

SOME WEIGHTED APPROXIMATION PROPERTIES OF NONLINEAR DOUBLE INTEGRAL OPERATORS

GÜMRAH UYSAL, VISHNU NARAYAN MISHRA*, AND
SEVILAY KIRCI SERENBAY

ABSTRACT. In this paper, we present some recent results on weighted pointwise convergence and the rate of pointwise convergence for the family of nonlinear double singular integral operators in the following form:

$$T_{\eta}(f; x, y) = \iint_{\mathbb{R}^2} K_{\eta}(t-x, s-y, f(t, s)) ds dt, \quad (x, y) \in \mathbb{R}^2, \quad \eta \in \Lambda,$$

where the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is Lebesgue measurable on \mathbb{R}^2 and Λ is a non-empty set of indices. Further, we provide an example to support these theoretical results.

1. Introduction

Approximation by integral operators or the functions having good properties has broad applications in miscellaneous areas of mathematics including Fourier theory, convergence of orthogonal series, theory of differential equations and harmonic analysis theory. Therefore, integral operators have various applications in many academic disciplines such as physics, engineering and medicine. The following integral operator is

Received June 8, 2018. Revised August 22, 2018. Accepted September 4, 2018.

2010 Mathematics Subject Classification: 41A35, 41A25, 28C10.

Key words and phrases: Generalized Lipschitz condition; Weighted pointwise convergence; Rate of convergence.

* Corresponding author.

© The Kangwon-Kyungki Mathematical Society, 2018.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

a well-known one:

$$(1.1) \quad L_n(f; y) = \int_{-\pi}^{\pi} f(s) K_n(s, y) ds, \quad y \in [-\pi, \pi], \quad n \in \mathbb{N},$$

where $K_n : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_0^+$ denotes a 2π -periodic kernel as a function of s satisfying some conditions similar to well-known approximate identities. In fact, magnetic resonance imaging, face recognition and computer aided geometric design are some of the application areas in which the integral operators are used. There are famous integral operators which are named after their constructors, such as Gauss-Weierstrass, Riesz-Bessel, Jackson-Stechkin, Fejer, Picard and Calderon-Zygmund singular integral operators.

In [17], Taberski, who mentioned the importance of singular integrals in Fourier analysis in his works, handled the pointwise approximation of 2π -periodic functions which are integrable on $\langle -\pi, \pi \rangle$, where $\langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed or open interval. The mentioned study used a two-parameter family of convolution-type singular integral operators in the following form:

$$(1.2) \quad L_\lambda(f; y) = \int_{-\pi}^{\pi} f(s) K_\lambda(s - y) ds, \quad y \in \langle -\pi, \pi \rangle, \quad \lambda \in \Lambda,$$

where $K_\lambda : \mathbb{R} \rightarrow \mathbb{R}_0^+$ denotes a 2π -periodic kernel function satisfying suitable conditions and Λ is a non-empty set of numbers with accumulation point λ_0 .

Taberski [18] advanced his previous analysis by using the double singular integral operators of the form:

$$(1.3) \quad L_\lambda(f; x, y) = \iint_Q f(t, s) K_\lambda(t - x, s - y) ds dt, \quad (x, y) \in Q,$$

where $Q = \langle -\pi, \pi \rangle \times \langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed or open rectangular region, $K_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}_0^+$ stands for a kernel function comprising appropriate features and $\lambda \in \Lambda$, Λ is a non-empty set of numbers with accumulation point λ_0 . Those results were later used by Siudut [14] presenting considerable theorems. It should be noted that, Rydzewska [13] also improved her previous analysis [12] by using the results of [18] and obtained the rate of convergence of the operators of

type (1.3) at μ -generalized d -point of integrable functions which are 2π -periodic with respect to each variables, separately.

Later on, Musielak [10] used the Lipschitz condition for $K_\lambda : G \times \mathbb{R} \rightarrow \mathbb{R}$ for the first time to prove some theorems for nonlinear integral operators of the form:

$$(1.4) \quad T_\lambda(f; s) = \int_G K_\lambda(t - s, f(t)) dt, \quad s \in G, \quad \lambda \in \Lambda,$$

where G is a locally compact Abelian group equipped with Haar measure and Λ is a non-empty index set with a topology. Therefore, well-known solution technics are used in nonlinear problems by the aid of indicated Lipschitz condition. Afterwards, Swiderski and Wachnicki [16] investigated the pointwise convergence of two-parameter setting of integrals of type (1.4) to the functions which are integrable on some locally compact Abelian groups with Haar measure.

Nowadays, the usage of nonlinear integral operators in sampling theory is very common. On the other hand, signal and image processing are two main research fields around sampling theory. We recommend the reader to see the book by Bardaro et al. [3] for further studies regarding the convergence of nonlinear integral operators and sampling type operators considered in some function spaces. As concerns the study of integral operators in various settings, the reader may see also [1–6, 8–10, 12–21].

One may also consider the pointwise approximation by singular integral operators in weighted Lebesgue spaces like in usual Lebesgue spaces. Therefore, we shall mention the papers [1], [9] and [19] which are important works on weighted approximation by singular integral operators. Those contain detailed information regarding the characterization of the weight functions with examples.

As a continuation of [20, 21], the aim of this paper is to investigate the pointwise convergence and the rate of pointwise convergence of nonlinear double singular integral operators in the following form:

$$(1.5) \quad T_\eta(f; x, y) = \iint_{\mathbb{R}^2} K_\eta(t - x, s - y, f(t, s)) ds dt, \quad (x, y) \in \mathbb{R}^2, \quad \eta \in \Lambda,$$

where Λ is a non-empty set of indices with a topology and $K_\eta : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is a kernel function. Also, $f \in L_{p,\varphi}(\mathbb{R}^2)$, where the symbol $L_{p,\varphi}(\mathbb{R}^2)$ denotes the space of all measurable functions f for which $\left| \frac{f(t,s)}{\varphi(t,s)} \right|^p$ ($1 \leq p < \infty$) is integrable and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is a locally bounded weight (positive and measurable) function. The norm formula for the space $L_{p,\varphi}(\mathbb{R}^2)$ (see, e.g., [9, 15]) is given by

$$\|f\|_{L_{p,\varphi}(\mathbb{R}^2)} = \left(\iint_{\mathbb{R}^2} \left| \frac{f(t,s)}{\varphi(t,s)} \right|^p ds dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The paper is organized as follows: In Section 2, we give some preliminary concepts. In Section 3, the existence of the operators of type (1.5) is explored. In Section 4, main result is presented. In Section 5, the rate of pointwise convergence of the operators of type (1.5) is established. In Section 6, we summarize the results of this paper.

2. Preliminaries

Now, we give main definitions used in the manuscript.

DEFINITION 2.1. Let δ_1 be a fixed positive real number such that $0 < h, k < \delta_1$. A p - μ -generalized Lebesgue point of a locally p -integrable function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a point $(x_0, y_0) \in \mathbb{R}^2$ satisfying

$$(2.1) \quad \lim_{(h,k) \rightarrow (0,0)} \left(\frac{1}{\mu(h,k)} \int_0^h \int_0^k |g(x_0 \pm t, y_0 \pm s) - g(x_0, y_0)|^p ds dt \right)^{\frac{1}{p}} = 0, \quad 1 \leq p < \infty,$$

where $\mu : \mathbb{R}^2 \rightarrow \mathbb{R}$ is absolutely continuous in two-dimensional sense on $[0, \delta_1] \times [0, \delta_1]$ (see [13]), increasing with respect to each variables separately on $[0, \delta_1]$ and $\mu(t, s) = 0$ whenever $ts = 0$.

REMARK 2.2. The given definition is an updated version of the μ -generalized d -point definition given in [13] depending on the requirements of the new problem (see also [6]).

DEFINITION 2.3. (Class A_φ) Let $\Lambda \subset \mathbb{R}_0^+$ be a non-empty set of indices. We allow the symbol η_0 to be either accumulation point of Λ or ∞ . Let $\varphi > 0$ be a locally bounded weight function defined on \mathbb{R}^2 , that

is to say φ is bounded on arbitrary bounded subsets of \mathbb{R}^2 , such that the following inequality:

$$(2.2) \quad \varphi(t+x, s+y) \leq \varphi(t, s)\varphi(x, y)$$

holds for every $(t, s) \in \mathbb{R}^2$ and $(x, y) \in \mathbb{R}^2$.

A family $(K_\eta)_{\eta \in \Lambda}$ consisting of the functions $K_\eta : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is called *Class* A_φ , if the following conditions hold:

- (a) $K_\eta(t, s, 0) = 0$ for every $(t, s) \in \mathbb{R}^2$ and for each $\eta \in \Lambda$, and $K_\eta(\cdot, \cdot, u) \in L_1(\mathbb{R}^2)$ for all values of $u \in \mathbb{R}$ and for any $\eta \in \Lambda$.
- (b) There exists a family $(L_\eta)_{\eta \in \Lambda}$ consisting of the (globally) integrable functions $L_\eta \geq 0$ defined on \mathbb{R}^2 such that the Lipschitz inequality:

$$|K_\eta(t, s, u) - K_\eta(t, s, v)| \leq L_\eta(t, s) |u - v|$$

holds for every $(t, s) \in \mathbb{R}^2$, $u, v \in \mathbb{R}$, and for any $\eta \in \Lambda$.

- (c) $\lim_{(x, y, \eta) \rightarrow (x_0, y_0, \eta_0)} \left| \iint_{\mathbb{R}^2} K_\eta \left(t-x, s-y, \frac{u}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - u \right| = 0$ for every $u \in \mathbb{R}$ and for any $(x_0, y_0) \in \mathbb{R}^2$.
- (d) $\lim_{\eta \rightarrow \eta_0} \left[\sup_{\xi \leq \sqrt{t^2+s^2}} [\varphi(t, s)L_\eta(t, s)] \right] = 0$ for every $\xi > 0$.
- (e) $\lim_{\eta \rightarrow \eta_0} \left[\iint_{\xi \leq \sqrt{t^2+s^2}} \varphi(t, s)L_\eta(t, s) dsdt \right] = 0$ for every $\xi > 0$.
- (f) $\|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \leq M < \infty$ for every $\eta \in \Lambda$ (value of M is independent of η).
- (g) For a given positive real number δ_0 satisfying $\delta_0 \geq \delta_1 > 0$, L_η is monotonically increasing on $(-\delta_0, 0]$ and monotonically decreasing on $[0, \delta_0)$ with respect to t and similarly, L_η is monotonically increasing on $(-\delta_0, 0]$ and monotonically decreasing on $[0, \delta_0)$ with respect to s , for any $\eta \in \Lambda$. Analogously, L_η is bimonotonically increasing with respect to (t, s) on $[0, \delta_0) \times [0, \delta_0)$ and $(-\delta_0, 0] \times (-\delta_0, 0]$ and similarly, L_η is bimonotonically decreasing with respect to (t, s) on $[0, \delta_0) \times (-\delta_0, 0]$ and $(-\delta_0, 0] \times [0, \delta_0)$ for any $\eta \in \Lambda$.

Throughout this paper the kernel function K_η belongs to *Class* A_φ .

REMARK 2.4. For the motivation of above definition which is used except condition (g) in also [20], we refer the reader to see [4, 9, 19].

REMARK 2.5. If the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bimonotonic on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2] \subset \mathbb{R}^2$ then the equality given as

$$\begin{aligned} V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) &= \bigvee_{\alpha_1}^{\alpha_2} \bigvee_{\beta_1}^{\beta_2} (g(t, s)) \\ &= |g(\alpha_1, \beta_1) - g(\alpha_1, \beta_2) - g(\alpha_2, \beta_1) + g(\alpha_2, \beta_2)| \end{aligned}$$

holds [7,18]. Here, V , which is given as $V(g; [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]) = \bigvee_{\alpha_1}^{\alpha_2} \bigvee_{\beta_1}^{\beta_2} (g(t, s))$, denotes bivariation of g on $[\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$.

3. Existence of the Operators

This section starts with the following theorem which gives the existence of the operators of type (1.5).

Main result in this work is based on the following theorem.

THEOREM 3.1. *If $f \in L_{p,\varphi}(\mathbb{R}^2)$, then $T_\eta(f) \in L_{p,\varphi}(\mathbb{R}^2)$ and*

$$\|T_\eta f\|_{L_{p,\varphi}(\mathbb{R}^2)} \leq \|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_{p,\varphi}(\mathbb{R}^2)},$$

holds for every $\eta \in \Lambda$.

Proof. Let $p = 1$. Using the norm formula for the space $L_{1,\varphi}(\mathbb{R}^2)$ (see, e.g., [19]), we have

$$\|T_\eta(f; x, y)\|_{L_{1,\varphi}(\mathbb{R}^2)} = \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left| \iint_{\mathbb{R}^2} K_\eta(t - x, s - y, f(t, s)) dsdt \right| dydx.$$

Using conditions (a) and (b), we have

$$\|T_\eta(f; x, y)\|_{L_{1,\varphi}(\mathbb{R}^2)} \leq \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left(\iint_{\mathbb{R}^2} |f(t + x, s + y)| L_\eta(t, s) dsdt \right) dydx.$$

Now, applying inequality (2.2) and Fubini's theorem (see, e.g., [5]) to the last inequality, we obtain

$$\begin{aligned}
& \|T_\eta(f; x, y)\|_{L_1, \varphi(\mathbb{R}^2)} \\
& \leq \iint_{\mathbb{R}^2} \frac{1}{\varphi(x, y)} \left(\iint_{\mathbb{R}^2} |f(t+x, s+y)| L_\eta(t, s) dsdt \right) dydx \\
& = \iint_{\mathbb{R}^2} L_\eta(t, s) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(t+x, s+y)} \right| \frac{\varphi(x+t, s+y)}{\varphi(x, y)} dydx \right) dsdt \\
& \leq \iint_{\mathbb{R}^2} L_\eta(t, s) \left(\iint_{\mathbb{R}^2} \left| \frac{f(t+x, s+y)}{\varphi(t+x, s+y)} \right| \frac{\varphi(x, y)\varphi(t, s)}{\varphi(x, y)} dydx \right) dsdt \\
& = \|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_1, \varphi(\mathbb{R}^2)}.
\end{aligned}$$

In view of condition (f), the assertion follows. Thus, the proof is completed for this case.

Let $1 < p < \infty$. Using the norm formula for the space $L_{p, \varphi}(\mathbb{R}^2)$ (see, e.g., [15]), we have

$$\begin{aligned}
& \|T_\eta(f; x, y)\|_{L_{p, \varphi}(\mathbb{R}^2)} \\
& = \left(\iint_{\mathbb{R}^2} \frac{1}{[\varphi(x, y)]^p} \left| \iint_{\mathbb{R}^2} K_\eta(t-x, s-y, f(t, s)) dsdt \right|^p dydx \right)^{\frac{1}{p}}.
\end{aligned}$$

Using conditions (a) and (b), we have

$$\begin{aligned}
& \|T_\eta(f; x, y)\|_{L_{p, \varphi}(\mathbb{R}^2)} \\
& \leq \left(\iint_{\mathbb{R}^2} \frac{1}{[\varphi(x, y)]^p} \left(\iint_{\mathbb{R}^2} |f(t+x, s+y)| L_\eta(t, s) dsdt \right)^p dydx \right)^{\frac{1}{p}}.
\end{aligned}$$

Now, applying inequality (2.2), and generalized Minkowski inequality (see, e.g., [15]) to the last inequality, we obtain the desired result, that is

$$\|T_\eta\|_{L_{p, \varphi}(\mathbb{R}^2)} \leq \|\varphi L_\eta\|_{L_1(\mathbb{R}^2)} \|f\|_{L_{p, \varphi}(\mathbb{R}^2)}.$$

Hence, the assertion follows from condition (f). Thus the proof is completed. \square

4. Convergence at Characteristic Points

THEOREM 4.1. *If $(x_0, y_0) \in \mathbb{R}^2$ is a common p - μ -generalized Lebesgue point of functions $f \in L_{p,\varphi}(\mathbb{R}^2)$ ($1 \leq p < \infty$) and φ , then*

$$\lim_{(x,y,\eta) \rightarrow (x_0,y_0,\eta_0)} |T_\eta(f; x, y) - f(x_0, y_0)| = 0$$

on any set Z on which the function $\Delta(x, y, \eta, \delta)$ defined by

$$\begin{aligned} \Delta(x, y, \eta, \delta) : &= \int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} L_\eta(t - x, s - y) |d_t d_s \mu(|t - x_0|, |s - y_0|)| \\ &+ 2 \int_{x_0 - \delta}^{x_0 + \delta} L_\eta(t - x, 0) |d_t \mu(|t - x_0|, |y - y_0|)| \\ &+ 2 \int_{y_0 - \delta}^{y_0 + \delta} L_\eta(0, s - y) |d_s \mu(|x - x_0|, |s - y_0|)| \\ &+ 4L_\eta(0, 0) \mu(|x - x_0|, |y - y_0|), \end{aligned}$$

where $0 < \delta < \delta_1$, is bounded as (x, y, η) tends to (x_0, y_0, η_0) .

Here, $|d_t d_s \mu(|t - x_0|, |s - y_0|)|$, $|d_t \mu(|t - x_0|, |y - y_0|)|$ and $|d_s \mu(|x - x_0|, |s - y_0|)|$ denote Lebesgue-Stieltjes measures.

Proof. The proof of theorem will be given for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar and it is skipped.

Now, set $I(x, y, \eta) := |T_\eta(f; x, y) - f(x_0, y_0)|$. Using condition (c), we obtain

$$\begin{aligned} I(x, y, \eta) &= \left| \iint_{\mathbb{R}^2} K_\eta(t-x, s-y, f(t, s)) dsdt - f(x_0, y_0) \right| \\ &= \left| \iint_{\mathbb{R}^2} K_\eta(t-x, s-y, f(t, s)) dsdt \right. \\ &\quad \left. - \iint_{\mathbb{R}^2} K_\eta\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt \right. \\ &\quad \left. + \iint_{\mathbb{R}^2} K_\eta\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|. \end{aligned}$$

Using condition (b), it is easy to see that the following inequality holds:

$$\begin{aligned} I(x, y, \eta) &\leq \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) L_\eta(t-x, s-y) dsdt \\ &\quad + \left| \iint_{\mathbb{R}^2} K_\eta\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|. \end{aligned}$$

Since the inequality $(m+n)^p \leq 2^p(m^p + n^p)$ holds provided that m and n are positive numbers (see, e.g., [11]), we have

$$\begin{aligned} [I(x, y, \eta)]^p &\leq 2^p \left(\iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) L_\eta(t-x, s-y) dsdt \right)^p \\ &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\eta\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|^p. \end{aligned}$$

Now, applying Hölder’s inequality (see, e.g., [11]), to the first integral of the resulting inequality, we have

$$\begin{aligned}
 & [I(x, y, \eta)]^p \\
 & \leq 2^p \rho(x, y, \eta) \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\
 & + 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p,
 \end{aligned}$$

where

$$\rho(x, y, \eta) = \left(\iint_{\mathbb{R}^2} \varphi(t, s) L_\eta(t - x, s - y) dsdt \right)^{\frac{p}{q}}.$$

Let $(x_0, y_0) \in \mathbb{R}^2$ be a common $p - \mu$ -generalized Lebesgue point of the functions $f \in L_{p,\varphi}(\mathbb{R}^2)$ and φ . In view of one of the limit relations (2.1), for every $\varepsilon > 0$, there exists $\delta > 0$, we have the following inequality:

$$(4.1) \quad \int_{x_0}^{x_0+h} \int_{y_0-k}^{y_0} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p dsdt < \varepsilon^p \mu(h, k)$$

for every h and k satisfying $0 < h, k \leq \delta < \delta_1$. Let $0 < x_0 - x < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$. Moreover, the following inequality holds:

$$\begin{aligned}
 & [I(x, y, \eta)]^p \\
 & \leq 2^p \rho(x, y, \eta) \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\
 & + 2^p \rho(x, y, \eta) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\
 & + 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p,
 \end{aligned}$$

where $B_\delta := \{(t, s) \in \mathbb{R}^2 : (t - x_0)^2 + (s - y_0)^2 < \delta^2\}$.

Now, applying the inequality given by $(m+n)^p \leq 2^p(m^p + n^p)$ once more to the first term of the above inequality, we obtain

$$\begin{aligned}
& [I(x, y, \eta)]^p \\
& \leq 2^{2p} \rho(x, y, \eta) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\mathbb{R}^2 \setminus B_\delta} \varphi(t, s) L_\eta(t-x, s-y) ds dt \\
& + 2^{2p} \rho(x, y, \eta) \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} \right|^p \varphi(t, s) L_\eta(t-x, s-y) ds dt \\
& + 2^p \rho(x, y, \eta) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t-x, s-y) ds dt \\
& + 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - f(x_0, y_0) \right|^p.
\end{aligned}$$

Now, we may define the following set as follows:

$$N_\delta = \left\{ (x, y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 < \frac{\delta^2}{4} \right\}.$$

Making comparison between the sets B_δ and N_δ gives the relation such that $\mathbb{R}^2 \setminus B_\delta \subseteq \mathbb{R}^2 \setminus A_\delta$, where

$$A_\delta = \left\{ (t, s) \in B_\delta : (t-x)^2 + (s-y)^2 < \frac{\delta^2}{4}, (x, y) \in N_\delta \right\}.$$

In view of definition of A_δ and inequality (2.2), the following inequality holds:

$$\begin{aligned}
& [I(x, y, \eta)]^p \\
& \leq 2^{2p} \varphi(x, y) \rho(x, y, \eta) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\mathbb{R}^2 \setminus A_\delta} \varphi(t-x, s-y) L_\eta(t-x, s-y) ds dt \\
& + 2^{2p} \varphi(x, y) \rho(x, y, \eta) \sup_{(t,s) \in \mathbb{R}^2 \setminus A_\delta} [\varphi(t-x, s-y) L_\eta(t-x, s-y)] \|f\|_{L_{p,\varphi}(\mathbb{R}^2)}^p
\end{aligned}$$

$$\begin{aligned}
 &+ 2^p \rho(x, y, \eta) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\
 &+ 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p.
 \end{aligned}$$

Rearranging and rewriting the last inequality, we obtain

$$\begin{aligned}
 &[I(x, y, \eta)]^p \\
 &\leq 2^{2p} \varphi(x, y) \rho(x, y, \eta) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\frac{\delta}{2} \leq \sqrt{u^2+v^2}} \varphi(u, v) L_\eta(u, v) dvdu \\
 &+ 2^{2p} \varphi(x, y) \rho(x, y, \eta) \sup_{\frac{\delta}{2} \leq \sqrt{u^2+v^2}} [\varphi(u, v) L_\eta(u, v)] \|f\|_{L_{p,\varphi}(\mathbb{R}^2)}^p \\
 &+ 2^p \rho(x, y, \eta) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\
 &+ 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p \\
 &= I_1(x, y, \eta) + I_2(x, y, \eta) + \rho(x, y, \eta) 2^p I_3(x, y, \eta) + I_4(x, y, \eta).
 \end{aligned}$$

In view of inequality (2.2) and change of variables, the following inequality holds:

$$\begin{aligned}
 \rho(x, y, \eta) &= \left(\iint_{\mathbb{R}^2} \varphi(t, s) L_\eta(t - x, s - y) dsdt \right)^{\frac{p}{q}} \\
 &\leq (\varphi(x, y))^{\frac{p}{q}} \left(\iint_{\mathbb{R}^2} \varphi(t - x, s - y) L_\eta(t - x, s - y) dsdt \right)^{\frac{p}{q}} \\
 &= (\varphi(x, y))^{\frac{p}{q}} \left(\iint_{\mathbb{R}^2} \varphi(u, v) L_\eta(u, v) dvdu \right)^{\frac{p}{q}}.
 \end{aligned}$$

Thus, boundedness of the term $\varphi(x, y) \rho(x, y, \eta)$ as (x, y, η) tends to (x_0, y_0, η_0) follows from condition (f) and boundedness of $\varphi(x, y)$ around (x_0, y_0) . On the other hand, $I_4(x, y, \eta) \rightarrow 0$ as $(x, y, \eta) \rightarrow (x_0, y_0, \eta_0)$ by condition (c). Lastly, $I_1(x, y, \eta) \rightarrow 0$ and $I_2(x, y, \eta) \rightarrow 0$ as $(x, y, \eta) \rightarrow (x_0, y_0, \eta_0)$ by conditions (e) and (d), respectively. Also, for similar technic showing how the terms $I_1(x, y, \eta)$ and $I_2(x, y, \eta)$ tend to zero as $(x, y, \eta) \rightarrow (x_0, y_0, \eta_0)$, we refer the reader to [14].

Now, we may write the following inequality for the integral $I_3(x, y, \eta)$:

$$\begin{aligned} I_3(x, y, \eta) &= \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\eta(t - x, s - y) dsdt \\ &\leq \sup_{(t,s) \in Q_\delta} \varphi(t, s) \iint_{Q_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\eta(t - x, s - y) dsdt \\ &= \sup_{(t,s) \in Q_\delta} \varphi(t, s) I_{31}(x, y, \eta), \end{aligned}$$

where $Q_\delta = (x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)$.

It is easy to see that the following equality holds:

$$\begin{aligned} I_{31} &= \left\{ \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} + \int_{x_0-\delta}^{x_0} \int_{y_0-\delta}^{y_0} \right\} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\eta(t - x, s - y) dsdt \\ &\quad + \left\{ \int_{x_0-\delta}^{x_0} \int_{y_0}^{y_0+\delta} + \int_{x_0}^{x_0+\delta} \int_{y_0}^{y_0+\delta} \right\} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\eta(t - x, s - y) dsdt \\ &= I_{311} + I_{312} + I_{313} + I_{314}. \end{aligned}$$

Now, we consider I_{311} .

We define the new function such that

$$F(t, s) := \int_{x_0}^t \int_s^{y_0} \left| \frac{f(u, v)}{\varphi(u, v)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p dvdu$$

for all t and s satisfying $0 < t - x_0 \leq \delta$ and $0 < y_0 - s \leq \delta$, in view of (4.1) one may easily observe that

$$(4.2) \quad |F(t, s)| \leq \varepsilon^p \mu(t - x_0, y_0 - s).$$

The Lebesgue-Stieltjes integral form of the integral I_{311} is as follows:

$$|I_{311}| = \left| \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} L_\eta(t-x, s-y) d_t d_s [-F(t, s)] \right|.$$

Therefore, applying bivariate integration by parts method (see, e.g., [18]) to the integral I_{311} , we have

$$\begin{aligned} |I_{311}| \leq & \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} |F(t, s)| |d_t d_s [L_\eta(t-x, s-y)]| \\ & + \int_{x_0}^{x_0+\delta} |F(t, y_0-\delta)| |d_t [L_\eta(t-x, y_0-\delta-y)]| \\ & + \int_{y_0-\delta}^{y_0} |F(x_0+\delta, s)| |d_s [L_\eta(x_0+\delta-x, s-y)]| \\ & + |F(x_0+\delta, y_0-\delta)| L_\eta(x_0+\delta-x, y_0-\delta-y). \end{aligned}$$

In view of inequality (4.2), we obtain

$$\begin{aligned} |I_{311}| \leq & \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} \mu(t-x_0, y_0-s) |d_t d_s [L_\eta(t-x, s-y)]| \\ & + \varepsilon^p \int_{x_0}^{x_0+\delta} \mu(t-x_0, \delta) |d_t [L_\eta(t-x, y_0-\delta-y)]| \\ & + \varepsilon^p \int_{y_0-\delta}^{y_0} \mu_2(\delta, y_0-s) |d_s [L_\eta(x_0+\delta-x, s-y)]| \\ & + \varepsilon^p \mu(\delta, \delta) L_\eta(x_0+\delta-x, y_0-\delta-y). \end{aligned}$$

Let us give the bivariation and single variations as follows:

$$A_1(t, s) := \begin{cases} \bigvee_t^{x_0+\delta-x} \bigvee_{y_0-\delta-y}^s L_\eta(u, v), & x_0-x \leq t < x_0+\delta-x, \\ & y_0-\delta-y < s \leq y_0-y, \\ 0, & \text{otherwise,} \end{cases}$$

$$A_2(t) := \begin{cases} \bigvee_t^{x_0+\delta-x} L_\eta(u, y_0 - \delta - y), & x_0 - x \leq t < x_0 + \delta - x, \\ 0, & \text{otherwise,} \end{cases}$$

$$A_3(s) := \begin{cases} \bigvee_{y_0-\delta-y}^s L_\eta(x_0 + \delta - x, v), & y_0 - \delta - y < s \leq y_0 - y, \\ 0, & \text{otherwise.} \end{cases}$$

Taking above variations and Remark 2.5 into account and applying bivariate integration by parts to last inequality for I_{311} , we have

$$|I_{311}| \leq \left| -\varepsilon^p \int_{x_0-x}^{x_0-x+\delta} \int_{y_0-y-\delta}^{y_0-y} [A_1(t, s) + A_2(t) + A_3(s) + L_\eta(x_0 + \delta - x, y_0 - \delta - y)] d_t d_s \mu(t - x_0 + x, y_0 - s - y) \right|.$$

For the similar situation, the reader may see also [13, 18].

Clearly, using Remark 2.5 and condition (g), the following inequality holds for I_{311} :

$$|I_{311}| \leq \varepsilon^p \int_{x_0}^{x_0+\delta} \int_{y_0-\delta}^{y_0} L_\eta(t - x, s - y) |d_t d_s \mu(|t - x_0|, |s - y_0|)| + 2\varepsilon^p \int_{x_0}^{x_0+\delta} L_\eta(t - x, 0) d_t \mu(|t - x_0|, |y - y_0|).$$

Evaluating the integrals I_{312} , I_{313} , I_{314} with the same method and combining the respective inequalities, we obtain

$$I_{31} \leq \varepsilon^p \Delta(x, y, \eta, \delta).$$

The remaining part of the proof is obvious by the hypothesis. Hence, I_{31} tends to 0 as (x, y, η) tends to (x_0, y_0, η_0) . Note that, if we reverse each of the assumptions what we supposed at the beginning of the proof, such as $0 < x - x_0 < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$, then we obtain the same inequality. Thus, the proof is completed. \square

5. Rate of Pointwise Convergence

THEOREM 5.1. *Suppose that the hypotheses of Theorem 4.1 are satisfied and the following conditions hold there:*

- (i) $\Delta(x, y, \eta, \delta)$ tends to 0 as (x, y, η) tends to (x_0, y_0, η_0) for some $\delta \in (0, \delta_1)$.
- (ii) For every $\xi > 0$, we have $\sup_{\xi \leq \sqrt{t^2+s^2}} [\varphi(t, s)L_\eta(t, s)] = o(\Delta(x, y, \eta, \delta))$ as (x, y, η) tends to (x_0, y_0, η_0) .
- (iii) For every $u \in \mathbb{R}$, we have $\left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{u}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - u \right|^p = o(\Delta(x, y, \eta, \delta))$ as (x, y, η) tends to (x_0, y_0, η_0) .
- (iv) For every $\xi > 0$, we have $\iint_{\xi \leq \sqrt{t^2+s^2}} \varphi(t, s)L_\eta(t, s) dsdt = o(\Delta(x, y, \eta, \delta))$ as (x, y, η) tends to (x_0, y_0, η_0) .

Then, at each common $p - \mu$ -generalized Lebesgue point of the functions $f \in L_{p,\varphi}(\mathbb{R}^2)$ ($1 \leq p < \infty$) and φ , we have

$$|T_\eta(f; x, y) - f(x_0, y_0)|^p = o(\Delta(x, y, \eta, \delta))$$

as (x, y, η) tends to (x_0, y_0, η_0) .

Proof. By the hypotheses of Theorem 4.1, we may write

$$\begin{aligned} & |T_\eta(f; x, y) - f(x_0, y_0)|^p \\ & \leq 2^{2p} \varphi(x, y) \rho(x, y, \eta) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\frac{\delta}{2} \leq \sqrt{u^2+v^2}} \varphi(u, v) L_\eta(u, v) dvdu \\ & \quad + 2^{2p} \varphi(x, y) \rho(x, y, \eta) \sup_{\frac{\delta}{2} \leq \sqrt{u^2+v^2}} [\varphi(u, v) L_\eta(u, v)] \|f\|_{L_{p,\varphi}(\mathbb{R}^2)}^p \\ & \quad + 2^p \varepsilon^p \rho(x, y, \eta) \sup_{(t,s) \in Q_\delta} \varphi(t, s) \Delta(x, y, \eta, \delta) \\ & \quad + 2^p \left| \iint_{\mathbb{R}^2} K_\eta \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p. \end{aligned}$$

Here, Q_δ denotes the set defined in the proof of Theorem 4.1. From (i)-(iv), and using *Class* A_φ conditions, the assertion follows. Thus, the proof is completed. \square

EXAMPLE 5.2. Define the kernel function such that

$$K_\eta(t, s, u) = \begin{cases} \frac{\eta u}{2} + \sin \frac{\eta u}{2}, & \text{if } (t, s) \in \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right], \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right], \end{cases}$$

where $\eta \in \mathbb{N}$ and $\eta_0 = \infty$. Let $u, v \in \mathbb{R}$. This kernel is the two dimensional analogue of the kernel given in [16].

Observe that $|K_\eta(t, s, u) - K_\eta(t, s, v)| \leq \eta |u - v|$ for $(t, s) \in \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right]$ and $|K_\eta(t, s, u) - K_\eta(t, s, v)| = 0$ otherwise. Hence, we have

$$L_\eta(t, s) = \begin{cases} \eta, & \text{if } (t, s) \in \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right], \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right] \times \left[\frac{-1}{\sqrt{2\eta}}, \frac{1}{\sqrt{2\eta}} \right]. \end{cases}$$

It is easy to see that the conditions of class A_φ are satisfied.

Now, let $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ be given by $\varphi(t, s) = (1 + |t|)(1 + |s|)$. Let us find the rate of convergence according to hypotheses of Theorem 5.1.

For the simplicity take $(x_0, y_0) = (1, 1)$, $\mu(t, s) = ts$ and $p = 1$. Denote

$$\begin{aligned} \Delta_1(x, y, \eta, \delta) &= \int_{x_0 - \delta y_0 - \delta}^{x_0 + \delta y_0 + \delta} \int L_\eta(t - x, s - y) |d_t d_s \mu(|t - x_0|, |s - y_0|)| \\ &= \int_{1 - \delta - \delta}^{1 + \delta + \delta} \int \eta ds dt. \end{aligned}$$

Let $\lim_{\eta \rightarrow \infty} 4\eta\delta^2 = 0$. Consequently, if we choose $\delta > 0$ such that $\delta = o\left(\frac{1}{\eta}\right)$, then $\lim_{\eta \rightarrow \infty} \Delta_1(x, y, \eta, \delta) = 0$. The evaluations of the remaining terms of $\Delta(x, y, \eta, \delta)$ yield the same conclusion. Thus, we have

$$|T_\eta(f; x, y) - f(x_0, y_0)| = o\left(\frac{1}{\eta}\right).$$

6. Conclusion

In this paper, the pointwise convergence of the convolution type non-linear double integral operators depending on three parameters is investigated. The main result is presented as Theorem 4.1. By using main result, the rate of pointwise convergence of the indicated operators is investigated.

Acknowledgement: The authors thank the referees for their careful reading, valuable comments and suggestions for the improvement of the manuscript.

References

- [1] G. Alexits, *Convergence Problems of Orthogonal Series*, Translated from the German by I. Földer. International Series of Monographs in Pure and Applied Mathematics, Vol. 20 Pergamon Press, New York, Oxford, Paris, 1961.
- [2] C. Bardaro, *On approximation properties for some classes of linear operators of convolution type*, Atti Sem. Mat. Fis. Univ. Modena **33** (2) (1984), 329–356.
- [3] C. Bardaro, J. Musielak, G. Vinti, *Nonlinear Integral Operators and Applications*, De Gruyter Ser. Nonlinear Anal. Appl. 9, Walter de Gruyter, Berlin, 2003.
- [4] C. Bardaro, G. Vinti and H. Karsli, *Nonlinear integral operators with homogeneous kernels: pointwise approximation theorems*, Appl. Anal. **90** (3-4) (2011), 463–474.
- [5] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Vol. I. Academic Press, New York, London, 1971.
- [6] A. D. Gadjev, *The order of convergence of singular integrals which depend on two parameters*, Special problems of Functional Analysis and Their Appl. to the Theory of Diff. Eq. and the Theory of Func., Izdat. Akad. Nauk Azerbaïdajan. SSR. (1968), 40–44.
- [7] S. R. Ghorpade and B. V. Limaye, *A Course in Multivariable Calculus and Analysis*, Springer, New York, 2010.
- [8] H. Karsli, *Convergence and rate of convergence by nonlinear singular integral operators depending on two parameters*, Appl. Anal. **85** (6-7) (2006), 781–791.
- [9] R. G. Mamedov, *On the order of convergence of m -singular integrals at generalized Lebesgue points and in the space $L_p(-\infty, \infty)$* , Izv. Akad. Nauk SSSR Ser. Mat. **27** (2) (1963), 287–304.
- [10] J. Musielak, *On some approximation problems in modular spaces*, Constructive Function Theory 1981, (Proc. Int. Conf., Varna, June 1-5, 1981), Sofia, Publ. House Bulgarian Acad. Sci. (1983), 455–461.
- [11] W. Rudin, *Real and Complex Analysis*, Mc-Graw Hill Book Co., London, 1987.
- [12] B. Ryzewska, *Approximation des fonctions par des intégrales singulières ordinaires*, Fasc. Math. **7** (1973), 71–81.
- [13] B. Ryzewska, *Approximation des fonctions de deux variables par des intégrales singulières doubles*, Fasc. Math. **8** (1974), 35–45.
- [14] S. Siudut, *On the convergence of double singular integrals*, Comment. Math. **28** (1) (1988), 143–146.
- [15] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, New Jersey, 1970.
- [16] T. Swiderski and E. Wachnicki, *Nonlinear singular integrals depending on two parameters*, Comment. Math. **40** (2000), 181–189.
- [17] R. Taberski, *Singular integrals depending on two parameters*, Prace Mat. **7** (1962), 173–179.
- [18] R. Taberski, *On double integrals and Fourier series*, Ann. Polon. Math. **15** (1964), 97–115.
- [19] R. Taberski, *On double singular integrals*, Prace Mat. **19** (1976), 155–160.

- [20] G. Uysal, *A new approach to nonlinear singular integral operators depending on three parameters*. Open Math. **14** (2016), 897–907.
- [21] G. Uysal, M. M. Yilmaz and E. Ibikli, *On pointwise convergence of bivariate nonlinear singular integral operators*, Kuwait J. Sci. **44** (2) (2017), 46–57.

Gümrah Uysal

Department of Computer Technologies
Division of Technology of Information Security
Karabük University
Karabük 78050, Turkey
E-mail: fgumrahuysal@gmail.com

Vishnu Narayan Mishra

¹Department of Mathematics
Indira Gandhi National Tribal University
Lalpur, Amarkantak 484 887, Madhya Pradesh, India
²L. 1627 Awadh Puri Colony, Beniganj, Phase III
Opposite Industrial Training Institute (I.T.I.)
Ayodhya main road, Faizabad 224 001, Uttar Pradesh, India
E-mail: vishnunarayanmishra@gmail.com

Sevilay Kirci Serenbay

Department of Mathematics
Harran University
Şanlıurfa, 63200, Turkey
E-mail: sevilaykirci@gmail.com