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UNIQUENESS OF CERTAIN TYPES OF DIFFERENCE POLYNOMIALS†

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Abstract. In this paper, we investigate the uniqueness problems of certain types of difference polynomials sharing a small function. The results of the paper improve and generalize the recent results due to H.P. Waghamore [Tbilisi Math. J. 11(2018), 1-13], P. Sahoo and B. Saha [App. Math. E-Notes. 16(2016), 33-44].

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1. Introduction and main results

By a meromorphic function we shall always mean a meromorphic function in the complex plane. Let k be a positive integer or infinity and $a \in C \cup \{\infty\}$. Set $E(a, f) = \{z : f(z) - a = 0\}$, where a zero point with multiplicity k is counted k times in the set. If these zeros points are only counted once, then we denote the set by $\overline{E}(a, f)$. Let f and g be two nonconstant meromorphic functions. If $E(a, f) = E(a, g)$, then we say that f and g share the value a CM; if $\overline{E}(a, f) = \overline{E}(a, g)$, then we say that f and g share the value a IM. We denote by $E_{k)}(a, f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{k)}(a, f)$ the set of distinct a-points of f with multiplicities not greater than k. We denote by $N_{k)}(r, 1/(f - a))$ the counting function for zeros of $f - a$ with multiplicity less than or equal to k, and by $\overline{N}_{k)}(r, 1/(f - a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f - a))$ be the counting function for zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f - a))$ the corresponding one for which multiplicity is not counted. It is assumed that

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the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance, in [7][26].

Around 2001, I Lahiri introduced the notion of weighted sharing, which measures how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 1.1. [10] For a complex number $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point with mutiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. For a complex number $a \in C \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n. We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

A lot of research works on entire and meromorphic funct ions whose differential polynomials share certain value or fixed points have been done by many mathematicians(see[1][2][3][4][6][11][12][13][14][15][16][17][18][19][21][22][25]). Recently, uniqueness problem in difference analogue has became a subject of great interest among the complex analysis researchers. In 2006, R.G. Halburd and R.J. Korhonen [8] established a version of Nevanlinna theory based on difference operators. They also gave the difference logarithmic derivative lemma [9]. With this development many researchers paid their attention to the uniqueness of different types of difference polynomials. In 2010, Zhang proved the following result.

Theorem 1.2. [27] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f \equiv g$.*

In 2014, Meng improved the above result with the notion of weakly weighted sharing and proved the following theorem.

Theorem 1.3. [20] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share "$(\alpha(z), 2)$", then $f \equiv g$.*

In 2016, P. Sahoo and B. Saha studied the uniqueness of certain type of difference polynomial sharing a small function with finite weight and obtained the following results.

Theorem 1.4. [23] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant, $n(\geq 1)$, $m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)}$ and $[g^n(z)(g^m(z)-1)g(z+c)]^{(k)}$ share $(\alpha(z), 2)$, then $f = tg$, where $t^m = 1$.*

Theorem 1.5. [23] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a non-zero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$ when $m \leq k+1$ and $n \geq 4k-m+10$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+c))^{(k)}$ share $(\alpha(z), 2)$, then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z+c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z+c)$.*

Very recently, H.P. Waghamore studied the uniqueness of difference polynomial of the form $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ where $c_j(j = 1, 2, ..., d)$ are complex constants, $v_j(j = 1, 2, ..., d)$ are non-negative integers and $\sigma = v_1 + v_2 + ... + v_d$ and obtained the following results.

Theorem 1.6. [24] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that $c_j(j = 1, 2, ..., d)$ are non-zero complex constants, $v_j(j = 1, 2, ..., d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+\sigma+5$. If $[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ and $[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ share $(\alpha(z), 2)$, then $f = tg$, where $t^m = 1$.*

Theorem 1.7. [24] *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that $c_j(j = 1, 2, ..., d)$ are non-zero complex constants, $v_j(j = 1, 2, ..., d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k + m + \sigma + 5$ when $m \leq k+1$ and $n \geq 4k - m + \sigma + 9$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}$ and $[g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}$ share $(\alpha(z), 2)$, then either $f = tg$, or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \prod_{j=1}^{d} \omega_1(z+c_j)^{v_j} - \omega_2^n(\omega_2 - 1)^m \prod_{j=1}^{d} \omega_2(z+c_j)^{v_j}$.*

Regarding Theorem 1.4-1.7, a natural question to ask is what can be said if we study the uniqueness of difference polynomials without the notion of weighted sharing ?

In the paper, our main concern is to find the possible answer of the above question. We prove the following results.

Theorem 1.8. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that $c_j(j = 1, 2, ..., d)$ are non-zero complex constants, $v_j(j = 1, 2, ..., d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq \max\{2k + m + \sigma + 5, 2d + \sigma + 2\}$. If $E_{3)}(\alpha(z), [f^n(z)(f^m(z) - 1)\prod_{j=1}^{d} f(z + c_j)^{v_j}]^{(k)}) = E_{3)}(\alpha(z), [g^n(z)(g^m(z) - 1)\prod_{j=1}^{d} g(z + c_j)^{v_j}]^{(k)})$, then $f = hg$, where h is a constant and $h^m = 1$.*

Theorem 1.9. *Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that $c_j(j = 1, 2, ..., d)$ are non-zero complex constants, $v_j(j = 1, 2, ..., d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+\sigma+5$ when $m \leq k+1$ and $n \geq 4k-m+\sigma+9$ when $m > k+1$. If $E_{3)}(\alpha(z), [f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}) = E_{3)}(\alpha(z), [g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)})$, then either $f = g$, or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by $R(\omega_1, \omega_2) = \omega_1^n(\omega_1-1)^m\prod_{j=1}^{d}\omega_1(z+c_j)^{v_j} - \omega_2^n(\omega_2-1)^m\prod_{j=1}^{d}\omega_2(z+c_j)^{v_j}$.*

2. Preliminary Lemmas

In this section, we present some lemmas which will be needed in the sequel. We will denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1. [27] *Let f be a meromorphic function of finite order and c is a non-zero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Arguing in a similar manner as in [5], we obtain the following lemma.

Lemma 2.2. *Let f be an entire function of finite order. Then $T(r, f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z + c_j)^{v_j}) = (n + m + \sigma)T(r, f) + S(r, f)$.*

Lemma 2.3. [24] *Let f be an entire function of finite order. Then $T(r, f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z + c_j)^{v_j}) = (n + m + \sigma)T(r, f) + S(r, f)$.*

Lemma 2.4. [28] *Let f be a non-constant meromorphic functions and p, k be two positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f),$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Lemma 2.5. [11] *If F and G are two non-constant meromorphic functions and $E_{3)}(1,F)=E_{3)}(1,G)$, then one of the following cases holds:*

$$(1)\quad T(r,F)+T(r,G)\le 2N_2\left(r,\frac{1}{F}\right)+2N_2(r,F)+2N_2\left(r,\frac{1}{G}\right)$$
$$+2N_2(r,G)+S(r,F)+S(r,G)\,,$$
$$(2)\quad F\equiv G,\quad (3)\quad FG\equiv 1\,.$$

Lemma 2.6. *Let h be a transcendental meromorphic function of finite order. Then we have*

$$T\left(r,h^{n+m}(z)\prod_{j=1}^{d}h(z+c_j)^{v_j}\right)\ge (n+m-\sigma)T(r,h)+S(r,f)\,,$$

where $\sigma=v_1+v_2+...+v_d$.

Proof. From Lemma 2.1, we have

$$(n+m+\sigma)T(r,h)=T(r,h^{n+m}(z)h^{\sigma})+S(r,h)$$
$$=m(r,h^{n+m}(z)h^{\sigma})+N(r,h^{n+m}(z)h^{\sigma})+S(r,h)$$
$$=m\left(r,h^{n+m}(z)\prod_{j=1}^{d}h(z+c_j)^{v_j}\frac{h^{\sigma}}{\prod_{j=1}^{d}h(z+c_j)^{v_j}}\right)$$
$$+N\left(r,h^{n+m}(z)\prod_{j=1}^{d}h(z+c_j)^{v_j}\frac{h^{\sigma}}{\prod_{j=1}^{d}h(z+c_j)^{v_j}}\right)+S(r,h)$$
$$\le T\left(r,h^{n+m}(z)\prod_{j=1}^{d}h(z+c_j)^{v_j}\right)+2\sigma T(r,h)+S(r,h)\,.$$

Thus, we get the conclusion.

3. Proof of Theorem 1.8

Let

$$F_1=f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j},\quad G_1=g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\,,$$

$$F=\frac{F_1^{(k)}}{\alpha(z)},\quad G=\frac{G_1^{(k)}}{\alpha(z)}\,.$$

Then F and G are transcendental meromorphic functions and $E_{3)}(1,F)=E_{3)}(1,G)$ except the zeros and poles of $\alpha(z)$. By Lemma 2.2 and Lemma 2.4 we have

$$N_2\left(r,\frac{1}{F}\right)\le N_2\left(r,\frac{1}{F_1^{(k)}}\right)+S(r,f)\le T(r,F_1^{(k)})$$

$$-T(r,F_1)+N_{2+k}\left(r,\frac{1}{F_1}\right)+S(r,f)$$

$$\leq T(r,F)-(n+m+\sigma)T(r,f)+N_{2+k}\left(r,\frac{1}{F_1}\right)+S(r,f)\,. \tag{1}$$

So we get

$$(n+m+\sigma)T(r,f)\leq T(r,F)+N_{2+k}\left(r,\frac{1}{F_1}\right)-N_2\left(r,\frac{1}{F}\right)+S(r,f)\,. \tag{2}$$

According to Lemma 2.4, we can deduce

$$N_2\left(r,\frac{1}{F}\right)\leq N_2\left(r,\frac{1}{F_1^{(k)}}\right)+S(r,f)\leq N_{2+k}\left(r,\frac{1}{F_1}\right)+S(r,f)\,. \tag{3}$$

Similarly we have

$$(n+m+\sigma)T(r,g)\leq T(r,G)+N_{2+k}\left(r,\frac{1}{G_1}\right)-N_2\left(r,\frac{1}{G}\right)+S(r,g)\,. \tag{4}$$

And

$$N_2\left(r,\frac{1}{G}\right)\leq N_{2+k}\left(r,\frac{1}{G_1}\right)+S(r,g)\,. \tag{5}$$

Suppose, if possible the (1) of Lemma 2.5 holds, that is

$$T(r,F)+T(r,G)\leq 2N_2\left(r,\frac{1}{F}\right)+2N_2(r,F)+2N_2\left(r,\frac{1}{G}\right)$$
$$+2N_2(r,G)+S(r,f)+S(r,g)\,. \tag{6}$$

By (2), (3), (4), (5) and (6), we have

$$(n+m+\sigma)(T(r,f)+T(r,g))\leq N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)$$
$$+N_{2+k}\left(r,\frac{1}{F_1}\right)+N_{2+k}\left(r,\frac{1}{G_1}\right)+S(r,f)+S(r,g)$$
$$\leq 2N_{2+k}\left(r,\frac{1}{F_1}\right)+2N_{2+k}\left(r,\frac{1}{G_1}\right)+S(r,f)+S(r,g)$$
$$\leq (2k+4+2m+2\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g)\,. \tag{7}$$

So

$$(n-2k-m-\sigma-4)(T(r,f)+T(r,g))\leq S(r,f)+S(r,g)\,, \tag{8}$$

which contradicts with the fact that $n\geq\max\{2k+m+\sigma+5,2d+\sigma+2\}$. Therefore, by Lemma 2.5 we have either $FG=1$ or $F=G$.

If $FG=1$, that is

$$[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}]^{(k)}\cdot[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}]^{(k)}$$

$$=\alpha^2 . \tag{9}$$

We can deduce from above that

$$N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{f-1}\right)=S(r,f)\,, \tag{10}$$

which is impossible. So we have $F=G$, that is

$$[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}=[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)}\,. \tag{11}$$

Integrating above, we deduce

$$\begin{aligned}&[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k-1)}\\&=[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k-1)}+c\,,\end{aligned} \tag{12}$$

where c is a constant. If $c\neq 0$, by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}T(r,F_1^{(k-1)})&\leq \overline{N}\left(r,\frac{1}{F_1^{(k-1)}}\right)+\overline{N}\left(r,\frac{1}{F_1^{(k-1)}-c}\right)+S(r,F)\\&\leq \overline{N}\left(r,\frac{1}{F_1^{(k-1)}}\right)+\overline{N}\left(r,\frac{1}{G_1^{(k-1)}}\right)+S(r,F)\,.\end{aligned} \tag{13}$$

By Lemma 2.4, we obtain

$$\begin{aligned}(n+m+\sigma)T(r,f)&\leq T(r,F_1^{(k-1)})-\overline{N}\left(r,\frac{1}{F_1^{(k-1)}}\right)\\&\qquad+N_k\left(r,\frac{1}{F_1}\right)+S(r,f)\\&\leq \overline{N}\left(r,\frac{1}{G_1^{(k-1)}}\right)+N_k\left(r,\frac{1}{F_1}\right)+S(r,f)\,,\\&\leq N_k\left(r,\frac{1}{F_1}\right)+N_k\left(r,\frac{1}{G_1}\right)+S(r,f)+S(r,g)\,,\\&\leq (k+m+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g)\,.\end{aligned} \tag{14}$$

Similarly,

$$(n+m+\sigma)T(r,g)\leq (k+m+\sigma)(T(r,f)+T(r,g))+S(r,f)+S(r,g)\,. \tag{15}$$

Combining (14) and (15), we obtain

$$(n-2k-m-\sigma)(T(r,f)+T(r,g)\leq S(r,f)+S(r,g)\,, \tag{16}$$

which contradicts with $n \geq 2k+m+\sigma+5$. Hence $c=0$. Integrating the (12) $k-1$ times, we can deduce

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j} = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\,. \tag{17}$$

Set $h=\frac{f}{g}$. If h is not a constant, from (17) we have

$$g^m(z) = \frac{h^n(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}\,. \tag{18}$$

If 1 is a Picard value of $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}$, then by the second fundamental theorem of Nevanlinna,

$$\begin{aligned} T\left(h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\right) &\leq \overline{N}\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\right) \\ &\quad +\overline{N}\left(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right) + S(r,h) \\ &\leq (2d+2)T(r,h)+S(r,h)\,. \end{aligned} \tag{19}$$

From the above inequality and $n \geq \max\{2k+m+\sigma+5, 2d+\sigma+2\}$, by Lemma 2.6, we can get a contradiction. Therefore, 1 is not a Picard value of $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}$. If $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j} \not\equiv 1$, from (18), we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}\right) &\leq \overline{N}\left(r, \frac{1}{h^m-1}\right) \\ &\leq mT(r,h)+S(r,h)\,. \end{aligned} \tag{20}$$

From the above inequality and by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}\right) &\leq \overline{N}\left(r, h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)\right) \\ &\quad +\overline{N}\left(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}}\right) \\ &\quad +\overline{N}\left(r, \frac{1}{h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j}-1}\right) + S(r,h) \\ &\leq (m+2d+2)T(r,h)+S(r,h)\,, \end{aligned} \tag{21}$$

which is a contradiction with $n \geq \max\{2k+m+\sigma+5, 2d+\sigma+2\}$.

If $h^{n+m}(z)\prod_{j=1}^{d} h(z+c_j)^{v_j} \equiv 1$, we have

$$(n+m)T(r,h) \leq \sigma T(r,h) + S(r,h)\,, \tag{22}$$

which is a contradiction with $n \geq \max\{2k+m+\sigma+5, 2d+\sigma+2\}$. Therefore, h is a constant. Substituting $f = gh$ into (17), we can get

$$\prod_{j=1}^{d} g(z+c_j)^{v_j}\left(g^{n+m}(z)(h^{n+m+\sigma}-1) + g^{n}(z)(h^{n+\sigma}-1)\right) = 0\,. \tag{23}$$

Since g is an entire function, we have $\prod_{j=1}^{d} g(z+c_j)^{v_j} \neq 0$. Thus

$$g^{n+m}(z)(h^{n+m+\sigma}-1) + g^{n}(z)(h^{n+\sigma}-1) = 0\,. \tag{24}$$

If $h^{n+\sigma} \neq 1$, by (24), we can deduce $T(r,g) = S(r,g)$, which contradicts with a transcendental function g. So $h^{n+\sigma} = 1$. We can also deduce that $h^{n+m+\sigma} = 1$. Then $h^{m} = 1$. This completes the proof of Theorem 1.8.

4. Proof of Theorem 1.9

Let

$$F_1 = f^{n}(z)(f(z)-1)^{m}\prod_{j=1}^{d} f(z+c_j)^{v_j}, \quad G_1 = g^{n}(z)(g(z)-1)^{m}\prod_{j=1}^{d} g(z+c_j)^{v_j}\,,$$

$$F = \frac{F_1^{(k)}}{\alpha(z)}, \quad G = \frac{G_1^{(k)}}{\alpha(z)}\,.$$

Then F and G are transcendental meromorphic functions and $E_{3)}(1,F) = E_{3)}(1,G)$ except the zeros and poles of $\alpha(z)$. By Lemma 2.3 and Lemma 2.4 we can get

$$(n+m+\sigma)T(r,f) \leq T(r,F) + N_{2+k}\left(r,\frac{1}{F_1}\right) - N_2\left(r,\frac{1}{F}\right) + S(r,f)\,, \tag{25}$$

$$N_2\left(r,\frac{1}{F}\right) \leq N_{2+k}\left(r,\frac{1}{F_1}\right) + S(r,f)\,, \tag{26}$$

$$(n+m+\sigma)T(r,g) \leq T(r,G) + N_{2+k}\left(r,\frac{1}{G_1}\right) - N_2\left(r,\frac{1}{G}\right) + S(r,g)\,, \tag{27}$$

$$N_2\left(r,\frac{1}{G}\right) \leq N_{2+k}\left(r,\frac{1}{G_1}\right) + S(r,g)\,. \tag{28}$$

Suppose, if possible the (1) of Lemma 2.5 holds, that is

$$\begin{aligned} T(r,F) + T(r,G) \leq{} & 2N_2\left(r,\frac{1}{F}\right) + 2N_2(r,F) + 2N_2\left(r,\frac{1}{G}\right) \\ & + 2N_2(r,G) + S(r,f) + S(r,g)\,. \end{aligned} \tag{29}$$

By (25), (26), (27), (28) and (29), we have

$$(n+m+\sigma)(T(r,f)+T(r,g)) \leq 2N_{2+k}\left(r,\frac{1}{F_1}\right) \\ +2N_{2+k}\left(r,\frac{1}{G_1}\right)+S(r,f)+S(r,g)\,. \tag{30}$$

If $m > k+1$, by (30) we obtain

$$(n+m-4k-\sigma-8)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)\,, \tag{31}$$

which contradicts with $n \geq 4k-m+\sigma+9$ when $m > k+1$. If $m \leq k+1$, by (30) we obtain

$$(n-2k-m-\sigma-4)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g)\,, \tag{32}$$

which contradicts with $n \geq 2k+m+\sigma+5$ when $m \leq k+1$. Therefore, by Lemma 2.5 we have either $FG=1$ or $F=G$.

If $FG=1$, that is

$$[f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}]^{(k)}\cdot[g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{v_j}]^{(k)} \\ =\alpha^2\,. \tag{33}$$

Proceeding in a like manner as in the proof of Theorem 1.8 we arrive at a contradiction.

If $F=G$, then applying the same technique as in the proof of Theorem 1.8 we obtain

$$f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j} = g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{v_j}\,. \tag{34}$$

Set $h=\frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (34), we deduce that

$$g^n(z)\prod_{j=1}^{d} g(z+c_j)^{v_j}[g^m(z)(h^{n+m+\sigma}-1)-C_m^1 g^{m-1}(z)(h^{n+m+\sigma-1}-1) \\ +...+(-1)^m(h^{n+\sigma}-1)]=0\,. \tag{35}$$

Since g is a transcendental entire function, we have $g^n(z)\prod_{j=1}^{d} g(z+c_j)^{v_j} \neq 0$. So we obtain

$$g^m(z)(h^{n+m+\sigma}-1)-C_m^1 g^{(m-1)}(z)(h^{n+m+\sigma}-1) \\ +...+(-1)^m(h^{n+\sigma}-1)=0\,, \tag{36}$$

which implies $h=1$. Hence $f=g$.

If h is not a constant, then it follows from (34) that f and g satisfy the algebraic equation $R(f,g)=0$, where $R(f,g)$ is given by

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\prod_{j=1}^{d}\omega_1(z+c_j)^{v_j}-\omega_2^n(\omega_2-1)^m\prod_{j=1}^{d}\omega_2(z+c_j)^{v_j}\,. \quad (37)$$

This completes the proof of Theorem 1.9.

References

1. A. Banerjee, *On uniqueness for nonlinear differential polynomials sharing the same 1-point*, Ann. Polon. Math. **89**(2006), 259-272.
2. A. Banerjee, *A uniqueness result on some differential polynomials saring 1-points*, Hiroshima Math. J. **37**(2007), 397-408.
3. A. Banerjee, *Nonlinear differential polynomials sharing one value*, Demonstratio Math. **40**(2007), 65-75.
4. A. Banerjee and S. Mukherjee, *Nonlinear differential polynomials sharing a small function*, Arch. Math.(Brno) **44**(2008), 41-56.
5. M.R. Chen and Z.X. Chen, *Properties of difference polynomials of entire functions with finite order*, Chinese Ann. Math. Ser A. **33**(2012), 359-374.
6. M.L. Fang and W. Hong, *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math. **32**(2001), 1343-1348.
7. W.K. Hayman, *Meromorphic Functions*, Clarendon, Oxford, 1964.
8. R.G. Halburd and R.J. Korhonen, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31**(2006), 463-478.
9. R.G. Halburd and R.J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with application to difference euqations*, J. Math. Anal. Appl. **314**(2006), 477-487.
10. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **161**(2001), 193-206.
11. I. Lahiri, *Uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points*, Ann. Polon. Math. **71**(1999), 113-128.
12. I. Lahiri, *Linear differential polynomials share the same 1-points with weight two*, Ann. Polon. Math. **79**(2002), 157-170.
13. I. Lahiri and R. Pal, *Non-linear differential polynomials sharing 1-points*, Bull. Korean Math. Soc. **43**(2006), 161-168.
14. I. Lahiri and A. Sarkar, *Nonlinear differential polynomials sharing 1-points with weighte two*, Chinese J. Contemp. Math. **25**(2004), 325-334.
15. W.C. Lin and H.X. Yi, *Uniqueness theorems for meromorphic functions*, Indian J. Pure Appl. Math. **35**(2004), 121-132.
16. W.C. Lin and H.X. Yi, *Uniqueness theorems for meromorphic functions concerning fixed-points*, Complex Var. Theory Appl. **49**(2004), 793-806.
17. C. Meng, *Uniqueness theorems for differential polynomials concerning fixed-points*, Kyungpook Math. J. **48**(2008), 25-35.
18. C. Meng, *On unicity of meromorphic functions when two differential polynomials share one value*, Hiroshima Math. J. **39**(2009), 163-179.
19. C. Meng, *Uniqueness for meromorphic functions concerning differential polynomials*, Georgian Math. J. **15**(2008), 731-738.
20. C. Meng, *Uniqueness of entire functions concerning difference polynomials*, Math. Bohem. **139**(2014), 89-97.
21. P. Sahoo, *Uniqueness and weighted sharing of entire functions*, Kyungpook Math. J. **51**(2011), 145-164.

22. P. Sahoo, *Entire functions that share fixed points with finite weights*, Bull. Belgian Math. Soc. Simon Stevin. **18**(2011), 883-895.
23. P. Sahoo and B. Saha, *Value distribution and uniqueness of certain type of difference polynomials*, App. Math. E-Notes. **16**(2016),33-44.
24. H.P. Waghamore, *Generalization of value distribution and uniqueness of certain types of difference polynomials*, Tbilisi Math. J. **11**(2018), 1-13.
25. C.C. Yang and X.H. Hua, *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **22**(1997), 395-406.
26. L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.
27. J.L. Zhang, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. **367**(2010), 401-408.
28. J.L. Zhang and L.Z. Yang, *Some results related to a conjecture of R. Brück*, J. Inequal. Pure Appl. Math. **8**(2007), Art. 18.

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