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Estimators Shrinking towards Projection Vector for Multivariate Normal Mean Vector under the Norm with a Known Interval

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Abstract

Consider the problem of estimating a $p\times 1$ mean vector $\theta\,(p-r\geq 3)$, $r=\mathrm{rank}(K)$ with a projection matrix K under the quadratic loss, based on a sample $Y_1, Y_2, \cdots, Y_n$. In this paper a James-Stein type estimator with shrinkage form is given when it's variance distribution is specified and when the norm $\|\boldsymbol{\theta}-K\boldsymbol{\theta}\|$ is constrain, where K is an idempotent and symmetric matrix and $\mathrm{rank}(K)=r$. It is characterized a minimal complete class of James-Stein type estimators in this case. And the subclass of James-Stein type estimators that dominate the sample mean is derived.

Keywords: James-Stein Type Estimators, Sample Mean, Projection Matrix.

1. Introduction

It is considered the problem of estimating a compound multinormal mean vector with quadratic loss and constraints on the norm $\|\boldsymbol{\theta}-K\boldsymbol{\theta}\|$. The class of estimation considered will consist of shrinkage James-Stein type estimators. James-Stein[1] and Lindley[2]1 introduced the class of estimators and they proved that some of estimators in the class dominate the sample mean vector in the multinormal distribution. Another result for the more extended case considered in this work of a mixed multinormal distribution is derived by Strawderman[3]. Amari[4], Kariya[5], Perron and Giri[6], Merchand and Giri[7], and Baek[8] among others derived the problem of estimation of a mean vector under restriction and it is focussed again in the context of curved model. The estimation of mean vectors in the variance mixed multivariate normal distributions was given by Berger[9].

In section 2, the general setting of our problem and develop necessary notations is given. In section 3, the estimation problem based on a James-Stein type estimators is examined when the norm $\|\boldsymbol{\theta}-K\boldsymbol{\theta}\|$ is constraint with a known interval. In this case, The subclass of James-Stein type estimators is given and the class dominates the sample mean vector under the norm with a known interval.

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2. Notation and Preliminaries

Let $\boldsymbol{Y}=(Y_1, \cdots, Y_p)'$, $p-r\geq 3$, be an random sample vector from a variance mixed multivariate normal distribution with unknown mean vector $\boldsymbol{\theta}\,(p\times 1)$ and mixture parameter $G(\bullet)$, where $G(\bullet)$ is a known distribution function on the interval $(0,\infty)$. The distribution of the random variable $\boldsymbol{Y}$ is represented as

$$\boldsymbol{L}(\boldsymbol{Y}\mid S=s)=N_p(\boldsymbol{\theta}, s\boldsymbol{I}_p),\ \forall\, s>0, \tag{1}$$

where S is the positive random variable with distribution function $G(\bullet)$.

The loss function is defined as

$$\boldsymbol{L}(\boldsymbol{\theta},\delta(\boldsymbol{Y}))=(\delta(\boldsymbol{Y})-\boldsymbol{\theta})'(\delta(\boldsymbol{Y})-\boldsymbol{\theta}),$$

with $\boldsymbol{\theta}\in\boldsymbol{\Theta}_{\lambda_2}^{\lambda_1}\subset R^p,\ \|\boldsymbol{\theta}-K\boldsymbol{\theta}\|\in[\lambda_1,\lambda_2], 0\leq\lambda_1\leq\lambda_2\leq\infty$, where K is an idempotent and symmetric matrix with $\mathrm{rank}(K)=r$ and the estimator δ, $\delta(\,\cdot\,):\boldsymbol{R}^p\to\boldsymbol{R}^p$, is of the form

$$\delta(Y) = KY + \left(1 - \frac{c}{(Y-KY)'(Y-KY)}\right)(Y-KY),\ c \in \boldsymbol{R}$$

By expression (2.1) the density of Y is

$$P_\theta(y) = \int_{(0,\infty)} (2\pi s)^{-p/2} \exp\left(\frac{\|y-\theta\|^2}{2s}\right) dG(s), \tag{2}$$

$y \in \boldsymbol{R}^p$, $\theta \in \Theta_{\lambda_2}^{\lambda_1}$, and $E(S) < \infty$. we assume the covariance matrix $\Sigma = Cov(Y) = E(S)I_p$, and the mean vector $E(Y) = \theta$. The risk function of the estimator δ is defined by

$$R(\theta, \delta) = E_\theta[L(\theta, \delta(Y))] = E_\theta[(\delta(Y) - \theta)'(\delta(Y) - \theta)],$$
$\theta \in \Theta_{\lambda_2}^{\lambda_1}$

Let

$$D_{JS} = \left\{\delta : \boldsymbol{R}^p \to \boldsymbol{R}^p \mid \delta^c(Y) = KY + \left(1 - \frac{c}{(Y-KY)'(Y-KY)}(Y-KY)\right), c \in \boldsymbol{R}\right\},$$

where the parameter space is of the form

$$\Theta_{\lambda_2}^{\lambda_1} = \Theta_\lambda = \{\theta \in \boldsymbol{R}^p \mid \|\theta - K\theta\| = \lambda\},\ \lambda \geq 0.$$

Using the method by Baek[8], If $\theta \in \Theta_\lambda$, $p - r \geq 3$ and $E[S] < \infty$, It can be shown that

$$R(\theta, \delta^c) = E_\theta[(\delta^c(Y) - \delta^c)'(\delta^c(Y) - \delta^c)]$$
$$= pE(S) + \left\{\int_{(0,\infty)} \left[\frac{c}{s} - 2c(p-r-2)\right] f_p(\lambda, s) dG(s)\right\} \tag{3}$$

By (2.3), the unique best estimator within the class D_{JS} is given by $\delta^{c^*(\lambda)}$ where

$$c^*(\lambda) = (p-r-2)\frac{\int_{(0,\infty)} f_p(\lambda, s) dG(s)}{\int_{(0,\infty)} f_p(\lambda, s)\frac{dG(s)}{s}} \tag{4}$$

and the risk is

$$R(\theta, \delta^{c^*(\lambda)}) = pE(S) - (p-r-2)^2 \frac{\left[\int_{(0,\infty)} f_p(\lambda, s) dH(s)\right]^2}{\int_{(0,\infty)} f_p(\lambda, s)\frac{dH(s)}{s}},$$

$\theta \in \Theta_\lambda$.

If $\|\theta - K\theta\| = \lambda$, the risk of this class of estimators is a strictly increasing function of distance $|c - c^*(\lambda)|$. Define $t(\lambda)$ such that $c = t(\lambda)c^*(\lambda)$ and using expression (2.3), we can express $R(\theta, \delta^c)$ as

$$pE(S) + (p-r-2)^2[t^2(\lambda) - 2t(\lambda)]\frac{\left[\int_{(0,\infty)} f_p(\lambda, s) dG(s)\right]^2}{\int_{(0,\infty)} f_p(\lambda, s)\frac{dG(s)}{s}}$$

Therefore,

$$R(\theta, \delta^c) - R(\theta, \delta^{c^*(\lambda)}) =$$
$$|c - c^*(\lambda)|^2 \int_{(0,\infty)} f_p(\lambda, s)\frac{dG(s)}{s}. \tag{6}$$

The usual estimator $\delta^0(Y) = Y$ is a member of the James-Stein type estimators and its risk is $pE(S)$. By the expression (2.5), it can be shown that the James-Stein type estimator δ^c dominates the usual estimator δ^0 if and only if $0 < c < 2 < c^*(\lambda)$ for $\theta \in \Theta_\lambda$.

3. Estimation under the Norm with a Known Interval

In this section, we deal with the case where the mean vector θ has a known interval $[\lambda_1, \lambda_2]$ case, no optimal James-Stein type estimator will exist when $\lambda_1 \leq \lambda_2$. We can also derive the subclass of James-Stein type estimators that dominate the usual estimator $\delta^0 = Y$ when $\theta \in \Theta_{\lambda_2}^{\lambda_1}$. Let

$\underline{c}^*[\lambda_1, \lambda_2] = inf_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda)$ and
$\bar{c}^*[\lambda_1, \lambda_2] = sup_{\lambda \in [\lambda_1, \lambda_2]} c^*(\lambda)$.

Theorem 3.1 Let Y be a single sample from a p-dimensional population with density of expression (2.1). Given assumptions $\theta \in \Theta_{\lambda_2}^{\lambda_1}$, $0 \leq \lambda_1 \leq \lambda_2 \leq \infty$; $p - r \geq 3$ and $E(S) < \infty$,

(a) the subclass $\{\delta^c \in D_{JS} \mid \underline{c}^*[\lambda_1, \lambda_2] \leq c \leq \bar{c}^*[\lambda_1, \lambda_2]\}$ is a minimal complete class in D_{JS} and

(b) the estimator δ^c will be dominate the usual estimator δ^0 when $0 < c < \underline{c}^*[\lambda_1, \lambda_2]$.

Proof. (a) Let c_0 be a real number such that $c_0 \notin [\underline{c}^*[\lambda_1, \lambda_2], \bar{c}^*[\lambda_1, \lambda_2]]$. Then, by expression (2.6), when $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$, it can be written the risk difference as

$$R(\theta, \delta^{c_0}) - R(\theta, \delta^{\underline{c}^*[\lambda_1,\lambda_2]})$$
$$= \left[R(\theta, \delta^{c_0}) - R(\theta, \delta^{c^*(\|\theta - K\theta\|)})\right] - \left[R(\theta, \delta^{\underline{c}^*[\lambda_1,\lambda_2]}) - R(\theta, \delta^{c^*(\|\theta - K\theta\|)})\right]$$
$$= \int_{(0,\infty)} f_p(\lambda, z)\frac{dH(z)}{z}\left\{ |c_0 - c^*(\|\theta - K\theta\|)|^2 - \left|\underline{c}^*[\lambda_1,\lambda_2] - c^*(\|\theta - K\theta\|)\right|^2 \right\}$$

this last expression being positive for all $\theta \in \Theta_{\lambda_2}^{\lambda_1}$ given that $c_0 < \underline{c}^*[\lambda_1, \lambda_2]$.

In a similar way, the estimator δ^c with $c = \bar{c}^*[\lambda_1, \lambda_2]$ will dominate the estimator δ^{c_0} if $c_0 > \bar{c}^*[\lambda_1, \lambda_2]$. By the intermediate value theorem we can derive that

$$R(\theta, \delta^c) - R(\theta, \delta^{c_0}) > 0, \ \forall c \neq c_0,$$

when $c^*(\|\theta - K\theta\|)) = c_0$. These last results show that all the estimator δ^c with $c \notin \left[\underline{c}^*[\lambda_1,\lambda_2], \bar{c}^*[\lambda_1,\lambda_2]\right]$ are inadmissible within the class D_{JS} and the estimator δ^c with c belonging to the interval $\left[\underline{c}^*[\lambda_1,\lambda_2], \bar{c}^*[\lambda_1,\lambda_2]\right]$ cannot be improved upon by another estimator of the class D_{JS}. Thus, the result of part (a) follows.

(b) Similar to last part in Section 2, the estimator δ^c will dominate the estimator δ^0 if

$$R(\theta, \delta^c) < R(\theta, \delta^0), \ \forall \theta \in \Theta_{\lambda_2}^{\lambda_1}$$
$$\Leftrightarrow 0 < c < 2c^*(\|\theta - K\theta\|), \ \forall \|\theta - K\theta\| \in [\lambda_1, \lambda_2]$$
$$\Leftrightarrow 0 < c < 2\underline{c}^*[\lambda_1, \lambda_2]$$

Hence, the estimator δ^c with $c = 2\underline{c}^*[\lambda_1, \lambda_2]$ will also dominate δ^0 under the conditions of the theorem when $\lambda_1 < \lambda_2$ and that all the estimators δ^c with $c > 2c^*[\lambda_1, \lambda_2]$ do not dominate δ^0 under the conditions of the theorem.

The case with no restrictions on the norm $\|\theta - K\theta\|$ (i. e. , $\lambda_1 = 0$, and $\lambda_2 = \infty$) can be expanded using by Strawderman's result[3] and it can be showed that the estimators δ^c with $0 \le c \le 2(p-r-2)E^{-1}(S^{-1})$ are minimax estimators by showing that their risk functions are uniformly less than or equal to the risk function $(= pE(S))$ of the minimax estimator δ^c. This result is derived below as a particular case of Theorem 3.1. To do so, we need to determine the quantity $\underline{c}^*[0, \infty]$. The following Lemmas will prove useful in determining $\underline{c}^*[0, \infty]$ and, also, $\underline{c}^*[\lambda_1, \lambda_2]$.

Lemma 3.2. Let Y be an random variable and let f and g be two real nondecreasing functions on the support of Y. Then, if the quantities $E[f(y)]$ and $E[g(y)]$ exist, $Cov(f(y), g(y)) \ge 0$ with the inequality being strict if f and g are strictly increasing and Y is nondegenerate.

Proof. See Chow and Wang[10].

Lemma 3.3. Let L be a Poisson random variable with mean $\gamma (>0)$ and $f_p^*(\gamma) = E^L[(p-r+2L-2)^{-1}]$, $p \ge 4$ then

(i) $f_{p-r}^*(\gamma) = e^{-\gamma}\int_{[0,1]} t^{p-r-3}e^{\gamma t^2}dt$ and

(ii) $f_{p-r+2}(\gamma) = (2\gamma)^{-1}[1-(p-r-2)f_{p-r}^*(\gamma)]$. (3.1)

Proof. See Egerton and Laycock[11].

Lemma 3.4. Let $f_p^*(\cdot)$, $p \ge 4$ be a function defined on $[0, \infty]$ and $f_p^*(\gamma) = E^L[(p-r+2L-2)^{-1}]$, $\gamma \ge 0$, where L is a Poisson random variable with mean γ. Then,

(i) $f_{p-r}^*(\cdot)$ is a strictly decreasing function,

(ii) $\lim_{\gamma\to 0^+} f_{p-r}^*(\gamma) = (p-r-2)^{-1}$, $\lim_{\gamma\to INF} f_{p-r}^*(\gamma) = 0$

(iii) if $p-r \ge 4$, $\gamma f_{p-r}^*(\gamma)$ is strictly increasing function for $\gamma \ge 0$.

Proof. (i) Using part (i) of Lemma 3.3, we have for $\gamma_2 > \gamma_1 > 0$,

$$f_{p-r}^*(\gamma_2) - f_{p-r}^*(\gamma_1) = \int_{[0,1]} t^{p-r-3}(e^{\gamma_2(t^2-1)} - e^{\gamma_1(t^2-1)})dt < 0$$

(ii) By the dominated convergence theorem,

$$\lim_{\gamma\to 0^+} f_{p-r}^*(\gamma) = \lim_{\gamma\to 0^+}\int_{[0,1]} t^{p-r-3}(e^{\gamma(t^2-1)})dt$$
$$= \int_{[0,1]} t^{p-r-3}(\lim_{\gamma\to 0^+}(e^{\gamma(t^2-1)}))dt$$
$$= \int_{[0,1]} t^{p-r-3}dt = (p-r-2)^{-1}$$

and

$$\lim_{\gamma\to\infty} f_{p-r}^*(\gamma) = \lim_{\gamma\to\infty}\int_{[0,1]} t^{p-r-3}e^{\gamma(t^2-1)}dt$$
$$= \int_{[0,1]} t^{p-r-3}(\lim_{\gamma\to\infty} e^{\gamma(t^2-1)})dt = 0$$

(iii) By Lemma 3.3, we can derive

$$\gamma f_5^*(\gamma) = \frac{1}{2}(1-e^{-\gamma}),$$

which can be shown to be strictly increasing. For $p-r \geq 5$ we have by the recurrence relation given by expression (3.1),

$$\gamma f_{p-r}^*(\gamma) = \frac{1}{2}(1-(p-r-4)f_{p-r-2}^*(\gamma)),\ \gamma > 0.$$

which must be strictly increasing given that function $f_{p-r-2}(\cdot)$ is strictly decreasing by part (i).

In the following, we will have $E^{-1}[S^{-1}]$ equal to zero when the expectation $E[S^{-1}] = \infty$.

Theorem 3.5. The function $c^*(\cdot)$ in (2.4) satisfies the next three properties :

(a) $\inf_{\lambda \geq 0} c^*(\lambda) = (p-r-2)E[S^{-1}]$

(b) $c^*(\gamma) = k \Rightarrow S$ is constant with probability one and,

(c) for $p-r \geq 4$,

Proof. (a) By expression (2.4),

$$c^*(\lambda) = (p-r-2)\frac{E^Z[f_{p-r}(\lambda, S)]}{E^Z[S^{-1}f_{p-r}(\lambda, S)]},\ \lambda \geq 0.$$

By using Lemma 3.2 to $f_{p-r}(\lambda, S)$ and S^{-1}, the function $f_{p-r}(\lambda, S)$ being an increasing function by part (i) of Lemma 3.4,

$$Cov(f_{p-r}(\lambda, S), -S^{-1}) \geq 0$$

$$\Rightarrow E^S[S^{-1}f_{p-r}(\lambda, S)] \geq E[S^{-1}]E^S[f_{p-r}(\lambda, S)]$$

$$\Rightarrow c^*(\lambda) \geq (p-r-2)E^{-1}[S^{-1}]$$

$$\Rightarrow \inf_{\lambda \geq 0} c^*(\lambda) \geq (p-r-2)E^{-1}[S^{-1}]$$

for $\lambda \geq 0$,

The reverse inequality is derived that $c^*(0) = (p-r-2)E^{-1}[S^{-1}]$.

(b) The constancy of $c^*(\lambda)$ implies $c^*(\lambda) = k = c^*(0) = (p-r-2)E^{-1}[S^{-1}]\ \forall \lambda > 0$, and

$$\int_{(0,\infty)} \left(p-r-1-\frac{k}{s}\right) f_{p-r}(\lambda, s)\, dG(s) = 0.$$

$f_{p-r}(\lambda, S)$ and $-ks^{-1}$ are strictly increasing function of s and then by Lemma 3.2, for nondegenerate S,

$$Cov(f_{p-r}(\lambda, S), p-r-2-kS^{-1}) \geq 0$$

$$\Rightarrow E[p-r-2-kS^{-1})f_{p-r}(\lambda, S)] >$$

$$E[p-r-2-kS^{-1})]\,E[f_{p-r}(\lambda, S)] = 0$$

which is a contradiction so that S is constant with probability one.

(c) By using Lemma 3.2 to $-z^{-1}f_{p-r}(\lambda, S)$ and s, the function $-s^{-1}f_{p-r}(\lambda, S)$ being an increasing function by (III) of Lemma 3.4, It can be shown that for $p-r \geq 4$ and $\lambda \geq 0$,

$$Cov(-S^{-1}f_{p-r}(\lambda, S),\ S) \geq 0$$

$$\Rightarrow E^Z[f_{p-r}(\lambda, S)] \leq E[S^{-1}f_{p-r}(\lambda, S)]\,E[S]$$

$$\Rightarrow c^*(\lambda) \leq (p-r-2)E[S]$$

$$\Rightarrow \sup_{\lambda \geq 0} c^*(\lambda) \geq (p-r-2)E(S)$$

The reverse inequality can be obtained by showing that $\lim_{\lambda \to \infty} c^*(\lambda) = (p-r-2)E[S]$ whenever $p-r \geq 4$.

$$c^*(\lambda) = (p-r-2)\frac{\displaystyle\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^{j+1}}{j!(p-r+2y-2)}\, s\, dG(s)}{\displaystyle\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^{j+1}}{j!(p-r+2y-2)}\, dG(s)}$$

$$= (p-r-2)\frac{\displaystyle\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2s}\right)^{j+1}}{j!} \frac{2j}{p-r+2y-4}\, s\, dG(s)}{\displaystyle\int_{(0,\infty)} \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^{j+1}}{j!} \frac{2j}{p-r+2y-4}\, dG(s)},$$

$\lambda > 0$

Hence, it can be written that

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-r-2)$$

$$\frac{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^j}{j!}\frac{2j}{p-r+2y-4}\,sdG(s)\right\}}{\lim_{\lambda\to\infty}\left\{\int_{(0,\infty)}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2s}}\left(\frac{\lambda^2}{2s}\right)^j}{j!}\frac{2j}{p-r+2y-4}\,dG(s)\right\}}$$

when two limits exist and the denominator is not equal to zero. By the dominated converge theorem, it can be written that

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-r-2)$$

$$\frac{\int_{(0,\infty)}\lim_{\lambda\to\infty} E^{L_s}\left[\frac{2L_s}{p-r+2L_s-4}1_{(1,2,\cdots)}(L_s)\right] sdG(s)}{\int_{(0,\infty)}\lim_{\lambda\to\infty} E^{L_s}\left[\frac{2L_s}{p-r+2L_s-4}1_{(1,2,\cdots)}(L_s)\right] dG(s)}$$

where, for $s>0$, L_s is a Poisson random variable with mean $\lambda^2/2s$. And,

$$\forall s>0,\ \lim_{\lambda\to\infty} E^{L_s}\left[\frac{2L_s}{p-r+2L_s-4}1_{(1,2,\cdots)}(L_s)\right] = 1$$

because the integrand tends $2L_s(p-q+2L_s-4)^{-1}$ tends to one if $L_s\to\infty$ it can be obtained that

$$\lim_{\lambda\to\infty} c^*(\lambda) = (p-r-2)$$

$$\frac{\int_{(0,\infty)} sdG(s)}{\int_{(0,\infty)} dH(s)} = (p-r-2)E(S).$$

Theorem 3.1 with the quantities $\underline{c}^*[0,\infty]$ and $\bar{c}^*[0,\infty]$ derives the following result.

Corollary 3.6. Let Y be a single sample vector from a p-dimensional population with density of expression (2.1). with $p-r\geq 3$, and under the assumption $\theta\in R^p$ and $E[S]<\infty$,

(a) the subclass $\left\{\delta^c\in D_{JS} \mid (p-r-2)E^{-1}[S^{-1}] \leq c \leq (p-r-2)E[S]\right\}$ is a minimal complete class D_{JS} for $p-r\geq 4$,

(b) the estimator δ^c will dominate the usual estimator δ^0 if $0<c<2(p-r-2)E^{-1}[S^{-1}]$.

Proof. It can be proved the results by using application of Theorem 3.1 and 3.5.

Remark 3.1. Under the conditions of Corollary 3.6, the estimator δ^c is a minimax estimator if and only if $0\leq c\leq 2(p-r-2)E^{-1}[S^{-1}]$. We can obtain this condition using part (a) of Theorem 3.5 and similar to last part in Section 2 it can be derived that

$$R(\boldsymbol{\theta},\delta^c)\leq p \iff 0\leq c\leq 2c^*(\|\boldsymbol{\theta}-K\boldsymbol{\theta}\|).$$

Note that the usual estimator δ^0 is the only minimax estimator within the class D_{JS} when the quantity $E[S^{-1}]$ does not exist.

Remark 3.2. The results above of Theorem 3.1 and Corollary 3.6 can be generalized to the case where the experimental information consist of a sample $\boldsymbol{Y}_1,\cdots,\boldsymbol{Y}_n$ with density of the form in (2.1) and the class of estimators considered is of the estimators of the form

$$\delta^c(\boldsymbol{Y}_1,\cdots,\boldsymbol{Y}_n) = K\overline{\boldsymbol{Y}} + \left(1-\frac{c}{(\overline{\boldsymbol{Y}}-K\overline{\boldsymbol{Y}})'(\overline{\boldsymbol{Y}}-K\overline{\boldsymbol{Y}})}\right)(\overline{\boldsymbol{Y}}-K\overline{\boldsymbol{Y}}),\ c\in R$$

where $\overline{\boldsymbol{Y}}$ is the sample mean vector and K is an idempotent and symmetric matrix. This can be seen by nothing that the probability law of sample mean vector $\overline{\boldsymbol{Y}}=n^{-1}\sum \boldsymbol{Y}_i$; $\boldsymbol{Y}_1,\cdots,\boldsymbol{Y}_n$ being n independently and identically distributed random vectors.

$$L(\boldsymbol{Y}_j \mid S_j=s_j) = N_p(\boldsymbol{\theta}, s_j I_p),\ j=1,\cdots,n$$

for all observations $s_1,\cdots,s_n$ of n independent sample $S_1,\cdots,S_n$ of a positive random variable S;

$$L(S\mid S_1=s_1,\cdots,S_n=s_n) = N_p(\boldsymbol{\theta}, n^{-2}\sum_{j=1}^{n}s_j I_p),$$

or

$$L(\overline{\boldsymbol{Y}}\mid W=w) = N_p(\boldsymbol{\theta}, wI_p),\ \forall w>0$$

where W is a random variable such that

$$L(W) = L(n^{-2}\sum_{j=1}^{n}S_j). \tag{3.2}$$

Hence the optimal James-Stein type estimator with the conditions $\theta \in \Theta_\lambda$, $E[Z] < \infty$ and, $p-r \geq 3$ is

$$\delta^{c_n^*(\lambda)} = K\overline{Y} + \left(1 - \frac{c_n^*(\lambda)}{(\overline{Y} - K\overline{Y})'(\overline{Y} - K\overline{Y})}\right)(\overline{Y} - K\overline{Y})$$

where

$$c_n^*(\lambda) = (p-r-2)\frac{\int_{(0,\infty)} f_p(\lambda, w)\, dH_n^*(w)}{\int_{(0,\infty)} f_p(\lambda, w)\frac{dH_n^*(w)}{w}},$$

with the distribution function $H_n^*(\cdot)$ of the random variable W defined by expression (3.2). Then, we can find a minimal complete class within the class

$$D_{JS} = \left\{\delta^c : R^p \to R^p \mid \delta^c(\overline{Y}) = K\overline{Y} + \left(1 - \frac{c}{(\overline{Y} - K\overline{Y})'(\overline{Y} - K\overline{Y})}\right)(Y - K\overline{Y})\right\}$$

which the result gives a subclass of James-Stein type estimators that dominate the sample mean vector $\delta^0(\overline{Y}) = \overline{Y}$. Futhermore, Part (b) Corollary 3.6 can be proved by Bravo and MacGibbon[12] under a more general cases.

Corollary 3.7. Let $Y_1, \cdots, Y_n$ be a sample vector from a p-dimensional population with density of the form in (2.1). If $\theta \in R^p$, $p-r \geq 3$, and $E[S] < \infty$, then

(a) for $p-r \geq 4$, the subclass

$$\left\{\delta^c \in D_{JS} \mid n^{-1}(p-r-2) E^{-1}\left[\left(\sum_{i=1}^n S_i\right)^{-1}\right] \leq c \leq n^{-1}(p-r-2)E[S]\right\}$$

is a minimal complete class with the class D_{JS}, and

(b) the estimator δ^c will dominate the sample mean vector when

$$0 < c < 2n^{-2}(p-r-2)E^{-1}\left[\left(\sum_{i=1}^n S_i\right)^{-1}\right]. \qquad (3.3)$$

Proof. By Corollary 3.6 and expression (3.2) we can prove this corollary.

However, the results concerning the minimax criteria given by Strawerman[3] cannot be applied to the estimators $\delta^c(\overline{Y})$ since the statistic $\overline{Y}$ does not represent in general a sufficient statistic. And then note that,

$$E^{-1}\left[\left(\sum_{i=1}^n S_i\right)^{-1}\right] \leq E\left[\sum_{i=1}^n S_i\right] = nE[S],$$

(the above inequality is a result of Lemma 3.2), implying that the interval

$$\left(0, 2n^{-1}(p-r-2)E^{-1}\left[\left(\sum_{i=1}^n S_i\right)^{-1}\right]\right) \to \varnothing \text{ as } n \to \infty$$

which, by expression (3.3), implies that the subclass of James-Stein type estimators dominating the sample mean vector can be made arbitrarily small as the sample size n increases.

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