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GEODESIC SEMI E-PREINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. Several classes of functions, named as semi E-preinvex functions and semilocal E-preinvex functions and their properties are studied by various authors. In this paper we introduce the geodesic concept over two types of problems first is semi E-preinvex functions and another is semilocal E-preinvex functions on Riemannian manifolds and study some of their properties.

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1. Introduction

Convexity and invexity play important and essential role in various fields like management science, engineering, mathematical economics, optimization theory and Riemannian manifolds etc. Therefore, it is essential to consider a large class of generalized convex functions and also look for practical criteria of convexity or invexity. Mangasarian [22] studied the concepts of convexity and concavity.

The concept of invex function introduced by Hanson [10] and named by Craven [5] is a significant generalization of the notion of convexity. This work inspired a great deal of subsequent works, which has greatly expanded the role and application of invexity in nonlinear optimization and other branches of pure and applied sciences. Ben-Israel and Mond [3] introduced a new generalization of convex sets and convex functions, later, termed as invex sets and preinvex functions, by Weir and Mond [31] and Weir and Jeyakumar [30].

In 1977, Ewing [8] developed a generalized convexity known as semilocal convexity by reducing the width of the line segment, where the concept is applied to provide sufficient optimality conditions in variational and control problems. Generalizations of semilocal convex functions and their properties have been

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studied by Kaul and Kaur [15, 16] and Kaur [17]. In 1996, Preda et al. [25] established optimality conditions and duality results for nonlinear programming involving semilocal preinvex and related functions. Later these results are extended in [26] for a multiple-objective programming problems. These results have many applications.

Youness [34] introduced the concepts of E-convex sets and E-convex functions, and studied their properties. Later, Chen [4] introduced a class of semi E-convex functions and studied their properties. Furthermore, on the basis of the concepts of semi E-convexity and semilocal convexity, Hu et al. [13] put up the concept of semilocal E-convexity.

Latterly, Fulga and Preda [9] generalized the concept of E-convexity to E-preinvexity and local E-preinvexity and studied some of its properties. Moreover, Luo and Jian [21] introduced the concept of semi E-preinvex maps in Banach spaces and studied their properties.

The initial results of Youness [34] inspired a great deal of subsequent work which has greatly expanded the role of E-convexity in optimization theory; see [6, 7, 28, 33]. Syau and Lee [28] introduced the concept of E-quasiconvex functions and discussed some properties of E-convex and E-quasiconvex functions.

In linear topological spaces, the notion of convex sets relies on connecting any two points of the space by a line segment. In several real- world applications, it is not possible to connect the points through a line segment. This lead to the idea of the generalization of the classical notion of convex sets. Udriste [29] and Rapcsak [27] proposed a generalization of the convexity notion by replacing the linear space by Riemannian manifolds, the line segments by a geodesic segments between any two points, and the convex function by the positiveness of their Hessian. A geodesic on the Riemannian manifolds is a curve, that locally minimize the arc length. Udriste [29] generalization is based on the fact that many of the properties of convex programs on Euclidean space carry over to the case of a complete Riemannian manifolds. Following Udriste [29], several other generalizations of convex sets and convex functions have been proposed on the Riemannian manifolds. In order to extend the validity of the results to a larger classes of optimization problems, these concepts have been generalized and extended in several directions using novel and innovative techniques. Several authors have studied the properties of generalized convex functions on Riemannian manifolds.

Pini [24] introduced the notion of invex function on Riemannian manifold and Mititelu [23] investigated its generalizations. Barani and Pouryayevali [1] defined the geodesic invex set, geodesic η-invex function and geodesic η-preinvex function on Riemannian manifold and discussed the relation between them. In [2] Barani and Pouryayevali introduced generalized invariant monotone vector fields on Riemannian manifolds and discussed their relationship with generalized invexities. Li *et al.* [20] studied the weak sharp minima for constrained optimization problems on Riemannian manifolds and their characterizations.

Iqbal *et al*. [11] generalized convexity and introduced geodesic E-convex sets and geodesic E-convex functions on Riemannian manifolds.

Motivated by earlier research works of [34, 4, 13, 9, 21, 12] and references therein, in this paper, we introduce the concept of geodesic on semi E-preinvex functions and semilocal E-preinvex function and study some of its important properties on Riemannian manifolds.

2. Preliminaries

In this section, we gave some preliminary notations about Riemannian manifolds which will be used throughout this paper. The preliminary part of this section, is taken from [1]. For further details on differential and Riemannian geometry, we refer to [19, 18].

Let $\mathbb{R}^n$ be n-dimensional Euclidean space and $\mathbb{R}$ be the set of real numbers. Let M be a C^∞ smooth manifold together with a Riemannian metric $\langle .,. \rangle_p$ on the tangent space T_pM and corresponding norm is denoted by $\|.\|_p$, which yields the Riemannian manifold M. Let us recall that the length of a piecewise differentiable curve $\gamma : [a, b] \to M$ joining p to q such that $\gamma(a) = p$ and $\gamma(b) = q$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\|_{\gamma(t)}\, dt.$$

Minimizing this length functional on the set of all piecewise differentiable curves joining p and q in M, we get a distance function $d(p, q)$. This distance function d induces the original topology on M. Let $\chi(M)$ denote the space of all vector fields on M. The metric induces a map $f \mapsto grad\ f \in \chi(M)$, which associates to each f its gradient via the rule $\langle df, X\rangle = df(X)$ for each $X \in \chi(M)$. On every Riemannian manifold there exists exactly one covariant derivative called Levi-Civita connection denoted by $\nabla_X Y$ for any vector fields $X, Y \in M$. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, that is, γ satisfies the equation

$$\nabla_{\frac{d\gamma(t)}{dt}} \frac{d\gamma(t)}{dt} = 0.$$

Any path γ joining p to q in M such that $L(\gamma) = d(p, q)$ is a geodesic and it is called a minimal geodesic.

3. Definitions and Properties of Geodesic Semi E-Preinvex Functions

Throughout this section E is a map from M to M then, we introduce the concepts of geodesic semi E-preinvex functions on Riemannian manifolds and study some of their properties.

Definition 3.1. Let $S \subset M$ be a geodesic E-invex set. A function $f : S \to \mathbb{R}$ is said to be geodesic E-preinvex on S if

$$f(\gamma_{E(x),E(y)}(\lambda)) \le \lambda f(E(x)) + (1-\lambda) f(E(y)), \forall x, y \in S, \lambda \in [0, 1].$$

Definition 3.2. Let $S \subset M$ be a geodesic E-invex set. A function $f : S \to \mathbb{R}$ is said to be geodesic semi E-preinvex on S, if

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y), \forall x, y \in S, \lambda \in [0,1] .$$

Definition 3.3. Let $S \subseteq M \times \mathbb{R}$ and $E : M \to M$, then the set S is said to be geodesic G-E-invex if and only if for all $(x, \alpha), (x, \beta) \in S$,

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha + (1-\lambda)\beta) \in S, \ \forall \lambda \in [0,1] .$$

Theorem 3.4. *Let $f : S \to \mathbb{R}$ be a geodesic E-preinvex function on a geodesic E-invex set $S \subset M$, then f is a geodesic semi E-preinvex function if and only if $f(E(x)) \leq f(x), \ \forall x \in S$.*

Proof. Suppose that f is a geodesic semi E-preinvex function on S, then for each pair of points $x, y \in S$ and $0 \leq \lambda \leq 1$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y), \ \forall \lambda \in [0,1] .$$

By putting $\lambda = 1$, we have $f(E(x)) \leq f(x), \ \forall x \in S$.

Conversely, if f is a geodesic E-preinvex function on a geodesic E-invex set S, then for any $x, y \in S$ and $0 \leq \lambda \leq 1$,

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f E((x)) + (1-\lambda) f E((y)), \ \forall \lambda \in [0,1] .$$

Since, $f(E(x)) \leq f(x), \ \forall x \in S$, then

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y), \ \forall \lambda \in [0,1] .$$

Hence, proved. □

Theorem 3.5. *Let $S \subset M$ be a geodesic E-invex set. Then f is a geodesic semi E-preinvex function on S, if and only if its epigraph $G_f = \{(x, \alpha) : x \in S, f(x) \leq \alpha, \alpha \in \mathbb{R}\}$ is a geodesic G-E-invex set corresponding to S.*

Proof. Assume that, f is geodesic semi E-preinvex function on S and $(x, \alpha_1), (y, \alpha_2) \in G_f$, then $x, y \in S, f(x) \leq \alpha_1, f(y) \leq \alpha_2$. Since S is a geodesic E-invex set, then

$$\gamma_{E(x),E(y)}(\lambda) \in S, \ \forall \lambda \in [0,1] .$$

Further, in view of f being a geodesic semi E-preinvex function on S, then

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda\alpha_1 + (1-\lambda)\alpha_2, \ \forall \lambda \in [0,1] .$$

i.e.

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha_1 + (1-\lambda)\alpha_2) \in G_f, \forall \lambda \in [0,1] .$$

Therefore, $G_f = \{(x, \alpha) : x \in K, f(x) \leq \alpha, \alpha \in \mathbb{R}\}$ is a geodesic G-E-invex set corresponding to S.

Conversely, if G_f is a geodesic G-E-invex set corresponding to S, then for any points $(x, f(x)), (y, f(y)) \in G_f$, we have

$$(\gamma_{E(x),E(y)}(\lambda), \lambda f(x) + (1-\lambda)f(y)) \in G_f, \ \forall \lambda \in [0,1] .$$

i.e.

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y),\ \forall \lambda \in [0,1]\,.$$

Thus, S is a geodesic E-invex set and f is a geodesic semi E-preinvex function on S. □

Theorem 3.6. *If f is a geodesic semi E-preinvex function on a geodesic E-invex set $S \subset M$, then the lower section of f defined by*

$$S_\alpha = \{x \in S : f(x) \leq \alpha\}$$

is a geodesic E-invex set for any $\alpha \in \mathbb{R}$.

Proof. For any $\alpha \in \mathbb{R}$ and $x, y \in S_\alpha$, then $f(x) \leq \alpha, f(y) \leq \alpha$. Since S is a geodesic E-invex set, then

$$\gamma_{E(x),E(y)}(\lambda) \in S,\ \forall \lambda \in [0,1]\,.$$

Moreover, due to the geodesic semi E- preinvexity of f, we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y) \leq \lambda\alpha + (1-\lambda)\alpha = \alpha,\ \forall \lambda \in [0,1]\,.$$

i.e.

$$\gamma_{E(x),E(y)}(\lambda) \in S_\alpha,\ \forall \lambda \in [0,1]\,.$$

Therefore, by definition S_α is a geodesic E-invex set for any $\alpha \in R$. □

Theorem 3.7. *Let $f : S \to \mathbb{R}$ be a function defined on a geodesic E-invex $S \subset M$. Then f is a geodesic semi E-preinvex function if and only if for each pair of points $x, y \in S$ and $0 \leq \lambda \leq 1$ such that*

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0,1]\,,$$

whenever $f(x) < \alpha, f(y) < \beta$.

Proof. Let $x, y \in S$ and $\alpha, \beta \in \mathbb{R}$ such that $f(x) < \alpha, f(y) < \beta$. By definition of geodesic E-invexity of S, we have

$$\gamma_{E(x),E(y)}(\lambda) \in S,\ \forall \lambda \in [0,1]\,.$$

Moreover, in view of f being a geodesic semi E-preinvex function on S, such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda)f(y) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0,1]\,.$$

Conversely, let $(x, \alpha), (y, \beta) \in G_f$ (by Theorem 3.5), then $x, y \in S, f(x) \leq \alpha, f(y) \leq \beta$. Hence, $f(x) < \alpha + \varepsilon$ and $f(y) < \beta + \varepsilon$ hold for any $\varepsilon > 0$. According to the hypothesis, for $x, y \in S$ and $0 \leq \lambda \leq 1$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta + \varepsilon,\ \forall \lambda \in [0,1]\,.$$

Let $\varepsilon \to 0^+$, then

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0,1]\,.$$

i.e.

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha + (1-\lambda)\beta) \in G_f,\ \forall \lambda \in [0,1]\,.$$

Therefore, G_f is a geodesic G-E-invex set corresponding to S. From Theorem 3.5, it follows that f is geodesic semi E-preinvex on S. □

4. Definitions and Properties of Geodesic Semilocal E-Preinvex

In this section, we also consider E is a map from M to M then, introduce the concepts of geodesic in semilocal E-preinvex functions on Riemannian manifolds and study some of its properties.

Definition 4.1. A set $S \subset M$ is said to be geodesic local invex set if for any $x, y \in S$, there exist a positive maximal number $0 < a(x, y) \leq 1$ and an unique geodesic $\gamma : [0, a(x, y)] \to M$ such that

$$\gamma_{x,y}(\lambda) \in S, \forall \lambda \in [0, a(x, y)] .$$

Definition 4.2. A set $S \subset M$ is said to be geodesic local E-invex if there is a map E such that corresponding to any points $x, y \in S$ and there is a maximal positive number $a(x, y) \leq 1$ satisfying

$$\gamma_{E(x),E(y)}(\lambda) \in S, \ \forall \lambda \in [0, a(x, y)] . \tag{1}$$

Definition 4.3. A function $f : S \to \mathbb{R}$ is said to be geodesic local E-preinvex on a geodesic local E-invex set $S \subset M$ if for any $x, y \in S$ (with maximal positive number $a(x, y) \leq 1,$) there exists $0 < b(x, y) \leq a(x, y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(E(x)) + (1 - \lambda) f(E(y)), \ \forall \lambda \in [0, b(x, y)] .$$

Definition 4.4. A function $f : S \to \mathbb{R}$ is said to be geodesic semilocal E-preinvex on a geodesic local E-invex set $S \subset M$ if for any $x, y \in S$ (with maximal positive number $a(x, y) \leq 1,$) there exists $0 < b(x, y) \leq a(x, y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1 - \lambda) f(y), \ \forall \lambda \in [0, b(x, y)] .$$

If the above inequality is strict for any $x, y \in S$ and $x \neq y$, then f is called a strict geodesic semilocal E-preinvex function.

Definition 4.5. Let $S \subseteq M \times \mathbb{R}$, is said to be geodesic local G-E-invex set with respect to η such that for any pair of points $(x, \alpha), (y, \beta) \in S$, then, we have

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha + (1 - \lambda)\beta) \in S, \quad \forall \lambda \in [0, a((x, \alpha), (y, \beta))]$$

Theorem 4.6. *Let $f : S \to \mathbb{R}$ be a geodesic local E-preinvex functions on a geodesic local E-invex set $S \subseteq M$, then f is a geodesic semilocal E-preinvex function if and only if $f(E(x)) \leq f(x), \ \forall x \in S$.*

Proof. Suppose that f is a geodesic semilocal E-preinvex function on S, then for each pair of points $x, y \in S$ (with a maximal positive number $a(x, y) \leq 1$ satisfying (1)), there exists a positive number $b(x, y) \leq a(x, y)$ we have

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1 - \lambda) f(y), \ \forall \lambda \in [0, b(x, y)] .$$

By setting $\lambda = 1$, we have $f(E(x)) \leq f(x), \ \forall x \in S$.

Conversely, assume that f is a geodesic local E-preinvex function on a geodesic local E-invex set S, then for any $x, y \in S$, there exist a maximal positive number $a(x,y) \leq 1$ satisfying (1) and a positive number $b(x,y) \leq a(x,y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f E((x)) + (1-\lambda) f E((y)),\ \forall \lambda \in [0, b(x,y)]\,.$$

Since, $f(E(x)) \leq f(x),\ \forall x \in S$, then

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda) f(y),\ \forall \lambda \in [0, b(x,y)]\,.$$

Hence, proved. □

Theorem 4.7. *Let $S \subset M$ be a geodesic local E-invex set. Then, f is a geodesic semilocal E-preinvex function on S if and only if its epigraph $G_f = \{(x,\alpha) : x \in S, f(x) \leq \alpha, \alpha \in \mathbb{R}\}$ is a geodesic local G-E-invex set corresponding to S.*

Proof. Assume that function f is geodesic semilocal E-preinvex on S and (x,α_1), $(y,\alpha_2) \in G_f$, then $x, y \in S, f(x) \leq \alpha_1, f(y) \leq \alpha_2$. Since S is a geodesic local E-invex set, there is a maximal positive number $a(x,y) \leq 1$ such that

$$\gamma_{E(x),E(y)}(\lambda) \in S,\ \forall \lambda \in [0, a(x,y)]\,.$$

Moreover, in view of f being a geodesic semilocal E-preinvex function on S, there is a positive number $b(x,y) \leq a(x,y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda\alpha_1 + (1-\lambda)\alpha_2,\ \forall \lambda \in [0, b(x,y)]\,.$$

i.e.

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha_1 + (1-\lambda)\alpha_2) \in G_f, \forall \lambda \in [0, b(x,y)]\,.$$

Therefore, $G_f = \{(x,\alpha) : x \in K, f(x) \leq \alpha, \alpha \in \mathbb{R}\}$ is a geodesic local G-E-invex set corresponding to S.

Conversely, if G_f is a geodesic local G-E-invex set corresponding to S, then for any points $(x, f(x)), (y, f(y)) \in G_f$, there is a maximal positive number $a((x, f(x)), (y, f(y))) \leq 1$ such that

$$(\gamma_{E(x),E(y)}(\lambda), \lambda f(x) + (1-\lambda) f(y)) \in G_f,\ \forall \lambda \in [0, a((x, f(x)), (y, f(y)))]\,.$$

i.e.

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda) f(y),\ \forall \lambda \in [0, a((x, f(x)), (y, f(y)))]\,.$$

Thus, S is a geodesic local E-invex set and f is a geodesic semilocal E-preinvex function on S. □

Theorem 4.8. *If f is a geodesic semilocal E-preinvex function on a geodesic local E-invex set $S \subset M$, then the lower section of f defined by*

$$S_\alpha = \{x \in S : f(x) \leq \alpha\}$$

is a geodesic local E-invex set for any $\alpha \in \mathbb{R}$.

Proof. For any $\alpha \in \mathbb{R}$ and $x, y \in S_\alpha$, then $f(x) \leq \alpha, f(y) \leq \alpha$. Since S is a geodesic local E-invex set, there is a maximal positive number $a(x,y) \leq 1$ such that

$$\gamma_{E(x),E(y)}(\lambda) \in S,\ \forall \lambda \in [0, a(x,y)]\,.$$

Moreover, due to the geodesic semilocal E-preinvexity of f, there is a positive number $b(x,y) \leq a(x,y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda) f(y) \leq \lambda\alpha + (1-\lambda)\alpha = \alpha,\ \forall \lambda \in [0, b(x,y)]\,.$$

i.e.

$$\gamma_{E(x),E(y)}(\lambda) \in S_\alpha,\ \forall \lambda \in [0, b(x,y)]\,.$$

Therefore, by definition S_α is a geodesic local E-invex set for any $\alpha \in \mathbb{R}$. □

Theorem 4.9. *Let $f : S \to \mathbb{R}$ be a function defined on a geodesic local E-invex $S \subset M$. Then f is a geodesic semilocal E-preinvex function if and only if for each pair of points $x, y \in S$ (with a maximal positive number $a(x,y) \leq 1$ satisfying(1)), there exists a positive number $b(x,y) \leq a(x,y)$ such that*

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0, b(x,y)]\,,$$

whenever $f(x) < \alpha, f(y) < \beta$.

Proof. Let $x, y \in S$ and $\alpha, \beta \in \mathbb{R}$ such that $f(x) < \alpha, f(y) < \beta$. Due to geodesic local E-invexity of S, there is a maximal positive number $a(x,y) \leq 1$ such that

$$\gamma_{E(x),E(y)}(\lambda) \in S,\ \forall \lambda \in [0, a(x,y)]\,.$$

Moreover, in view of f being a geodesic semilocal E-preinvex function on S, there is a positive number $b(x,y) \leq a(x,y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) \leq \lambda f(x) + (1-\lambda) f(y) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0, b(x,y)]\,.$$

Conversely, let $(x,\alpha), (y,\beta) \in G_f$ (see epigraph G_f in Theorem 4.7), then $x, y \in S, f(x) \leq \alpha, f(y) \leq \beta$. Hence, $f(x) < \alpha + \varepsilon$ and $f(y) < \beta + \varepsilon$ hold for any $\varepsilon > 0$. According to the hypothesis, for $x, y \in S$ (with a maximal positive number $a(x,y) \leq 1$ satisfying(1)), there exists a positive number $b(x,y) \leq a(x,y)$ such that

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta + \varepsilon,\ \forall \lambda \in [0, b(x,y)]\,.$$

Let $\varepsilon \to 0^+$, then

$$f(\gamma_{E(x),E(y)}(\lambda)) < \lambda\alpha + (1-\lambda)\beta,\ \forall \lambda \in [0, b(x,y)]\,.$$

i.e.

$$(\gamma_{E(x),E(y)}(\lambda), \lambda\alpha + (1-\lambda)\beta) \in G_f,\ \forall \lambda \in [0, b(x,y)]\,.$$

Therefore, G_f is a geodesic local G-E-invex set corresponding to S. From Theorem 4.7, it follows that f is geodesic semilocal E-preinvex on S. □

5. Conclusion

In this paper, we introduced two new classes of functions called geodesic semi E-preinvex functions and geodesic semilocal E-preinvex functions on Riemannian manifolds. In section 3, we have shown that a if and only if relationship with geodesic semi E-preinvex functions. After this, we established a relation between geodesic semi E-preinvex functions and lower section of f. Finally, we characterize some properties of geodesic semi E-preinvex functions in terms of their epigraphs. In section 4, we have studied same results in terms of geodesic semilocal E-preinvex functions on Riemannian manifolds. The results of the paper, generalizes and extends some earlier results from Jiao [14] and study some of their properties on Riemannian manifolds.

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