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IMAGE DEBLURRING USING GLOBAL PCG METHOD WITH KRONECKER PRODUCT PRECONDITIONER†

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Abstract. We first show how to construct the linear operator equations corresponding to Tikhonov regularization problems for solving image deblurring problems with nearly separable point spread functions. We next propose a Kronecker product preconditioner which is suitable for the global PCG method. Lastly, we provide numerical experiments of the global PCG method with the Kronecker product preconditioner for several image deblurring problems to evaluate its effectiveness.

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1. Introduction

Image deblurring is the fundamental problem in image processing which recovers a true image from a blurry and noisy image. The problem of image deblurring usually reduces to solving the following *Tikhonov regularization problem* which is to solve a minimization problem of the form

$$\min_{x \in \mathbb{R}^N} \left\{ \|Ax - b\|_2^2 + \lambda^2 \|Dx\|_2^2 \right\}, \tag{1}$$

where $\lambda > 0$ is a regularization parameter which controls a balance between the data-fitting term $\|Ax - b\|_2^2$ and the regularization term $\|Dx\|_2^2$, $x \in \mathbb{R}^N$ and $b \in \mathbb{R}^N$ represent the original and observed images respectively, $A \in \mathbb{R}^{N\times N}$ is a blurring matrix, and $D \in \mathbb{R}^{N\times N}$ is a regularization matrix which is a finite difference approximation of the first or second order partial derivative operators [2, 4, 5].

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The purpose of this paper is to propose how to solve the Tikhonov regularization problem (1) using the global preconditioned conjugate gradient (Gℓ-PCG) method [14] when A and D are nearly separable. This paper is organized as follows. In Section 2, we introduce some definitions and properties which are used in this paper. In Section 3, we first show how to construct linear operator equation corresponding to a regularization matrix D in the Tikhonov regularization problem (1) when A and D can be represented or well approximated by Kronecker products of two small matrtices [7, 9], and then we propose a *Kronecker product preconditioner* which is required for accelerating the global conjugate gradient (Gℓ-CG) method [12]. In Section 4, we provide numerical experiments of the Gℓ-PCG with the Kronecker product preconditioner for several image deblurring problems, and its performance is evaluated by comparing its numerical results with those of the Gℓ-CG and PCGLS [1, 10] methods. Lastly, some conclusions are drawn.

2. Preliminaries

Let $X \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$ represent the original and observed images, respectively. We first introduce the operator *vec* which transforms a matrix $C \in \mathbb{R}^{m\times n}$ into a long vector $c \in \mathbb{R}^{m\,n}$ by stacking the columns of C from left to right, that is,

$$c = vec(C) = (c_1^T, c_2^T, \ldots, c_n^T)^T \in \mathbb{R}^{m\,n}$$

where c_i is the ith column of C. Then the Tikhonov regularization problem (1) is mathematically equivalent to solving the following linear equation

$$\left(A^T A + \lambda^2 D^T D\right) x = A^T b, \tag{2}$$

where $x = vec(X)$ and $b = vec(B)$. Since the size of the original image X is $m \times n$, the size of blurring matrix A is $mn \times mn$, which is very large and sparse when m and n are large. So, the linear system (2) is usually solved using iterative methods such as PCGLS, PLSQR and other variants of iterative method based on least squares problem [1, 3, 10, 11].

Notice that the blurring matrix A is determined by the point spread function (PSF) and the boundary condition imposed outside of the image. In this paper, we only consider the reflexive boundary condition which usually represents real situations well. If a PSF P can be expressed as an outer product of two vectors, then the PSF P is called *separable*. If a PSF is separable, then A can be represented by the Kronecker product of A_r and A_c, i.e. $A = A_r \otimes A_c$, where $A_r \in \mathbb{R}^{n\times n}$ and $A_c \in \mathbb{R}^{m\times m}$. Here, the matrix A satisfying $A = A_r \otimes A_c$ is also called separable. If A and D are separable, then the large sparse linear system (2) can be transformed into the small size of matrix equations which are generated from A_r and A_c. For this reason, we want to study how to solve the small size of matrix equations instead of solving the large sparse linear system (2). Constructing such small size of matrix equations will be discussed in the next section.

For matrices X and $Y \in \mathbb{R}^{m\times n}$, the *Frobenius inner product* of X and Y is defined by $\langle X,\ Y\rangle_F = tr(X^TY)$, and the corresponding *Frobenius norm* of $X \in \mathbb{R}^{m\times n}$ is defined by $\|X\|_F = \sqrt{\langle X, X\rangle_F}$, where $tr(X^TY)$ denotes the *trace* of X^TY. It is well-known that if $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{n\times n}$, then $tr(AB) = tr(BA)$ and $tr(C) = \sum_{i=1}^n \lambda_i$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of C [8].

Let H be a Hilbert space. A bounded linear operator $\mathcal{T} : H \to H$ is called *self-adjoint* if $\mathcal{T}^* = \mathcal{T}$, where $\mathcal{T}^*$ is the *adjoint operator* of $\mathcal{T}$. It is well-known that a bounded linear operator $\mathcal{T} : H \to H$ is self-adjoint if and only if $\langle \mathcal{T}x, y\rangle = \langle x, \mathcal{T}y\rangle$ for all $x, y \in H$ [6]. A self-adjoint operator $\mathcal{T} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is said to be *positive definite on a subset S of $\mathbb{R}^{m\times n}$* if $\langle X, \mathcal{T}(X)\rangle_F > 0$ for all nonzero $X \in S$.

3. Linear Operator equation for D corresponding to $\|x_{ss}\|_2^2 + \|x_{tt}\|_2^2$

We consider the Tikhonov regularization problem (1) for the case where D is a finite difference approximate matrix corresponding to $\|x_{ss}\|_2^2 + \|x_{tt}\|_2^2$, where s and t denote the variables in the vertical direction and the horizontal direction, respectively. We first introduce how to construct linear operator equation corresponding to the regularization matrix D, and then we propose a *Kronecker product preconditioner* which is suitable for the global preconditioned conjugate gradient (Gℓ-PCG) method. We assume that the blurring matrix A is nearly separable, that is, $A = A_r \otimes A_c$ or $A \approx A_r \otimes A_c$, where $A_r \in \mathbb{R}^{n\times n}$ and $A_c \in \mathbb{R}^{m\times m}$. For simplicity of exposition, we only consider the case of $A = A_r \otimes A_c$.

Let $D_{2,m}$ and $D_{2,n}$ be $m \times m$ and $n \times n$ matrices obtained by finite difference approximations to the second order partial derivatives x_{ss} and x_{tt} [5]. That is, when $m = 4$, the matrix $D_{2,m}$ is given by

$$D_{2,m} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Consider the matrix D in the Tikhonov regularization problem (1) such that $\|Dx\|_2^2 = \|x_{ss}\|_2^2 + \|x_{tt}\|_2^2$. Then we can easily obtain

$$\begin{aligned}\|Dx\|_2^2 &= \left\| \begin{pmatrix} x_{ss} \\ x_{tt} \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} (I_n \otimes D_{2,m})\, x \\ (D_{2,n} \otimes I_m)\, x \end{pmatrix} \right\|_2^2 \\ &= \|(I_n \otimes D_{2,m})\, x\|_2^2 + \|(D_{2,n} \otimes I_m)\, x\|_2^2.\end{aligned}$$

Thus, (1) can be transformed into the following form

$$\begin{aligned}&\min_{x\in\mathbb{R}^N} \left\{\|Ax - b\|_2^2 + \lambda^2\|(I_n \otimes D_{2,m})x\|_2^2 + \lambda^2\|(D_{2,n} \otimes I_m)\, x\|_2^2\right\} \\ &= \min_{x\in\mathbb{R}^N} \left\{ \left\| \begin{pmatrix} A \\ \lambda D_s \\ \lambda D_t \end{pmatrix} x - \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \right\|_2^2 \right\},\end{aligned} \tag{3}$$

where $D_s = I_n \otimes D_{2,m}$ and $D_t = D_{2,n} \otimes I_m$. It is easy to show that the minimization problem (3) is equivalent to solving the following equation

$$\left(A^T A + \lambda^2 D_s^T D_s + \lambda^2 D_t^T D_t\right) x = A^T b. \tag{4}$$

Since $A = A_r \otimes A_c$, $D_s = I_n \otimes D_{2,m}$ and $D_t = D_{2,n} \otimes I_m$, the linear system (4) can be rewritten as

$$\left\{A_r^T A_r \otimes A_c^T A_c + \lambda^2 (I_n \otimes D_{2,m}^T D_{2,m}) + \lambda^2 (D_{2,n}^T D_{2,n} \otimes I_m)\right\} x = (A_r^T \otimes A_c^T) b. \tag{5}$$

From (5), the following matrix equation can be obtained

$$(A_c^T A_c) X (A_r^T A_r) + \lambda^2 \left((D_{2,m}^T D_{2,m}) X + X (D_{2,n}^T D_{2,n})\right) = A_c^T B A_r, \tag{6}$$

where $B \in \mathbb{R}^{m\times n}$ is a matrix such that $b = vec(B)$. Let us define the linear operator $\mathcal{A}_2 : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ given by

$$\mathcal{A}_2(X) = (A_c^T A_c) X (A_r^T A_r) + \lambda^2 \left((D_{2,m}^T D_{2,m}) X + X (D_{2,n}^T D_{2,n})\right). \tag{7}$$

Then (6) can be expressed as the following operator equation

$$\mathcal{A}_2(X) = A_c^T B A_r. \tag{8}$$

The following theorem shows that $\mathcal{A}_2$ is self-adjoint and positive definite on a subset of $\mathbb{R}^{m\times n}$.

Theorem 3.1. *Suppose that at least one column of X is not a constant vector. Then the operator $\mathcal{A}_2$ is self-adjoint and $\langle X, \mathcal{A}_2(X)\rangle_F > 0$.*

Proof. For all X and Y in $\mathbb{R}^{m\times n}$,

$$\begin{aligned}
\langle \mathcal{A}_2(X), Y\rangle_F &= \left\langle A_c^T A_c X A_r^T A_r + \lambda^2 (D_{2,m}^T D_{2,m} X + X D_{2,n}^T D_{2,n}), Y \right\rangle_F \\
&= tr\left(A_r^T A_r X^T A_c^T A_c Y + \lambda^2 (X^T D_{2,m}^T D_{2,m} Y + D_{2,n}^T D_{2,n} X^T Y)\right) \\
&= tr\left(X^T A_c^T A_c Y A_r^T A_r\right) + \lambda^2 \left(tr(X^T D_{2,m}^T D_{2,m} Y) + tr(X^T Y D_{2,n}^T D_{2,n})\right) \\
&= tr\left(X^T (A_c^T A_c Y A_r^T A_r + \lambda^2 (D_{2,m}^T D_{2,m} Y + Y D_{2,n}^T D_{2,n}))\right) \\
&= \left\langle X,\, A_c^T A_c Y A_r^T A_r + \lambda^2 (D_{2,m}^T D_{2,m} Y + Y D_{2,n}^T D_{2,n}) \right\rangle_F \\
&= \langle X, \mathcal{A}_2(Y)\rangle_F .
\end{aligned}$$

Hence, the operator $\mathcal{A}_2$ is self-adjoint. For each $X \in \mathbb{R}^{m\times n}$ and $X \neq O$,

$$\begin{aligned}
\langle X, \mathcal{A}_2(X)\rangle_F &= tr\left(X^T (A_c^T A_c X A_r^T A_r + \lambda^2 (D_{2,m}^T D_{2,m} X + X D_{2,n}^T D_{2,n}))\right) \\
&= tr\left(X^T A_c^T A_c X A_r^T A_r\right) + \lambda^2 \left(tr(X^T D_{2,m}^T D_{2,m} X) + tr(X^T X D_{2,n}^T D_{2,n})\right) \\
&= tr(A_c X A_r^T A_r X^T A_c^T) + \lambda^2 \left(tr(X^T D_{2,m}^T D_{2,m} X) + tr(X D_{2,n}^T D_{2,n} X^T)\right) \\
&= tr\left((A_r X^T A_c^T)^T (A_r X^T A_c^T)\right) \\
&\quad + \lambda^2 \left\{tr\left((D_{2,m} X)^T (D_{2,m} X)\right) + tr\left((D_{2,n} X^T)^T (D_{2,n} X^T)\right)\right\} \\
&= \|A_r X^T A_c^T\|_F^2 + \lambda^2 \left(\|D_{2,m} X\|_F^2 + \|D_{2,n} X^T\|_F^2\right)
\end{aligned}$$

Since at least one column of X is not a constant vector, $D_{2,m}X \neq O$ and $D_{2,n}X^T \neq O$. Hence $\|D_{2,m}X\|_F^2 > 0$ and $\|D_{2,n}X^T\|_F^2 > 0$. It follows that $\langle X, \mathcal{A}_2(X)\rangle_F > 0$, which completes the proof. □

In order to accelerate the convergence of the Gℓ-CG, a good choice of preconditioner corresponding to the operator equation (8) is required. From the left side of the linear system (5), one can obtain the following approximate relation

$$\begin{aligned}&\left(A_r^TA_r \otimes A_c^TA_c + \lambda^2(I \otimes D_{2,m}^TD_{2,m}) + \lambda^2(D_{2,n}^TD_{2,n} \otimes I)\right)x\\ &\approx \left(A_r^TA_r + \lambda(I_n + D_{2,n}^TD_{2,n})\right) \otimes \left(A_c^TA_c + \lambda(D_{2,m}^TD_{2,m} + I_m)\right)x. \qquad (9)\end{aligned}$$

From (9), we can choose a Kronecker product preconditioner of the form

$$M_2 = M_r \otimes M_c,$$

where

$$M_r = A_r^T A_r + \lambda(I_n + D_{2,n}^TD_{2,n}) \text{ and } M_c = A_c^T A_c + \lambda(I_m + D_{2,m}^TD_{2,m}).$$

Then it is clear that $M_r \in \mathbb{R}^{n\times n}$ and $M_c \in \mathbb{R}^{m\times m}$. Now we define a preconditioner operator $\mathcal{M}_2 : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ by

$$\mathcal{M}_2(X) = M_cXM_r^T. \qquad (10)$$

To apply Gℓ-PCG to the operator equation (8), it is required that the preconditioner operator $\mathcal{M}_2$ should be self-adjoint and positive definite.

Theorem 3.2. *The preconditioner operator $\mathcal{M}_2$ in (10) is self-adjoint and positive definite.*

Proof. Since $(I_n+D_{2,n})^T(I_n+D_{2,n})$ and $(I_m+D_{2,m})^T(I_m+D_{2,m})$ are symmetric positive definite, M_r and M_c are symmetric positive definite. For all $X, Y \in \mathbb{R}^{m\times n}$,

$$\begin{aligned}\langle \mathcal{M}_2(X), Y\rangle_F &= \left\langle M_cXM_r^T, Y\right\rangle_F = tr\left(M_rX^TM_c^TY\right)\\ &= tr\left(X^TM_c^TYM_r\right) = \langle X, \mathcal{M}_2(Y)\rangle_F .\end{aligned}$$

Hence the preconditioner operator $\mathcal{M}_2$ is self-adjoint. To show the positive definiteness of $\mathcal{M}_2$, for each $X \in \mathbb{R}^{m\times n}$ and $X \neq O$

$$\langle \mathcal{M}_2(X), X\rangle_F = tr\left(M_rX^TM_c^TX\right) = tr\left(X^TM_c^TXM_r\right) = tr\left(X^TM_cXM_r\right).$$

Since $X^TM_cXM_r$ is similar to $M_r^{\frac{1}{2}}X^TM_cXM_r^{\frac{1}{2}}$ which is symmetric positive semi-definite, all eigenvalues of $X^TM_cXM_r$ are nonnegative. It follows that $tr\left(X^TM_cXM_r\right) \geq 0$. Since M_r and M_c are symmetric positive definite, $XM_r^{\frac{1}{2}} \neq 0$ and thus

$$M_r^{\frac{1}{2}}X^TM_cXM_r^{\frac{1}{2}} = (XM_r^{\frac{1}{2}})^TM_c(XM_r^{\frac{1}{2}})$$

is a non-zero symmetric matrix. It follows that all eigenvalues of $X^TM_cXM_r$ are not zero. Hence $tr(X^TM_cXM_r) > 0$, which implies that $\mathcal{M}_2$ is positive definite. □

Remark 3.1. Theorems 3.1 assumed that at least one column of $X \in \mathbb{R}^{m\times n}$ is not a constant vector. Most cases of practical images X satisfy these assumptions, so we may say that $\mathcal{A}_2$ is positive definite operators in real situation.

4. Numerical experiments

The Gℓ-PCG algorithm with Kronecker product preconditioners for solving the linear operator equations has been introduced in [14]. In this section, we provide numerical experiments of the Gℓ-PCG method for the linear operator equation $\mathcal{A}_2(X) = \mathcal{B}$ with the Kronecker product preconditioner $\mathcal{M}_2$ discussed in Section 3. Performance of the Gℓ-PCG algorithm is evaluated by comparing its numerical results with those of the Gℓ-CG and preconditioned CGLS (PCGLS) methods (see Tables 1 and 2).

All numerical tests have been performed using Matlab R2016b on a personal computer equipped with Intel Core i5-4570 3.2GHz CPU and 8GB RAM. For numerical experiments, we have used 3 types of PSFs (point spread functions) which are S-blur, Motion blur and Disk blur of size 7×7. The PSF array P for S-blur of size 7×7 is given by

$$K = (1\ 2\ 3\ 4\ 3\ 2\ 1)^T \otimes \left(\frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ 1\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{8}\right) \quad \text{and} \quad P = \frac{K}{\sum_{i,j=1}^{7} k_{ij}},$$

where k_{ij} denotes the (i, j)-component of K. The PSF array P for Disk blur of size 7×7 is generated by the Matlab function $fspecial('disk', 3)$, and the PSF array P for Motion blur of size 7×7 is generated by the Matlab function

$$P = zeros(7);\ P(3:5, :) = fspecial('motion', 7, 1).$$

Notice that S-blur is separable, but Disk blur and Motion blur are nonseparable. For a nonseparable PSF, we have used a separable PSF which is a rank-1 approximation to the nonseparable PSF using Kronecker product approximation techniques proposed in [7, 9]. So A can be expressed or approximated as $A_r \otimes A_c$ for all PSFs.

The blurred and noisy image B is generated by $vec(B) = A \cdot vec(X) + vec(E)$, where A stands for the blurring matrix which can be generated by the original PSF array P according to the boundary condition to be used, and E is a Gaussian white noise. In this paper, we have used the reflexive boundary condition and the Gaussian white noise E with mean 0 and standard deviation 0.95 which can be generated using Matlab function $E = 0.95 \times randn(m, n)$, where (m, n) denotes the size of the true image X.

The initial image X_0 is set to the blurred and noisy image B. The stopping criterion for iterative methods at the k-th iterate is

$$\frac{\|R_k\|_F}{\|R_0\|_F} \le 5 \cdot 10^{-3},$$

where R_k represents the k-th residual matrix corresponding to the k-th iteration matrix X_k of iterative methods with R_0 the initial residual matrix corresponding to X_0. A restored image G is measured by the PSNR (Peak Signal to Noise Ratio) which is defined by

$$\text{PSNR} = 10 \log_{10} \left(\frac{\max\limits_{i,j} |x_{ij}|^2 \cdot m \cdot n}{\|X - G\|_F^2} \right)$$

where $X = (x_{ij})$ represents the true image.

We have used 2 test images called Cameraman and Jetplane for numerical experiments. The pixel size of Cameraman image is 256×256, and the pixel size of Jetplane image is 512×512. For the preconditioner of the PCGLS method, we have chosen the symmetric approximation matrix which can be easily obtained using the DCT2 (2-dimensional discrete Cosine transform) (see [5] for details). For the PCGLS method, the blurring matrix A whose size is large is not constructed since its construction is very time-consuming and matrix-vector multiplication with A can be performed without constructing A (see [4] for details). For the Gℓ-PCG method, the matrices A_r and A_c whose size is very small compared to the size of A are constructed.

In Tables 1 and 2, "PSNR" represents the PSNR values for the restored images, PSNR_0 represents the PSNR value for the blurred and noisy image, "Itime" denotes the elapsed CPU time in seconds required for iteration steps of Gℓ-CG, Gℓ-PCG, CGLS and PCGLS methods, "IT" denotes the number of iterations required for Gℓ-CG, Gℓ-PCG and PCGLS methods, and "λ" denotes a near optimal regularization parameter which is chosen by numerical tries. Notice that the near optimal values of λ vary from 0.01 to 0.05.

Numerical results are provided in Tables 1 and 2 and Figures 1 to 3. As can be seen in Tables 1 and 2, Gℓ-CG and Gℓ-PCG with Kronecker product preconditioners restore the true image as well as PCGLS except for the nonseparable Disk blur. The reason for worse performance for Disk blur is that Gℓ-CG and Gℓ-PCG use a rank-1 approximation to the Disk blur which is not a good approximation to the Disk blur.

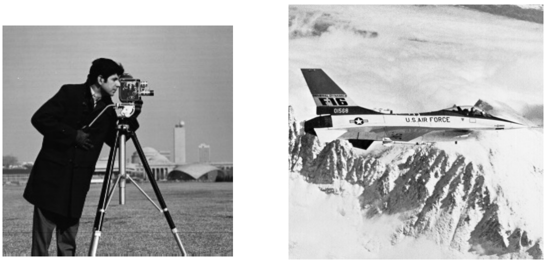
For all test problems, Gℓ-PCG with Kronecker product preconditioners yields a superior performance in terms of execution time. Notice that PCGLS has extremely faster convergence rate since the DCT2 type of symmetric preconditioner is a very good approximation to the original matrix. However PCGLS takes much more execution time than Gℓ-CG and Gℓ-PCG. The reason is as follows: Before the first iteration starts, PCGLS computes eigenvalues, using PSF and FFT2, which are required to perform matrix-vector multiplication with A without constructing A. So the first iteration takes much more execution time than the remaining iterations. As compared with the unpreconditioned Gℓ-CG, Kronecker product preconditioner for Gℓ-PCG proposed in this paper works extremely well in terms of convergence rate. This means that the Kronecker product preconditioner is a good approximation to the original matrix since $\lambda > 0$ is chosen to be a small number between 0.01 and 0.05.

Table 1. Numerical results for Cameraman image

PSF	Method	$PSNR_0$	PSNR	λ	Itime	IT
S	PCGLS	23.67	28.23	0.02	0.26	1
	Gℓ-PCG		28.20	0.02	0.06	8
	Gℓ-CG		28.16	0.015	0.07	32
Motion	PCGLS	22.81	30.48	0.02	0.29	2
	Gℓ-PCG		30.20	0.02	0.05	7
	Gℓ-CG		30.06	0.02	0.07	33
Disk	PCGLS	22.85	27.62	0.02	0.25	1
	Gℓ-PCG		26.15	0.03	0.06	8
	Gℓ-CG		26.22	0.025	0.07	33

Table 2. Numerical results for Jetplane image

PSF	Method	$PSNR_0$	PSNR	λ	Itime	IT
S	PCGLS	26.92	32.59	0.04	0.50	1
	Gℓ-PCG		32.51	0.035	0.27	6
	Gℓ-CG		32.53	0.035	0.39	21
Motion	PCGLS	26.97	34.67	0.045	0.66	2
	Gℓ-PCG		34.42	0.045	0.26	6
	Gℓ-CG		34.17	0.05	0.40	23
Disk	PCGLS	26.31	32.25	0.035	0.50	1
	Gℓ-PCG		30.61	0.04	0.26	6
	Gℓ-CG		30.71	0.035	0.40	23



(a) Cameraman (b) Jetplane

Fig. 1. True Images

5. Conclusions

In this paper, we have studied application of the Gℓ-PCG with Kronecker product preconditioners to image deblurring problems with nearly separable PSFs. Gℓ-CG and Gℓ-PCG with Kronecker product preconditioners restore the true image as well as PCGLS when the PSF is well approximated by a rank-1 approximation to the PSF. For all test problems, Gℓ-PCG with Kronecker product preconditioners yields a superior performance in terms of execution time. As compared with the unpreconditioned Gℓ-CG, Kronecker product preconditioners for Gℓ-PCG proposed in this paper work extremely well in terms of

(a) Blurred and noisy image

(b) PCGLS (PSNR = 28.23)

(c) $G\ell$-CG (PSNR = 28.16)

(d) $G\ell$-PCG (PSNR = 28.20)

Fig. 2. Cameraman images for S-blur ((b, c, d): restored images)

(a) Blurred and noisy image

(b) PCGLS (PSNR = 34.67)

(c) $G\ell$-CG (PSNR = 34.17)

(d) $G\ell$-PCG (PSNR = 34.42)

Fig. 3. Jetplane images for Motion blur ((b, c, d): restored images)

convergence rate. This means that the Kronecker product preconditioners are good approximations to the original matrices.

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