boldsymbol{A}^m_{1-\lambda,\lambda,0}(\alpha,\beta) \equiv \boldsymbol{D}^m_{\lambda}(\alpha,\beta)$ has been studied by Darus and İbrahim (see [9]),
- for $\delta = 0, \mu = 1-\lambda, \alpha = 0, \beta = 1$, the operator $A^m_{1-\lambda,\lambda,0}(0,1) \equiv \boldsymbol{D}^m_{\lambda}$ has been studied by Al-Oboudi (see [1]),
- for $\mu = 0, \lambda = 1, \delta = 0$ and $\alpha = 0, \beta = 1$ the operator $A^m_{0,1,0}(0,1) \equiv \boldsymbol{D}^m$ is the popular Salagean operator [22].

Firstly, we will state the Lemma 1.1 to obtain our result.

Lemma 1.1. *(Carathèodory Lemma) (see [20]) If $c \in P$, then $|c_i| \leq 2$ for each i, where P is the family all functions c, analytic in Δ, for which*

$$\Re\{c(z)\} > 0,$$

where

$$c(z) = 1 + c_1 z + c_2 z^2 + \cdots .$$

Through out this paper it is assumed that Ψ be an analytic function with positive real part in Δ, with $\Psi(0) = 1$ and $\Psi'(0) > 0$. Also, let $\Psi(\Delta)$ be starlike with respect to 1 and symmetric with respect to the real axis. Such a function has a series expansion of the following form:

$$\Psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0,\ z \in \Delta)\,. \tag{3}$$

Making use of the differential operator $\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)$, we introduce a new class of analytic bi-univalent functions as follows:

Definition 1.2. *A function $f \in \Sigma$ given by (1) belongs to the class*

$$\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\Psi)$$

$$(\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq t < 1;\ m \in \mathbb{N}_0,\ \alpha,\delta \geq 0,\ \beta,\lambda,\mu > 0,\ \alpha \neq \lambda;\ z,w \in \Delta)$$

if the following subordinations are satisfied:

$$1+\frac{1}{\gamma}\left[\frac{z\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{(1-t)\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)+tz\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}-1\right]\prec\Psi(z)$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)g(w)\right)'}{(1-t)\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)g(w)+tw\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)g(w)\right)'}-1\right]\prec\Psi(w),$$

where the function g is given by (2).

Example 1.3. *For $t=0$ and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$ given by (1) is said to be in the class $\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\Psi)$, if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}\left[\frac{z\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-1\right]\prec\Psi(z)$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}-1\right]\prec\Psi(w),$$

where $m\in\mathbb{N}_0,\ \alpha,\delta\geq 0,\ \beta,\lambda,\mu>0,\ \alpha\neq\lambda;\ z,w\in\Delta$ and the function g is given by (2).

Example 1.4. *(see [19]) For $t=m=0$ and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$ given by (1) is said to be in the class $S^{*}_{\Sigma}(\gamma,\Psi)$, if the following conditions are satisfied:*

$$1+\frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)}-1\right]\prec\Psi(z)$$

and

$$1+\frac{1}{\gamma}\left[\frac{wg'(w)}{g(w)}-1\right]\prec\Psi(w),$$

where $z,w\in\Delta$ and the function g is given by (2).

2. Main result

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\Psi)$ proposed by Definition 1.2.

Theorem 2.1. *Let f of the form (1) be in the class $\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\Psi)$. Then*

$$|a_2| \le \frac{|\gamma|\, B_1\sqrt{B_1}}{\sqrt{\left|\left[2(1-t)Y-(1-t^2)X^2\right]\gamma B_1^2-(1-t)^2X^2(B_2-B_1)\right|}}$$

and

$$|a_3| \le \frac{|\gamma|^2\, B_1^2}{(1-t)^2X^2} + \frac{|\gamma|\, B_1}{2(1-t)\,|Y|},$$

where

$$X = \left[1+\frac{(\lambda-\alpha)\beta+2\delta}{\lambda+\mu}\right]^m \tag{4}$$

and

$$Y = \left[1+\frac{2(\lambda-\alpha)\beta+4\delta}{\lambda+\mu}\right]^m. \tag{5}$$

Proof. Since $f \in \boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\Psi)$, from Definition 1.2 we get

$$1+\frac{1}{\gamma}\left[\frac{z\left(\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}{(1-t)\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)+tz\left(\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'}-1\right] = \Psi(u(z)) \tag{6}$$

and

$$1+\frac{1}{\gamma}\left[\frac{w\left(\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)g(w)\right)'}{(1-t)\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)g(w)+tw\left(\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta)g(w)\right)'}-1\right] = \Psi(v(w)), \tag{7}$$

where u, v are analytic functions satisfying $u, v : \Delta \to \Delta$ with $u(0) = v(0) = 0$, $|u(z)| < 1, |v(w)| < 1$.

Now let us determine the functions c_1 and c_2 in P given by

$$c_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_1z + c_2z^2 + \cdots$$

and

$$c_2(w) = \frac{1+v(w)}{1-v(w)} = 1 + d_1w + d_2w^2 + \cdots .$$

Thus,

$$u(z) = \frac{c_1(z)-1}{c_1(z)+1} = \frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right] \tag{8}$$

and

$$v(w) = \frac{c_2(w)-1}{c_2(w)+1} = \frac{1}{2}\left[d_1w+\left(d_2-\frac{d_1^2}{2}\right)w^2+\cdots\right]. \tag{9}$$

The fact that c_1 and c_2 are analytic in Δ with $c_1(0) = c_2(0) = 1$. Since $u, v : \Delta \to \Delta$, the functions c_1, c_2 have a positive real part in Δ, and the relations $|c_i| \leq 2$ and $|d_i| \leq 2$ are true. Using (8) and (9) together with (3) in the right hands of the relations (6) and (7), we obtain

$$\Psi\left(u(z)\right) = 1 + \frac{1}{2}B_1 c_1 z + \left\{\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right\} z^2 + \cdots$$

and

$$\Psi\left(v(w)\right) = 1 + \frac{1}{2}B_1 d_1 w + \left\{\frac{1}{2}B_1\left(d_2 - \frac{d_1^2}{2}\right) + \frac{1}{4}B_2 d_1^2\right\} w^2 + \cdots . \tag{10}$$

In the light of (6) and (7), we get

$$\frac{(1-t)X}{\gamma} a_2 = \frac{B_1 c_1}{2}, \tag{11}$$

$$\frac{2(1-t)Y a_3 - (1-t^2)X^2 a_2^2}{\gamma} = \frac{B_1}{2}\left(c_2 - \tfrac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} \tag{12}$$

and

$$-\frac{(1-t)X}{\gamma} a_2 = \frac{B_1 d_1}{2}, \tag{13}$$

$$\frac{2(1-t)Y(2a_2^2 - a_3) - (1-t^2)X^2 a_2^2}{\gamma} = \frac{B_1}{2}\left(d_2 - \tfrac{d_1^2}{2}\right) + \frac{B_2 d_1^2}{4}. \tag{14}$$

Now, (11) and (13) give

$$c_1 = -d_1 \tag{15}$$

and

$$8(1-t)^2 X^2 a_2^2 = \gamma^2 B_1^2\left(c_1^2 + d_1^2\right). \tag{16}$$

Adding (12) and (14), we get

$$\frac{4(1-t)Y - 2(1-t^2)X}{\gamma} a_2^2 = \frac{B_1\left(c_2 + d_2\right)}{2} + \frac{\left(B_2 - B_1\right)\left(c_1^2 + d_1^2\right)}{4}. \tag{17}$$

and thus, by using (15), (16) and Lemma 1.1 in (17), we obtain

$$|a_2| \leq \frac{|\gamma|\, B_1\sqrt{B_1}}{\sqrt{\left|\left[2(1-t)Y - (1-t^2)X^2\right]\gamma B_1^2 - (1-t)^2 X^2 (B_2 - B_1)\right|}}.$$

The first inequality of the conclusion is proved.

Next, to find the bound on $|a_3|$, by using subtracting (14) and (12), we deduce

$$\frac{4(1-t)Y}{\gamma}\left(a_3-a_2^2\right)=\frac{B_1\left(c_2-d_2\right)}{2}. \tag{18}$$

It follows from (15), (16) and (18) that

$$a_3=\frac{\gamma^2 B_1^2\left(c_1^2+d_1^2\right)}{8(1-t)^2X^2}+\frac{\gamma B_1\left(c_2-d_2\right)}{8(1-t)Y}.$$

Finally, taking modulus on both sides and applying Lemma 1.1, we readily get

$$|a_3|\leq\frac{|\gamma|^2 B_1^2}{(1-t)^2X^2}+\frac{|\gamma| B_1}{2(1-t)\,|Y|}.$$

□

Remark 2.2. *Letf of the form (1) be in the class* $\boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,\Psi)$. *Then*

$$|a_2|\leq\frac{|\gamma|\,B_1\sqrt{B_1}}{\sqrt{\left|\left[2Y-X^2\right]\gamma B_1^2-X^2(B_2-B_1)\right|}}$$

and

$$|a_3|\leq|\gamma|\,B_1\left(\frac{|\gamma|\,B_1}{X^2}+\frac{1}{2\,|Y|}\right),$$

where X,Y *are given by (4) and (5), respectively.*

Remark 2.3. *Letf of the form (1) be in the class* $S^*_\Sigma\left(\gamma,\Psi\right)$. *Then*

$$|a_2|\leq\frac{|\gamma|\,B_1\sqrt{B_1}}{\sqrt{\left|\gamma B_1^2-(B_2-B_1)\right|}}$$

and

$$|a_3|\leq|\gamma|^2 B_1^2+\frac{|\gamma|\,B_1}{2}.$$

3. Applications of the main result

Various choices of Ψ as mentioned above and suitably choosing the values of B_1 and B_2, we state some interesting results analogous to Theorem 2.1 and the Corollaries 3.1 to 3.3.

For example, the function Ψ is given by

$$\Psi\left(z\right)=\left(\frac{1+z}{1-z}\right)^\theta=1+2\theta z+2\theta^2z^2+\cdots\qquad\left(0<\theta\leq1\right),$$

which gives

$$B_1 = 2\theta \text{ and } B_2 = 2\theta^2.$$

Corollary 3.1. *Let$f \in \boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\theta)$ be of the form (1). Then*

$$|a_2| \leq \frac{2\,|\gamma|\,\theta}{\sqrt{|[4(1-t)Y - 2(1-t^2)X^2]\,\gamma\theta + (1-t)^2X^2(1-\theta)|}}$$

and

$$|a_3| \leq \frac{2\,|\gamma|\,\theta}{(1-t)}\left(\frac{2\,|\gamma|\,\theta}{(1-t)X^2} + \frac{1}{2\,|Y|}\right),$$

where X, Y are given by (4) and (5), respectively.

By taking

$$\Psi(z) = \frac{1+(1-2\eta)\,z}{1-z} = 1+2\,(1-\eta)\,z+2\,(1-\eta)\,z^2+\cdots \quad (0 \leq \eta < 1)\,,$$

we obtain immediately that

$$B_1 = B_2 = 2\,(1-\eta)\,.$$

Corollary 3.2. *Let $f \in \boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,\eta)$ be of the form (1). Then*

$$|a_2| \leq \frac{|\gamma|\,\sqrt{2\,(1-\eta)}}{\sqrt{|[2(1-t)Y - (1-t^2)X^2]\,\gamma|}}$$

and

$$|a_3| \leq \frac{2\,|\gamma|\,(1-\eta)}{(1-t)}\left(\frac{2\,|\gamma|\,(1-\eta)}{(1-t)X^2} + \frac{1}{2\,|Y|}\right),$$

where X, Y are given by (4) and (5), respectively.

On the other hand, for $-1 \leq B < A \leq 1$, if we let

$$\Psi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + \cdots\,,$$

then we have

$$B_1 = (A-B) \text{ and } B_2 = -B(A-B).$$

Corollary 3.3. *Let $f \in \boldsymbol{A}^m_{\mu,\lambda,\delta}(\alpha,\beta,\gamma,t,A,B)$ be of the form (1). Then*

$$|a_2| \leq \frac{|\gamma|\,(A-B)}{\sqrt{|[2(1-t)Y - (1-t^2)X^2]\,\gamma(A-B) + (1-t)^2X^2(1+B)\,|}}$$

and

$$|a_3| \leq \frac{|\gamma|\,(A-B)}{(1-t)}\left(\frac{|\gamma|\,(A-B)}{(1-t)X^2} + \frac{1}{2\,|Y|}\right),$$

where X, Y are given by (4) and (5), respectively.

By suitably specializing the parameters, the class $\boldsymbol{A}^{m}_{\mu,\lambda,\delta}(\alpha, \beta, \gamma, t, \Psi)$ reduces to the various subclasses of bi-univalent functions. The details involved may be left as an exercise for the interested reader.

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