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ON A CLASS OF QUANTUM ALPHA-CONVEX FUNCTIONS

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Abstract. Let $f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the open unit disc E. Then f is said to belong to the class M_α of alpha-convex functions, if it satisfies the condition

$$\Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > 0, \quad (z \in E).$$

In this paper, we introduce and study q-analogue of the class M_α by using concepts of Quantum Analysis. It is shown that the functions in this new class $M(q, \alpha)$ are q-starlike. A problem related to q-Bernardi operator is also investigated.

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1. Introduction

Let A be the class of analytic functions f defined in the open unit disc $E = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let C, S^* and M_α be the subclasses of A which consist of convex, starlike and α-convex functions, respectively. These classes are defined as follows.

$$\begin{aligned} C &= \left\{f \in A : \Re\left\{\frac{(zf'(z))'}{f'(z)}\right\} > 0,\ z \in E\right\} \\ S^* &= \left\{f \in A : \Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0,\ z \in E\right\} \\ M_\alpha &= \left\{f \in A : \Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > 0,\ \alpha \geq 0,\ z \in E\right\}. \end{aligned}$$

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The q-analogues of the classes C and S^* have been introduced and studied previously, see [2, 11, 13]. In this paper, we define q-analogue of a certain subclass of M_α and investigate some of its properties.

Quantum or q-calculus is ordinary calculus without limit. Recently it has attracted attention of many researchers due to its vast applications in many branches of mathematics and physics. Ismail et. al. [2] used q-derivative concept to introduce the class S_q^*, $0 < q < 1$, which is a generalization of the class S^*. It is shown that $\cap_{0<q<1} S_q^* = S^*$. For geometric properties of some classes of analytic functions involving q-calculus, see [6, 7, 8, 9, 10, 11, 12] and the references therein.

We recall some basic concepts from q-calculus which will be used in our discussion and refer to [3, 4] for more details.

The q-derivative of a function $f \in A$ is defined by

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0,$$

and $D_q f(0) = f'(0)$, where $q \in (0,1)$, see [3].

For a function $g(z) = z^n$, the q-derivative is

$$D_q g(z) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

We note that, as $q \to 1^-$, $D_q f(z) \to f'(z)$ and $[n]_q \to n$.
Thus, for $f \in A$ and given by (1), we have

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Also, as an inverse of q-derivative, Jackson [4] introduced the q-integral of $f \in A$ given by

$$\int_0^z f(t) \mathrm{d}_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(q^n z),$$

provided the series converges.
Under the hypothesis of the definition, the q-difference operator D_q satisfies certain algebraic properties and for details we refer to [1, 8, 10].

Let $f, g \in A$. Then f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, $z \in E$, if there exists a Schwartz function $w(z)$ analytic in E with $w(0) = 0$ and

$|w(z)| < 1$ for $z \in E$ such that $f(z) = g(w(z))$. If g is univalent in E, then $f \prec g$, if and only if, $f(0) = g(0)$ and $f(E) \subset g(E)$.

We recall the following definitions:

$$\begin{aligned} C_q(\gamma) &= \left\{ f \in A : \Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)} \right) > \gamma,\ 0 \le \gamma < 1,\ z \in E \right\} \\ S^*(\gamma) &= \{F \in A : F = zD_qf,\ f \in C_q(\gamma),\ 0 \le \gamma < 1,\ z \in E\}. \end{aligned}$$

Here and throughout this paper, it is assumed that $q \in (0,1)$, $z \in E$, unless otherwise stated.

Definition 1.1. Let $f \in A$, $q \in (0,1)$. Then f is said to belong to the class $ST(q)$ if it satisfies the following condition, for $z \in E$

$$\left| \frac{\left\{ \frac{zD_qf(z)}{f(z)} - 1 \right\}}{\left\{ \frac{zD_qf(z)}{f(z)} + 1 \right\}} \right| < q. \tag{2}$$

When $q \to 1^-$, the class $ST(q)$ coincides with the class S^* of starlike functions.

Similarly, $f \in A$ is said to belong to the class $CV(q)$ if, for $z \in E$

$$\left| \frac{\left\{ \frac{D_q(zD_qf(z))}{D_qf(z)} - 1 \right\}}{\left\{ \frac{D_q(zD_qf(z))}{D_qf(z)} + 1 \right\}} \right| < q \tag{3}$$

For $q \to 1^-$, $CV(q) \to C$, the class of convex functions.

Definition 1.2. Let $f \in A$ and let, for $\alpha \ge 0$, $\quad z \in E$

$$J_q(\alpha, f) = \alpha \left\{ \frac{D_q(zD_qf(z))}{D_qf(z)} \right\} + (1-\alpha) \left\{ \frac{zD_qf(z)}{f(z)} \right\}. \tag{4}$$

Then $f \in M(q,\alpha)$, if the following condition is satisfied. That is,

$$\left| \frac{\{J_q(\alpha, f) - 1\}}{\{J_q(\alpha, f) + 1\}} \right| < q.$$

When $q \to 1^-$, $M(q,\alpha)$ reduces to the class M_α of α-convex functions.

We note that $M(q,0) = ST(q)$ and $M(q,1) = CV(q)$.

2. Main Results

Theorem 2.1. *Let $f \in M(q,\alpha)$, $\alpha \ge 0$. Then $f \in ST(q)$.*

Proof. The case $\alpha = 0$ is trivial. We suppose $\alpha > 0$. To prove that $f \in ST(q)$, we have to show that f satisfies condition (1), which is equivalent to

$$\frac{zD_qf(z)}{f(z)} \prec \frac{1-qz}{1+qz},\ q \in (0,1).$$

Let

$$\frac{zD_qf(z)}{f(z)} \prec \frac{1-qw(z)}{1+qw(z)}. \tag{5}$$

Clearly $w(0)=0$ and $1+qw(z)\neq 0$. We shall show that $|z(z)|<1, \quad \forall z\in E$. We suppose on the contrary that there exists z_0, $z_\circ \in E$, such that $|w(z_0)|=1$. Then

$$J_q(\alpha, f(z_0)) = \frac{1-qw(z_\circ)}{1+qw(z_0)} - \frac{2\alpha qmw(z_0)}{(1+qw(z_0))(1-qw(z_0))}, \tag{6}$$

where we have used (5) and q-analogue of the well known Jack's Lemma for which we refer to [1]. It is shown that if $w(z)$ is analytic in E with $w(0)=0$, then $|w(z)|$ attains its maximum value on the circle $|z|=r$ at a point $z_o\in E$ and in this case $z_0D_qw(z_0)=mw(z_0), \quad m\geq 1$.

Now, from (6)

$$\left|\frac{J_q(\alpha, f(z_0))-1}{J_q(\alpha, f(z_0))+1}\right| \lesseqgtr q$$

if

$$|1+\alpha m - qw(z_0)|^2 \lesseqgtr |1-(1+\alpha m)qw(z_0)|^2,$$

or

$$(2\alpha m+\alpha^2m^2)(1-q^2)\lesseqgtr 0.$$

Since α and m are positive and $q\in(0,1)$, so the last expression is positive. This leads to conclude that $f\notin M(q,\alpha)$, which is a contradiction. Thus, $|w(z)|<1, \forall z\in E$. Hence $\frac{zD_qf(z)}{f(z)} \prec \frac{1-qz}{1+qz}$ and this completes the proof. □

Theorem 2.2. *For $0\leq\beta<\alpha$, $M(q,\alpha)\subset M(q,\beta)$.*

Proof. The case $\beta=0$ follows directly from Theorem 2.1. Therefore we suppose $\beta>0$ and $f\in M(q,\alpha)$. Then there exist $w_1(z), w_2(z)$ which are analytic in E with $w_i(0)=0$ and $|w_i(z)|<1$ for $i=1,2$ such that

$$\frac{zD_qf(z)}{f(z)} = \frac{1-qw_1(z)}{1+qw_1(z)} = p_1(z) \prec \frac{1-qz}{1+qz} \quad \text{by Theorem 2.1}$$

and

$$J_q(\alpha, f(z)) = \frac{1-qw_2(z)}{1+qw_2(z)} = p_2(z) \prec \frac{1-qz}{1+qz}.$$

For $\beta<\alpha$, we can write

$$\begin{aligned} J_q(\beta, f(z)) &= \frac{\beta}{\alpha}J_q(\alpha, f(z)) + \left(1-\frac{\beta}{\alpha}\right)\frac{zD_qf(z)}{f(z)} \\ &= \frac{\beta}{\alpha}p_1(z) + \left(1-\frac{\beta}{\alpha}\right)p_2(z) \end{aligned}$$

$$= \quad p(z).$$

Using subordination principle, it follows that $p(z) \prec \frac{1-qz}{1+qz}$.
Therefore,

$$J_q(\beta, f(z)) \prec \frac{1-qz}{1+qz}$$

and this proves $f \in M(q, \beta)$ in E. □

Corollary 2.3. *For $\alpha \geq \frac{1}{q}$, $M(\alpha, q) \subset CV(q)$.*

When $q \to 1^-$, we obtain the established result that α-convex functions are convex for $\alpha \geq 1$, see [5].

Remark 2.1. From Theorem 2.2, we have

$$M(q, \alpha) \subset M(q, \beta) \subset ST(q), \ 0 \leq \beta < \alpha. \tag{7}$$

In view of (7), it follows that, given a function in $ST(q)$, we can find the largest possible value of α such that $f \in M(q, \alpha)$, $\quad \alpha \geq 0$.

We define the following.

Definition 2.4. Let $f \in ST(q)$ and

$$\alpha = \alpha(f) = l.u.b\{\beta : f \in M(q, \beta), \ \beta \geq 0\}.$$

Then we say that f is q-starlike of order q and type α and we write $f \in M^*(q, \alpha)$, where α is nonnegative and may be infinite.

If $f \in M^*(q, \alpha)$, then $f \in M(q, \beta)$ for all β, $\quad 0 \leq \beta \leq \alpha$.
That is

$$J_q(\beta, f) = \frac{1 - qw(z)}{1 + qw(z)}, \quad 0 \leq \beta \leq \alpha,$$

where $w(z)$ is analytic in E, $w(0) = 0$ and $|w(z)| < 1$ in E. When $\beta \to \alpha$, $f \in M(q, \alpha)$.
Hence $f \in M^*(q, \alpha)$ for $\alpha < \infty$, if and only if,

$$f \in M(q, \beta), \quad \text{for} \quad 0 \leq \beta \leq \alpha$$

and $f \notin M(q, \beta)$ for $\beta > \alpha$. Thus, we write $ST(q)$ as a disjoint union

$$ST(q) = \cup_{\alpha \geq 0} M^*(q, \alpha).$$

Theorem 2.5. *Let $f \in M^*(q, \alpha)$, $\alpha > 0$. For $0 < \beta < \alpha$, choose the branch of $\{\frac{zD_q f(z)}{f(z)}\}^\beta$ which takes value 1 at the origin. Then $F_\beta \in ST(q)$, where*

$$F_\beta(z) = f(z) \left\{ \frac{zD_q f(z)}{f(z)} \right\}^\beta. \tag{8}$$

Proof. Let $f \in M^*(q,\alpha)$. This implies $f \in M(q,\beta)$ for all $\beta < \alpha$.
Now q-logarithmic differentiation of (8) yields

$$\begin{aligned} \frac{zD_qF_\beta(z)}{F_\beta(z)} &= \frac{zD_qf(z)}{f(z)} + \beta\left\{\frac{D_q(zD_qf(z))}{D_qf(z)} - \frac{zD_qf(z)}{f(z)}\right\} \\ &= (1-\beta)\frac{zD_qf(z)}{f(z)} + \beta\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) \\ &= J_q(\beta,f) \prec \frac{1-qz}{1+qz}. \end{aligned}$$

This implies $F_\beta \in ST(q)$, and the proof is complete. □

Remark 2.2. If we denote by $B(q,\alpha)$ the subclass of q-Bazilevic functions f defined by

$$f(z) = \left\{\alpha \int_0^z (F(t))^\alpha t^{-1} \mathrm{d}_q t\right\}^{\frac{1}{\alpha}},$$

where $F \in ST(q)$ for $\alpha > 0$, then it can easily be seen that

$$B\left(q, \frac{1}{\alpha}\right) = M(q,\alpha).$$

Theorem 2.6. *Let* $\frac{zD_qf(z)}{f(z)} \prec \frac{1}{1-qz}$, $g \in M(q,0)$ *and, for all* $m \in \mathbb{N} = \{1,2,3,...\}$, *define*

$$F_m(z) = \frac{[m+1]_q}{(g(z))^m} \int_0^z t^{m-1} f(t) \mathrm{d}_q t, \quad q > \frac{1}{2m}. \tag{9}$$

Then

$$\Re\left\{\frac{zD_qF_m(z)}{F_m(z)}\right\} > 0 \quad \textit{for} \quad |z| < \frac{1}{q}.$$

Proof. We can write (9) as

$$\begin{aligned} \left[F_m(z)\left(\frac{g(z)}{z}\right)^m\right] &= \frac{[m+1]_q}{z^m} \int_0^z t^{m-1} f(t) \mathrm{d}_q t \\ &= E_m(z). \end{aligned} \tag{10}$$

We note that right hand side of (10) represents q-Bernardi integral operator and it is shown in [**?**] that

$$\frac{zD_qE_m(z)}{E_m(z)} \prec \frac{1}{1-qz}, \tag{11}$$

if f satisfies the given condition in E.

Now differentiating (10) q-logarithmically, and with some computation, we have

$$\frac{zD_qF_m(z)}{F_m(z)} = -m\left[\frac{zD_qg(z)}{g(z)} - 1\right] + \frac{zD_qE_m(z)}{E_m(z)}$$

$$= \quad -mh_1(z) + m + h_2(z), \tag{12}$$

where

$$\begin{aligned} h_1(z) &= \frac{zD_q g(z)}{g(z)}, \\ h_2(z) &= \frac{zD_q E_m(z)}{E_m(z)}. \end{aligned}$$

Since $g \in M(q, 0)$, we have

$$\frac{1-qr}{1+qr} \leq |h_1(z)| \leq \frac{1+qr}{1-qr}. \tag{13}$$

Also, from (11), it follows that

$$\frac{1}{1+qr} \leq |h_2(z)| \leq \frac{1}{1-qr}. \tag{14}$$

Thus, using (13), (14), it follows from (12) that

$$\begin{aligned} \Re\left\{\frac{zD_qF_m(z)}{F_m(z)}\right\} &\geq -m\frac{1+qr}{1-qr} + m + \frac{1}{1+qr} \\ &= \frac{-m(1+qr)^2 + m(1-q^2r^2) + (1-qr)}{(1+qr)(1-qr)} \\ &= \frac{1 + q(1-2m)r - 2mq^2r^2}{(1+qr)(1-qr)} \\ &= \frac{T(r)}{(1+qr)(1-qr)}, \end{aligned} \tag{15}$$

where

$$T(r) = 1 - q(2m-1)r - 2mq^2r^2.$$

Clealry

$$T(0) = 1 > 0, \quad T(1) = 1 - q(2m-1) - 2mq^2 < 0, \quad \text{for} \quad q > \frac{1}{2m}.$$

Thus $T(r) = 0$ has a least positive root $r_q = \frac{1}{q}$ for which the right hand side of (15) is positive. This proves the required result. □

As a special case, we note that, for $q \to 1^-$, $f \in S^*(\frac{1}{2})$; $g \in S^*$. Then F_m defined by (9) is starlike in E.

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