J. Appl. Math. & Informatics Vol. **36***(2018), No. 5 - 6, pp. 505 - 519*
https://doi.org/10.14317/jami.2018.505

(p, q)-LAPLACE TRANSFORM†

YOUNG ROK KIM AND CHEON SEOUNG RYOO*

Abstract. In this paper we define a (p, q)-Laplace transform. By using this definition, we obtain many properties including the linearity, scaling, translation, transform of derivatives, derivative of transforms, transform of integrals and so on. Finally, we solve the differential equation using the (p, q)-Laplace transform.

AMS Mathematics Subject Classification : 33B15, 26A51, 26A48.
Key words and phrases : (p, q)-Laplace transform, (p, q)-Gamma function, (p, q)-derivative, (p, q)-integral, (p, q)-calculus, (p, q)-exponential functions, (p, q)-trigonometric functions.

1. Introduction

Let $f(x)$ be a given function that is defined for all $x \geq 0$. The Laplace transform $\mathscr{L}$ of a function $f(x)$ is given by

$$\mathscr{L}\{f(t)\}(s) = \int_0^\infty f(x)e^{-sx}dx,$$

where $s \in \mathbb{C}$, $\Re(s) > 0$, $\Re(s)$ denotes the real part of s. This Laplace transform plays a very important role in pure and applied analysis, especially in solving differential equations.

Many authors studied the extended version of the q-version of Laplace transform(see [1, 2, 3, 4, 12]). Hahn [12] defined the q-analogues of the Laplace transform $\mathscr{L}_q$ by

$$\mathscr{L}_q\{f(t)\}(s) = \frac{1}{1-q}\int_0^\infty f(x)e_q(-st)d_qt, \quad (\Re(s) > 0),$$

Received January 15, 2018. Revised August 7, 2018. Accepted August 14, 2018.
*Corresponding author.
†This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

and

$$L_q\{f(t)\}(s) = \frac{1}{1-q}\int_0^\infty f(x)E_q(-qst)d_qt, \quad (\Re(s) > 0),$$

where the q-analogues of the classical exponential function is defined as:

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} \quad (|t| < 1),$$

and

$$E_q(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} t^n}{(q;q)_n} = (t;q)_\infty \quad (t \in \mathbb{C}).$$

Recently, P. Njionou Sadjang [11] constructed the (p,q)-Laplace transform associated with the (p,q)-calculus involving (p,q)-exponential, (p,q)-integration, and (p,q)-differentiation.

For a given function $f(t)$, Sadjang define the (p,q)-Laplace transform by means of

$$\mathscr{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t)e_{p^{-1},q^{-1}}(-sqt), \ s > 0, \tag{1}$$

and

$$\mathscr{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t)e_{p,q}(-spt), \ s > 0, \tag{2}$$

where equation (1) is called the (p,q)-Laplace transform of the first kind, and equation (2) is called the (p,q)-Laplace transform of the second kind.

In this paper, we reconstruct the definition of (p,q)-Laplace transform by referring to Hahn's definition of q-Laplace transform and Sadjang's (p,q)-Laplace transform. We demonstrate several properties for the newly defined (p,q)-Laplace transform based on Equation (1).

2. Basic definitions and miscellaneous results

We introduce the following notations and definitions in [5, 7, 8, 9, 10, 11]. The (p,q)-number is defined by, for any number n,

$$[n]_{p,q} = \frac{p^n - q^n}{p-q},$$

which is a clear generalization of q-number when p approaches 1. That is, $\lim_{p\to 1}[n]_{p,q} = [n]_q$. For $n \in \mathbb{N}$, the (p,q)-factorial is defined by [5, 7, 11]

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \geq 1, \quad [0]_{p,q}! = 1.$$

We also introduce the so-called (p,q)-binomial coefficients in [7, 9, 10, 11].

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \quad 0 \leq k \leq n, n \in \mathbb{N}.$$

Note that as $p \to 1$, the (p, q)-binomial coefficients is the q-binomial coefficients. The following equation is obvious by definition.

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \begin{bmatrix} n \\ n-k \end{bmatrix}_{p,q}.$$

Definition 2.1 ([9, 11])**.** Let f be an any function and a be a real number. Then the (p, q)-integral of f is defined by

$$\int_0^a f(x) d_{p,q}x = (p-q) \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}} a\right) \quad \text{if} \quad \left|\frac{p}{q}\right| > 1.$$

Definition 2.2 ([9, 11])**.** The improper (p, q)-integral of $f(x)$ on $[0, \infty]$ is defined to be

$$\int_0^{\infty} f(x) d_{p,q}x = (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}} a\right), \quad 0 < \frac{p}{q} < 1,$$

where f is a function defined on the set of the complex numbers.

Definition 2.3 ([5, 6, 7, 8, 9, 11])**.** We define the (p, q)-derivative operator of any function f, also referred to as the Jackson derivative, as follows:

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $D_{p,q} f(0) = f'(0)$.

Proposition 2.4 ([6, 7, 9, 11])**.** *This operator $D_{p,q}$ has the following basic properties: (i) Derivative of a product*

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px) D_{p,q} g(x) + g(qx) D_{p,q} f(x) \\ &= g(px) D_{p,q} f(x) + f(qx) D_{p,q} g(x). \end{aligned}$$

(ii) Derivative of a ratio

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx) D_{p,q} f(x) - f(qx) D_{p,q} g(x)}{g(px) g(qx)} \\ &= \frac{g(px) D_{p,q} f(x) - f(px) D_{p,q} g(x)}{g(px) g(qx)}. \end{aligned}$$

Proposition 2.5 ([6, 9, 11])**.** *Let $F(x)$ be a (p, q)-antiderivative of $f(x)$ and $F(x)$ be continuous at $x = 0$. We get the following equation*

$$\int_a^b f(x) d_{p,q}x = F(b) - F(a), \quad 0 \le a < n \le \infty.$$

Corollary 2.6 ([6, 9, 11])**.** *Let $f'(x)$ exist in a neighborhood of $x = 0$ and be continuous at $x = 0$. If $f'(x)$ denotes the ordinary derivative of $f(x)$, then we obtain the following equation*

$$\int_a^b D_{p,q} f(x) d_{p,q}x = f(b) - f(a).$$

Proposition 2.7 ([6, 9, 11]). *Let $f(x)$ and $g(x)$ be two functions whose ordinary derivatives exist in a neighborhood of $x = 0$. If a and b are two real numbers such that $a < b$, then we have the following equation*

$$\int_a^b f(px)(D_{p,q}g(x))d_{p,q}x = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_{p,q}f(x))d_{p,q}x.$$

Definition 2.8 ([6, 7, 8, 11]). Let z be any complex numbers with $|z| < 1$. Then the two forms of (p,q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!},$$
$$E_{p,q}(z) = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}.$$

The useful relation of two forms of (p,q)-exponential functions was obtained by

$$e_{p,q}(z)E_{p,q}(-z) = 1, \qquad E_{p,q}(z) = e_{p^{-1},q^{-1}}(z).$$

Proposition 2.9 ([7, 8, 11]). *Let λ be a complex number. Then we have the following equations*

$$D_{p,q}e_{p,q}(\lambda x) = \lambda e_{p,q}(\lambda p x),$$
$$D_{p,q}e_{p^{-1},q^{-1}}(\lambda x) = \lambda e_{p^{-1},q^{-1}}(\lambda q x).$$

Proposition 2.10 ([7, 11]). *Let n be a nonnegative integer. Then we obtain the following equations*

$$D^n_{p,q}e_{p,q}(\lambda x) = \lambda^n p^{\binom{n}{2}} e_{p,q}(\lambda p^n x),$$
$$D^n_{p,q}e_{p^{-1},q^{-1}}(\lambda x) = \lambda^n q^{\binom{n}{2}} e_{p^{-1},q^{-1}}(\lambda q^n x).$$

Using Definition 2.8, we get the following proposition.

Proposition 2.11 ([7, 11]). *Let n be a nonnegative integer. Then we get the following equations*

$$\cos_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$
$$\sin_{p,q}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1},$$
$$Cos_{p,q}(z) = \cos_{p^{-1},q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$
$$Sin_{p,q}(z) = \sin_{p^{-1},q^{-1}}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}.$$

Using Proposition 2.11, we derive the following proposition.

Proposition 2.12 ([7, 11]). *The following relation equations hold true.*

$$\cos_{p,q}(z)Cos_{p,q}(z) + \sin_{p,q}(z)Sin_{p,q}(z) = 1,$$
$$\sin_{p,q}(z)Cos_{p,q}(z) - \cos_{p,q}(z)Sin_{p,q}(z) = 0.$$

The (p, q)-analogues of the hyperbolic functions can be defined in the same way as well-known Taylor expressions using exponential functions.

Proposition 2.13 ([7, 11]). *Let n be a nonnegative integer. Then we obtain*

$$\cosh_{p,q}(z) = \frac{e_{p,q}(z) + e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$
$$\sinh_{p,q}(z) = \frac{e_{p,q}(z) - e_{p,q}(-z)}{2} = \sum_{n=0}^{\infty} \frac{p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1},$$
$$Cosh_{p,q}(z) = \frac{e_{p^{-1},q^{-1}}(z) + e_{p^{-1},q^{-1}}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$
$$Sinh_{p,q}(z) = \frac{e_{p^{-1},q^{-1}}(z) - e_{p^{-1},q^{-1}}(-z)}{2} = \sum_{n=0}^{\infty} \frac{q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}.$$

Using Proposition 2.13, we get the following proposition.

Proposition 2.14 ([7, 11]). *The following relation equations hold true.*

$$\cosh_{p,q}(z)Cosh_{p,q}(z) - \sinh_{p,q}(z)Sinh_{p,q}(z) = 1,$$
$$\cosh_{p,q}(z)Sinh_{p,q}(z) - \sinh_{p,q}(z)Cosh_{p,q}(z) = 0.$$

Definition 2.15 ([7, 11]). For any $n \in \mathbb{N}$, we propose (p, q)-Gamma function as

$$\Gamma_{p,q}(n + 1) = p^{\frac{(n-1)(n-2)}{2}} \int_0^{\infty} x^{n-1} e_{p^{-1},q^{-1}}(-qx) d_{p,q}x.$$

Using Definition 2.15, we have the following Proposition 2.16 and Lemma 2.17.

Proposition 2.16 ([7, 11]). *For any $n \in \mathbb{N}$, we have*

$$\Gamma_{p,q}(n + 1) = [n]_{p,q}\Gamma_{p,q}(n).$$

Lemma 2.17 ([7, 11]). *For any $n \in \mathbb{N}$, it follows from equation in Proposition 2.16 that*

$$\Gamma_{p,q}(n + 1) = [n]_{p,q}!.$$

3. Properties of the (p, q)-Laplace Transform

In this section, we introduce the two types of definitions of the (p, q)-version of Laplace transforms and their properties.

Definition 3.1. For a given function $f(t)$, we define (p, q)-Laplace transform of the first kind as the function

$$F(s) = \mathscr{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t) e_{p^{-1},q^{-1}}(-st), \ s > 0, \tag{3}$$

and we define (p, q)-Laplace transform of the second kind as the function

$$\mathbb{F}(s) = \mathscr{L}_{p,q}\{f(t)\}(s) = \int_0^\infty f(t) e_{p,q}(-st), \ s > 0. \tag{4}$$

Note that our definition of the (p, q)-version of Laplace transforms is different form that of [11]. Now, we investigate the properties for (p, q)-Laplace transform of the first kind.

Theorem 3.2. (*Linearity*) *For any two complex numbers α and β, we obtain*

$$\mathscr{L}_{p,q}\{\alpha f(t) + \beta g(t)\}(s) = \alpha \mathscr{L}_{p,q}\{f(t)\}(s) + \beta \mathscr{L}_{p,q}\{g(t)\}(s). \tag{5}$$

Proof. Using (3) in Definition 3.1, we have

$$\begin{aligned}
&\mathscr{L}_{p,q}\{\alpha f(t) + \beta g(t)\}(s) \\
&= \int_0^\infty e_{p^{-1},q^{-1}}(-st)(\alpha f(t) + \beta g(t)) d_{p,q}t \\
&= \alpha \int_0^\infty e_{p^{-1},q^{-1}}(-st) f(t) d_{p,q}t + \beta \int_0^\infty e_{p^{-1},q^{-1}}(-st) g(x) d_{p,q}t \\
&= \alpha \mathscr{L}_{p,q}\{f(t)\}(s) + \beta \mathscr{L}_{p,q}\{g(t)\}(s).
\end{aligned}$$

□

Applying the Equation (5), we note that

$$\begin{aligned}
\mathscr{L}_{p,q}\{1\}(s) &= -\frac{q}{s} \int_0^\infty D_{p,q}\left(e_{p^{-1},q^{-1}}\left(-\frac{s}{q}t\right)\right) d_{p,q}t \\
&= -\frac{q}{s}\left[e_{p^{-1},q^{-1}}\left(-\frac{s}{q}t\right)\right]_0^\infty = \frac{q}{s}, \quad s > 0. \\
\mathscr{L}_{p,q}\{t\}(s) &= -\frac{q}{ps} \int_0^\infty (pt) D_{p,q}\left(e_{p^{-1},q^{-1}}\left(-\frac{s}{q}t\right)\right) d_{p,q}t \\
&= \frac{q}{ps} \int_0^\infty e_{p^{-1},q^{-1}}(-st) d_{p,q}t \\
&= \frac{q^2}{ps^2}, \quad s > 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}\mathscr{L}_{p,q}\{1+5t\}(s) &= \mathscr{L}_{p,q}\{1\}(s) + 5\mathscr{L}_{p,q}\{t\}(s)\\ &= \frac{q}{s} + \frac{5q^2}{ps^2}, \quad s > 0.\end{aligned}$$

Theorem 3.3. (*Scaling*) *If α is a non-zero complex number, then the following formula applies*

$$f(\alpha t) \rightleftharpoons_{p,q} \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right). \tag{6}$$

Proof. According to Equation (3) in Definition 3.1, we get

$$\begin{aligned}F\left(\frac{s}{\alpha}\right) &= \int_0^\infty e_{p^{-1},q^{-1}}\left(-\frac{s}{\alpha}t\right) f(t) d_{p,q}t\\ &= \alpha \int_0^\infty e_{p^{-1},q^{-1}}(-st) f(\alpha t) d_{p,q}t\end{aligned}$$

or

$$\begin{aligned}\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) &= \frac{1}{\alpha}\int_0^\infty e_{p^{-1},q^{-1}}\left(-\frac{s}{\alpha}t\right) f(t) d_{p,q}t\\ &= \int_0^\infty e_{p^{-1},q^{-1}}(-st) f(\alpha t) d_{p,q}t.\end{aligned}$$

Therefore we obtain

$$f(\alpha t) \rightleftharpoons_{p,q} \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right).$$

□

Theorem 3.4. (*Attenuation or Substitution*) *For a given function $f(t)$, we get*

$$e_{p^{-1},q^{-1}}(-st + s_0 t) e_{p,q}(st) f(t) \rightleftharpoons_{p,q} F(s - s_0). \tag{7}$$

Proof. By (3) in Definition 3.1, we have

$$\begin{aligned}F(s-s_0) &= \int_0^\infty e_{p^{-1},q^{-1}}(-(s-s_0)t) f(t) d_{p,q}t\\ &= \int_0^\infty e_{p^{-1},q^{-1}}(-st)[e_{p^{-1},q^{-1}}(-st+s_0t)e_{p,q}(st)f(t)] d_{p,q}t\\ &= \mathscr{L}_{p,q}\{e_{p^{-1},q^{-1}}(-st+s_0t)e_{p,q}(st)f(t)\}.\end{aligned}$$

Therefore we get

$$e_{p^{-1},q^{-1}}(-st + s_0 t) e_{p,q}(st) f(t) \rightleftharpoons_{p,q} F(s - s_0).$$

□

Theorem 3.5. (*Translation*) *Consider the function*

$$\eta(x) = \begin{cases} 0 & \textit{if } x < 0, \\ 1 & \textit{if } x \geq 0. \end{cases}$$

Hence we have

$$\mathscr{L}_{p,q}\{f(x-x_0)\}(s)=\mathscr{L}_{p,q}\{e_{p,q}(st)e_{p^{-1},q^{-1}}(-s(t+x_0))f(t)\}(s). \tag{8}$$

Proof. It is clear that $f(x)=f(x)\eta(x)$ for $x\geq 0$. Hence we have

$$\mathscr{L}_{p,q}\{f(x-x_0)\}(s)=\int_{x_0}^{\infty}e_{p^{-1},q^{-1}}(-sx)(f(x-x_0)\eta(x-x_0))d_{p,q}x.$$

By putting $x-x_0=t$, we have

$$\begin{aligned}&\mathscr{L}_{p,q}\{f(x-x_0)\}(s)\\&=\int_0^{\infty}e_{p^{-1},q^{-1}}(-s(t+x_0))f(t)d_{p,q}t\\&=\int_0^{\infty}e_{p^{-1},q^{-1}}(-st)e_{p,q}(st)e_{p^{-1},q^{-1}}(-s(t+x_0))f(t)d_{p,q}t\\&=\mathscr{L}_{p,q}\{e_{p,q}(st)e_{p^{-1},q^{-1}}(-s(t+x_0))f(t)\}(s).\end{aligned}$$

□

Theorem 3.6. (*Transform of derivatives*) *For $n\in\mathbb{N}$, we have*

$$D_{p,q}^n f(x)\rightleftharpoons_{p,q}\frac{1}{p^{\binom{n}{2}}}\left(\frac{s}{pq}\right)^nF\left(\frac{s}{p^n}\right)-\sum_{j=0}^{n-1}\frac{1}{p^{\binom{n-1-j}{2}}}\left(\frac{s}{pq}\right)^{n-1-j}D_{p,q}^jf(0). \tag{9}$$

Proof. Using Proposition 2.7 and Equation (3) in Definition 3.1, we obtain

$$\begin{aligned}&\mathscr{L}_{p,q}\{D_{p,q}f(x)\}(s)\\&=\int_0^{\infty}e_{p^{-1},q^{-1}}(-sx)[D_{p,q}f(x)]d_{p,q}x\\&=\left[e_{p^{-1},q^{-1}}\left(-\frac{s}{q}x\right)f(x)\right]_0^{\infty}-\int_0^{\infty}f(px)D_{p,q}\left(e_{p^{-1},q^{-1}}\left(-\frac{s}{q}x\right)\right)d_{p,q}x\\&=-f(0)+\frac{s}{q}\int_0^{\infty}f(px)e_{p^{-1},q^{-1}}(-sx)d_{p,q}x\\&=\frac{s}{pq}F\left(\frac{s}{p}\right)-f(0).\end{aligned}$$

Thus we get

$$D_{p,q}f(x)\rightleftharpoons_{p,q}\frac{s}{pq}F\left(\frac{s}{p}\right)-f(0). \tag{10}$$

As a consequence, we get

$$D_{p,q}^2f(x)\rightleftharpoons_{p,q}\frac{s}{pq}D_{p,q}\left(F\left(\frac{s}{p}\right)\right)-D_{p,q}f(0),$$

$$G\left(\frac{s}{p}\right)=\int_0^{\infty}e_{p^{-1},q^{-1}}\left(-\frac{s}{p}x\right)D_{p,q}f(x)d_{p,q}x$$

$$= \left[e_{p^{-1},q^{-1}}\left(-\frac{s}{pq}x\right) f(x) \right]_0^{\infty} - \int_0^{\infty} f(px) D_{p,q} \left(e_{p^{-1},q^{-1}}\left(-\frac{s}{pq}x\right)\right) d_{p,q}x$$
$$= -f(0) + \frac{s}{q}\int_0^{\infty} f(p^2x) e_{p^{-1},q^{-1}}(-sx) d_{p,q}x$$
$$= -f(0) + \frac{s}{q}\mathscr{L}_{p,q}\{f(p^2x)\}$$
$$= \frac{s}{p^2q}F\left(\frac{s}{p^2}\right) - f(0).$$

Thus we have

$$D^2_{p,q}f(x) \rightleftharpoons_{p,q} \frac{1}{p}\frac{s^2}{p^2q^2}F\left(\frac{s}{p^2}\right) - \frac{1}{p}\frac{s}{q}f(0) - D_{p,q}f(0),$$

$$D^3_{p,q}f(x) \rightleftharpoons_{p,q} \frac{1}{p^3}\frac{s^3}{p^3q^3}F\left(\frac{s}{p^3}\right) - \frac{1}{p}\frac{s^2}{p^2q^2}f(0) - \frac{s}{pq}D_{p,q}f(0) - D^2_{p,q}f(0),$$

$$\vdots$$

Therefore we obtain

$$D^n_{p,q}f(x) \rightleftharpoons_{p,q} \frac{1}{p^{\binom{n}{2}}}\left(\frac{s}{pq}\right)^n F\left(\frac{s}{p^n}\right) - \sum_{j=0}^{n-1} \frac{1}{p^{\binom{n-1-j}{2}}}\left(\frac{s}{pq}\right)^{n-1-j} D^j_{p,q}f(0).$$

□

Corollary 3.7. *For α is a non-zero complex number, we get*

$$\mathscr{L}_{p,q}\{D_{p,q}f(\alpha x)\}(s) = \frac{s}{\alpha^2 pq}F\left(\frac{s}{\alpha p}\right) - \frac{1}{\alpha}f(0).$$

Proof. We can prove the following by referring to proof of the Theorem 3.6.

$$\mathscr{L}_{p,q}\{D_{p,q}f(\alpha x)\}(s)$$
$$= \frac{1}{\alpha}\int_0^{\infty} e_{p^{-1},q^{-1}}\left(-\frac{s}{\alpha}x\right) D_{p,q}f(x) d_{p,q}x$$
$$= \frac{1}{\alpha}\left[e_{p^{-1},q^{-1}}\left(-\frac{s}{\alpha p}x\right) f(x)\right]_0^{\infty} - \frac{1}{\alpha}\int_0^{\infty} f(qx) D_{p,q}\left(e_{p^{-1},q^{-1}}\left(-\frac{s}{\alpha p}x\right)\right) d_{p,q}x$$
$$= -\frac{1}{\alpha}f(0) + \frac{s}{\alpha^2 pq}\mathscr{L}_{p,q}\{f(x)\}\left(\frac{s}{\alpha p}\right)$$
$$= \frac{s}{\alpha^2 pq}F\left(\frac{s}{\alpha p}\right) - \frac{1}{\alpha}f(0).$$

□

Theorem 3.8. (*Derivative of transforms*) *For $n \in \mathbb{N}$, we get*

$$D^n_{p,q,s}F(s) = \int_0^\infty (-t)^n q^{\binom{n-1}{2}} e_{p^{-1},q^{-1}}(-sq^n t) f(t) d_{p,q}t. \tag{11}$$

Proof. Again, let

$$F(s) = \int_0^\infty e_{p^{-1},q^{-1}}(-st) f(t) d_{p,q}t.$$

We calculate

$$\begin{aligned}
D_{p,q,s}e_{p^{-1},q^{-1}}(-st) &= -te_{p^{-1},q^{-1}}(-sqt),\\
D^2_{p,q,s}e_{p^{-1},q^{-1}}(-st) &= (-t)(-qt)e_{p^{-1},q^{-1}}(-sq^2t),\\
D^3_{p,q,s}e_{p^{-1},q^{-1}}(-st) &= (-t)(-qt)(-q^2t)e_{p^{-1},q^{-1}}(-sq^3t),\\
&\vdots\\
D^n_{p,q,s}e_{p^{-1},q^{-1}}(-st) &= (-t)(-qt)(-q^2t)\dots(-q^{n-1}t)e_{p^{-1},q^{-1}}(-sq^nt)\\
&= (-t)^n q^{\binom{n-1}{2}} e_{p^{-1},q^{-1}}(-sq^nt).
\end{aligned}$$

Therefore we get

$$D^n_{p,q,s}F(s) = \int_0^\infty (-t)^n q^{\binom{n-1}{2}} e_{p^{-1},q^{-1}}(-sq^n t) f(t) d_{p,q}t.$$

□

Theorem 3.9. (*Transform of integrals*) *We have*

$$\int_0^x f(x) d_{p,q}x \rightleftharpoons_{p,q} \frac{q}{s}F(ps). \tag{12}$$

Proof. Using (3) in Definition 3.1, we get

$$\begin{aligned}
&\mathscr{L}_{p,q}\Big\{\int_0^x f(x)d_{p,q}x\Big\}(s)\\
&= \int_0^\infty e_{p^{-1},q^{-1}}(-sx)\left(\int_0^x f(x)d_{p,q}x\right)d_{p,q}x\\
&= -\frac{q}{s}\int_0^\infty \left(\int_0^x f(x)d_{p,q}x\right) D_{p,q}\left(e_{p^{-1},q^{-1}}\left(-\frac{s}{q}x\right)\right)d_{p,q}x\\
&= \frac{q}{s}F(ps).
\end{aligned}$$

Therefore we have

$$\int_0^x f(x) d_{p,q}x \rightleftharpoons_{p,q} \frac{q}{s}F(ps).$$

□

Theorem 3.10. (*Integral of transforms*) *We get*

$$\int_s^\infty F(s) d_{p,q}s \rightleftharpoons_{p,q} q\frac{f(qx)}{x}. \tag{13}$$

Proof. Using (3) in Definition 3.1, we obtain

$$\begin{aligned}
&\mathscr{L}_{p,q}\Big\{\int_s^\infty F(s)d_{p,q}s\Big\}\\
&= q\int_0^\infty e_{p^{-1},q^{-1}}(-sx)\frac{f(qx)}{x}d_{p,q}x\\
&= \mathscr{L}_{p,q}\left\{q\frac{f(qx)}{x}\right\}.
\end{aligned}$$

Therefore we have

$$\int_s^\infty F(s)d_{p,q}s \rightleftharpoons_{p,q} q\frac{f(qx)}{x}.$$

□

Theorem 3.11. *For $\alpha > -1$, we obtain*

$$\mathscr{L}_{p,q}\{t^\alpha\}(s) = \frac{q^{\alpha+1}}{p^{\binom{\alpha+1}{2}}s^{\alpha+1}}\Gamma_{p,q}(\alpha+1). \tag{14}$$

Proof. Using (3) in Definition 3.1 and Definition 2.15, we obtain

$$\begin{aligned}
\mathscr{L}_{p,q}\{t^\alpha\}(s) &= \frac{q^{\alpha+1}}{p^{\binom{\alpha+1}{2}}s^{\alpha+1}}\int_0^\infty p^{\binom{\alpha+1}{2}}t^{(\alpha+1)-1}e_{p^{-1},q^{-1}}(-qt)d_{p,q}t\\
&= \frac{q^{\alpha+1}}{p^{\binom{\alpha+1}{2}}s^{\alpha+1}}\Gamma_{p,q}(\alpha+1).
\end{aligned}$$

□

The following theorem is a particular case of Theorem 3.11 when $\alpha = n$ is a nonnegative integer.

Theorem 3.12. *Let $n \in \mathbb{N}_0$. For $s > 0$, we have*

$$\mathscr{L}_{p,q}\{t^n\}(s) = \frac{q^{n+1}}{p^{\binom{n+1}{2}}s^{n+1}}[n]_{p,q}!. \tag{15}$$

Proof. We intend to demonstrate by induction of these results. The result for $n = 0$ is clear. For some nonnegative integer n, then we get by using (p,q)-integration by Proposition 2.7.

$$\begin{aligned}
&\mathscr{L}_{p,q}\{t^{n+1}\}(s)\\
&= \frac{q[n+1]_{p,q}}{p^{n+1}s}\int_0^\infty e_{p^{-1},q^{-1}}(-st)t^n d_{p,q}t\\
&= \frac{q^{n+2}}{p^{\binom{n+2}{2}}s^{n+2}}[n+1]_{p,q}!.
\end{aligned}$$

□

Theorem 3.13. *Let a be a real number. Then we get*

$$\mathscr{L}_{p,q}\{e_{p,q}(at)\}(s) = \frac{pq}{ps-aq}, \quad \frac{aq}{ps} < 1, \tag{16}$$

$$\mathscr{L}_{p,q}\{e_{p^{-1},q^{-1}}(at)\}(s) = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{\binom{n+1}{2}}\left(\frac{a}{s}\right)^{n}. \tag{17}$$

Proof. Using Proposition 2.10, (5), and (15), we have

$$\begin{aligned}\mathscr{L}_{p,q}\{e_{p,q}(at)\}(s) &= \sum_{n=0}^{\infty}\frac{a^n p^{\binom{n}{2}}}{[n]_{p,q}!}\frac{q^{n+1}}{p^{\binom{n+1}{2}}s^{n+1}}[n]_{p,q}! \\ &= \frac{q}{s}\sum_{n=0}^{\infty}\left(\frac{aq}{ps}\right)^n = \frac{pq}{ps-aq},\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{p,q}\{e_{p^{-1},q^{-1}}(at)\}(s) &= \sum_{n=0}^{\infty}\frac{a^n q^{\binom{n}{2}}}{[n]_{p,q}!}\frac{q^{n+1}}{p^{\binom{n+1}{2}}s^{n+1}}[n]_{p,q}! \\ &= \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{q}{p}\right)^{\binom{n+1}{2}}\left(\frac{a}{s}\right)^{n}.\end{aligned}$$

□

Theorem 3.14. *Let a be a real number. Then we obtain*

$$\mathscr{L}_{p,q}\{\cos_{p,q}(at)\}(s) = \frac{p^2qs}{(ps)^2+(aq)^2}, \tag{18}$$

$$\mathscr{L}_{p,q}\{\sin_{p,q}(at)\}(s) = \frac{pq^2a}{(ps)^2+(aq)^2}. \tag{19}$$

Proof. Using Proposition 2.11, and Equation (3), we have

$$\begin{aligned}\mathscr{L}_{p,q}\{\cos_{p,q}(at)\}(s) &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}p^{\binom{2n}{2}}}{[2n]_{p,q}!}\int_0^{\infty} e_{p^{-1},q^{-1}}(-st)t^{2n}d_{p,q}t \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}p^{\binom{2n}{2}}}{[2n]_{p,q}!}\frac{q^{2n+1}}{p^{\binom{2n+1}{2}}s^{2n+1}}[2n]_{p,q}! = \frac{p^2qs}{p^2s^2+a^2q^2},\end{aligned}$$

and

$$\begin{aligned}\mathscr{L}_{p,q}\{\sin_{p,q}(at)\}(s) &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}\int_0^{\infty} e_{p^{-1},q^{-1}}(-st)t^{2n+1}d_{p,q}t \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{2n+1}p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!}\frac{q^{2n+2}}{p^{\binom{2n+2}{2}}s^{2n+2}}[2n+1]_{p,q}!\end{aligned}$$

$$= \frac{q}{s}\frac{aq}{ps}\sum_{n=0}^{\infty}\left(-\frac{a^2q^2}{p^2s^2}\right)^n$$
$$= \frac{pq^2a}{p^2s^2 + a^2q^2}.$$

□

Theorem 3.15. *Let a be a real number. Then we get*

$$\mathscr{L}_{p,q}\{\cosh_{p,q}(at)\}(s) = \frac{p^2qs}{(ps)^2 - (aq)^2}, \quad \frac{aq}{ps} < 1, \tag{20}$$

$$\mathscr{L}_{p,q}\{\sinh_{p,q}(at)\}(s) = \frac{pq^2a}{(ps)^2 - (aq)^2}, \quad \frac{aq}{ps} < 1. \tag{21}$$

Proof. Using Proposition 2.13, and Equation (3), we have

$$\mathscr{L}_{p,q}\{\cosh_{p,q}(at)\}(s)\} = \frac{1}{2}\left(\mathscr{L}_{p,q}\{e_{p,q}(at)\}(s) + \mathscr{L}_{p,q}\{e_{p,q}(-at)\}(s)\right)$$
$$= \frac{p^2qs}{(ps)^2 - (aq)^2},$$

and

$$\mathscr{L}_{p,q}\{\sinh_{p,q}(at)\}(s)\} = \frac{1}{2}\left(\mathscr{L}_{p,q}\{e_{p,q}(at)\}(s) - \mathscr{L}_{p,q}\{e_{p,q}(-at)\}(s)\right)$$
$$= \frac{pq^2a}{(ps)^2 - (aq)^2}.$$

□

4. Application of (p, q)-Laplace transform to certain (p, q)-difference equations

In this section, we solve the differential equation using the (p, q)-Laplace transform. We consider the problem of finding $f(t)$, where $f(t)$ satifies (p, q)-Cauchy problem

$$D_{p,q}f(qt) + cqf(pqt) = 0, \ f(0) = 1,$$

where $c \in \mathbb{C}$. Applying the (p, q)-Laplace transform of the first kind to (3) and Corollary 3.7, we obtain

$$-\frac{1}{q}f(0) + \frac{s}{pq^3}\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) + cq\mathscr{L}_{p,q}\{f(pqt)\}(s) = 0.$$

Next, using Equation (6) and the initial condition $f(0) = 1$, we get

$$-\frac{1}{q} + \frac{s}{pq^3}\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) + \frac{c}{p}\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) = 0.$$

Hence we have

$$\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) = \frac{pq^2}{s+cq^3},$$

and so

$$\mathscr{L}_{p,q}\{f(t)\}(s) = \frac{pq}{ps+cq^2}. \tag{22}$$

It follows that $f(t) = e_{p,q}(-cqt)$.

In addition, the result of approaching q to 1 in Equation (22) is equivalent to the result of approaching q to 1 in equation (4.48) of [11]. Also, if $p \to 1$, then we can certainly see that it is the same as the solution of the classic Laplace transform.

Now, consider the (p,q)-differential equation

$$D_{p,q}f(qt) - \lambda q f(pqt) = qe_{p,q}(\lambda q^2 t)\ ,\ f(0) = 0.$$

Applying the (p,q)-Laplace transform of first kind to (3) and Corollary 3.7, it follows that

$$\mathscr{L}_{p,q}\{D_{p,q}f(qt)\}(s) - \lambda q \mathscr{L}_{p,q}\{f(pqt)\}(s) = \mathscr{L}_{p,q}\{qe_{p,q}(\lambda q^2 t)\},$$

$$-\frac{1}{q}f(0) + \frac{s}{pq^3}\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) - \frac{\lambda}{p}\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) = \frac{pq^2}{ps-\lambda q^3}.$$

If you simply look at the equation above, then we get

$$\mathscr{L}_{p,q}\{f(t)\}\left(\frac{s}{pq}\right) = \frac{p^2q^5}{(ps-\lambda q^3)(s-\lambda q^3)}.$$

Finally, if we replace s with pqs, then we have

$$\mathscr{L}_{p,q}\{f(t)\}(s) = \frac{p^2q^3}{(p^2s-\lambda q^2)(ps-\lambda q^2)}. \tag{23}$$

So, clear $f(t) = te_{p,q}(\lambda t)$. Also, we see that the result of approaching q to 1 in Equation (23) is equal to approaching q to 1 in the result of equation (4.49) in [11].

References

1. W.H. Abdi, *On q-Laplace transform*,Proc. Acad. Sci. India **29A** (1960), 389-408.
2. W.H. Abdi, *On certain q-difference equations and q-Laplace transform*, Proc. nat. inst. Sci. India Acad. **28A** (1962), 1-15.
3. W.H. Abdi, *Application of q-Laplace transforms to the solution of certain q-integer equations*, Rend. Circ. mat. Palermo **11** (1962), 1-13.
4. W.H. Abdi, *Certain inversion and representation formulae for q-Laplace transforms*, Math. Zeitschr. **83** (1964), 238-249.
5. A. Aral, *Applications of (p,q)-Gamma Function to Szasz Durrmeyer operators*, https://www.researchgate.net/publication/290446425.
6. I.M. Burban, A.U. Klimyk, *P,Q-differentiation, P,Q-integration, and P,Q-hypergeometric functions related to quantum groups, Integral Transforms and Special Functions*, **2:1** (1994), 15-36.

7. U. Duran, M. Acikgoz, S. Araci, *A Study on Some New Results Arising from* (p, q)*-Calculus*, Preprints 2018, 2018030072 (doi: 10.20944/preprints201803.0072.v1).
8. K.H. Roh, H.Y. Lee, Y.R. Kim and J.Y. Kang, *A reserch on nonlinear* (p, q)*-difference equation transformable to linear equations using* (p, q)*-derivative*, J. Appl. Math. & Informatics **36** (2018), 271 - 283.
9. P. Njionou Sadjang, *On the fundamental theorem of* (p, q)*-calculus and some (p,q)-Taylor formulas*, http://arxiv.org/abs/1309.3934.
10. P. Njionou Sadjang, *On the (p,q)-Gamma and the* (p, q)*-Beta functions*, http://arxiv.org/abs/1506.07394.
11. P. Njionou Sadjang, *On two* (p, q)*-analogues of the Laplace transform*, http://arxiv.org/abs/1703.01906.
12. Hahn W, *Math*, Nachr **2** (1949), 340 - 370.

Young Rok Kim received M.Sc. from Kongju National University. His research interests include number theory, p-adic functional analysis and scientific computing.

Department of Mathematics, Hannam University, Daejeon 34430, Korea.
e-mail: cantor73@naver.com

Cheon Seoung Ryoo received Ph.D. degree from Kyushu University. His research interests focus on the numerical verification method, scientific computing and p-adic functional analysis.

Department of Mathematics, Hannam University, Daejeon, 306-791, Korea
e-mail:ryoocs@hnu.kr