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UPPER BOUND OF THE CARDINALITY OF E-POWERED NUMBERS OF DIGITS

KYUNG HO JEONG, JISUK SO, SOEUN KIM* AND DAEYEOUL KIM

Abstract. Let us think of adding each digit of a natural number, in bases b, that are e-powered. By studying for a number becoming greater than or equal to oneself, we will consider values of $\lim_{b\to\infty} \frac{N(e,b)}{b^e}$.

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1. Introduction

Let $\mathbb{N} := \{1, 2, 3, \cdots\}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$.

If b is a natural number that is greater than or equal to 2 and e is also a natural number. Let $S_{e,b} : \mathbb{N} \to \mathbb{N}$ map each positive integer to the sum of the e-th powers of its base b digits. Then the natural number $x = \sum_{i=0}^{n} a_i b^i$, where a_i are integers $0 \le a_i \le b-1$, $a_n \ne 0$ with $n \in \mathbb{N}_0$. For $x = a_n a_{n-1} ... a_1 a_{0(b)}$, we define $S_{e,b}(x) = \sum_{i=0}^{n} a_i^e$. See [2]. We will give examples as follows Table 1.

A positive integer x is called the abundant (resp. stable) number according as $S_{e,b}(x) > x$, (resp. $= x$). Among the abundant number, the largest value of x is called the greatest abundant number. Moreover, among the abundant and stable number, the largest value of x is called the greatest number.

If e is given and b increases infinitely, then we consider the cardinality of abundant number and the ratio of b^e. In other words, when $N(e, b)$ is defined as the cardinality of abundant number in the operation for the addition of e-powered in base b, does the value of

$$\lim_{b\to\infty} \frac{(Cardinality\ of\ abundant\ number)}{b^e} = \lim_{b\to\infty} \frac{N(e,b)}{b^e}$$

exist?

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$b,\ e$	$x = 6$	$S_{e,b}(x)$
$b = 2,\ 1 \le e \le 6$	$x = 110_{(2)}$	$S_{e,2}(x) = 2$
$b = 3,\ 1 \le e \le 6$	$x = 20_{(3)}$	$S_{1,3}(x) = 2,\ S_{2,3}(x) = 4,\ S_{3,3}(x) = 8$ $S_{4,3}(x) = 16,\ S_{5,3}(x) = 32,\ S_{6,3}(x) = 64$
$b = 4,\ 1 \le e \le 6$	$x = 12_{(4)}$	$S_{1,4}(x) = 3,\ S_{2,4}(x) = 5,\ S_{3,4}(x) = 9$ $S_{4,4}(x) = 17,\ S_{5,4}(x) = 33,\ S_{6,4}(x) = 65$
$b = 5,\ 1 \le e \le 6$	$x = 11_{(5)}$	$S_{e,5}(x) = 2$
$b = 6,\ 1 \le e \le 6$	$x = 10_{(6)}$	$S_{e,6}(x) = 1$
$b = 7,\ 1 \le e \le 6$	$x = 6_{(7)}$	$S_{1,7}(x) = 6,\ S_{2,7}(x) = 36,\ S_{3,7}(x) = 216$ $S_{4,7}(x) = 1296,\ S_{5,7}(x) = 7776,\ S_{6,7}(x) = 46656$

TABLE 1. Examples of $S_{e,b}(x)$, $2 \le b \le 7, 1 \le e \le 6$

Our main goal of this article is to prove a upper bound of abundant number as $b \to \infty$. More precisely, we prove the following theorem.

Theorem 1.1. *Let $N(e,b)$ be the cardinality of x satisfying $x < S_{e,b}(x)$. Then,*

$$\lim_{b\to\infty} \frac{N(e,b)}{b^e} = \frac{1}{2}\sum_{a_e=0}^{e-2} \int_0^1 \int_0^1 \cdots \int_0^1 (|y-1| - y + 1)\, dx_1 dx_2 \cdots dx_{e-1}.$$

Here,

$$y = \left(\frac{|a_e + x_{e-1} - \sum_{i=1}^{e-1} x_i^e| + a_e + x_{e-1} - \sum_{i=1}^{e-1} x_i^e}{2} \right)^{\frac{1}{e}}.$$

This paper is organized as follows. In section 2, we give lemma for proving Theorem 1.1 and Theorem 2.2. In section 3, we give the proof of Theorem 1.1. A few note related references are [1], [3], [4], [5] and [6].

2. Case of the square

To prove our Theorem 1.1 and Theorem 2.2 , we need the following lemma.

Lemma 2.1 ([7, Lemma 3.1]). *Let $b > 2$, $e \in \mathbb{N} - \{1\}$ and $x = \sum_{i=0}^{n} a_i b^i$ with $0 \le a_i \le b-1$, $a_n \ne 0$. If $x \le S_{e,b}(x)$ then $x < (e-1)\, b^e$.*

Now, we consider the cardinality of abundant numbers when b tends to infinity with $e = 2$.

Theorem 2.2. *Let $N(2,b)$ be the cardinality of x satisfying $x < S_{2,b}(x)$. Then,*

$$\lim_{b\to\infty} \frac{N(2,b)}{b^2} = 1 - \frac{\pi}{8}.$$

Proof. By Lemma 2.1, if $x \leq S_{2,b}(x)$ then $x < b^2$. Take $x = kb + c$, for k and c are nonnegative integers that less than or equal to $b - 1$. We note that $x \leq S_{2,b}(x)$ if and only if $kb + c \leq k^2 + c^2$. Thus, we obtain

$$\left(k - \frac{b}{2}\right)^2 + \left(c - \frac{1}{2}\right)^2 \geq \left(\frac{b}{2}\right)^2 + \left(\frac{1}{2}\right)^2. \tag{1}$$

Naturally, we can picture of a square containing a diagonal line, which connects the origin and a point $(b-1, b-1)$ by assuming that k is the x-coordinate and c is the y-coordinate by (1) (see Figure 1). Then, the integer ordered pairs within the boundary or on the boundary of a square are overlapped with a circle, $\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{1}{2}\right)^2$. Among the ordered pairs, pairs that are out of the circle indicate the abundant number, $x < S_{2,b}(x)$ and the pairs is the stable number on the boundary of a circle except the origin indicate $x = S_{2,b}(x)$. Thus, the cardinality of abundant number is the amount of integer ordered pairs on the within the square and out of the circle excluding the boundary.

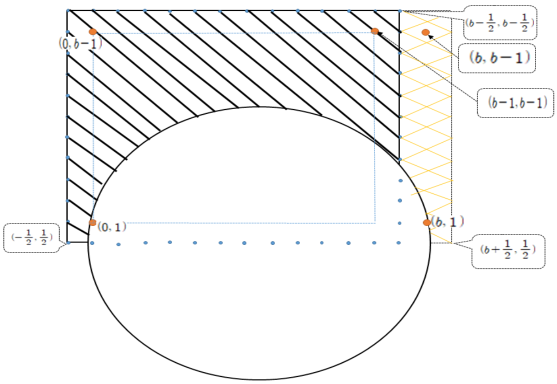

FIGURE 1. The area of $x \leq S_{2,b}(x)$

The area of $x < S_{2,b}(x)$ is

(Area of a square) $-$ (Area of half circle) $<$ (Area of $x < S_{2,b}(x)$) $<$ (Area of a square) $-$ (Area of half circle)$+(b-1)$.

Since

(Area of a square) $-$ (Area of half circle)$= b\,(b-1) - \frac{1}{2} \cdot \pi \cdot \left((\frac{b}{2})^2 + (\frac{1}{2})^2\right)$,

we obtain

$b\,(b-1) - \frac{1}{2} \cdot \pi \cdot \left((\frac{b}{2})^2 + (\frac{1}{2})^2\right) < N(2,b) < b\,(b-1) - \frac{1}{2} \cdot \pi \cdot \left((\frac{b}{2})^2 + (\frac{1}{2})^2\right) + (b-1)$.

Thus, if b is large enough, then

$$\lim_{b \to \infty} \frac{N(2,b)}{b^2} = 1 - \frac{\pi}{8}.$$

This is completed the proof of Theorem 2.2. □

Example 2.3. In base 10, if $x = 10k + c$, then

$$x < S_{2,10}(x) \Leftrightarrow 10k + c < k^2 + c^2$$
$$\Leftrightarrow (k-5)^2 + \left(c - \frac{1}{2}\right)^2 > 25 + \left(\frac{1}{4}\right).$$

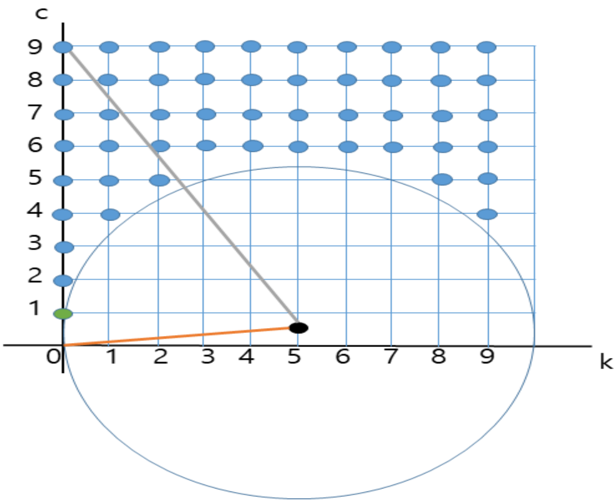

FIGURE 2. Sum of squares in base 10, when $x \leq S_{2,10}(x)$

With base 10, $(0, 9)$ is the farthest point from the origin of the circle and the amount of abundant number is 50 (see Figure 2).

Remark 2.1. By Theorem 1.1, in base b, if $x = kb + c$ then the inequality satisfied with $x < S_{2,b}(x)$ is $\left(k - \frac{b}{2}\right)^2 + \left(c - \frac{1}{2}\right)^2 > \left(\frac{b}{2}\right)^2 + \left(\frac{1}{2}\right)^2$.

At this time the farthest point from the origin of the circle is $(0, b-1)$ and we obtain that

$$S_{2,b}(x) - x = \left(k - \frac{b}{2}\right)^2 + \left(c - \frac{1}{2}\right)^2 - \left((\frac{b}{2})^2 + (\frac{1}{2})^2\right)$$

= Square of (distance between a point indicating oneself and the center of a circle) − Square of (length of radius of a circle).

3. Proof of Theorem 1.1

Now, let us look around more general cases.

Proof of Theorem 1.1

Let $x = \sum_{i=0}^{n} a_i b^i$ be an abundant number. It is known that $n \leq e$ by [7, Theorem 3.8]. The cardinality of abundant number $N(e, b)$ is the number of ordered pairs $(a_e, a_{e-1}, \cdots, a_1, a_0)$ that satisfy

$$\begin{aligned}S_{e,b}(x) - x &= \sum_{i=0}^{e} a_i^e - \sum_{i=0}^{e} a_i b^i \\ &= \sum_{i=1}^{e} \left(a_i^e - a_i b^i\right) + (a_0^e - a_0) > 0.\end{aligned}$$

In other words, $N(e,b)$ is the number of ordered pairs $(a_e, a_{e-1}, \cdots, a_1, a_0)$ that satisfy

$$a_0^e - a_0 > \sum_{i=1}^{e} \left(a_i b^i - a_i^e\right).$$

Let $f(x) = x^e - x$ with $e \geq 2$. We derive that $f(0) = f(1)$ and $f(x)$ increases at $x \geq 1$ (see Figure 3).

Consider $f_1(x) := f(x) - \sum_{i=1}^{e} \left(a_i b^i - a_i^e\right)$. If $f_1(x)$ have real roots then we call α is the largest real root of $f_1(x)$. It is easy checked $\alpha > 0$ (see Figure 3).

Otherwise, $f_1(x)$ have no real roots, we put $\alpha = 0$.

Denote $c = \#\{a_0 | a_0^e - a_0 > \sum_{i=1}^{e} \left(a_i b^i - a_i^e\right) \ \textit{with}\ a_0 \in \{0, 1, \cdots, b-1\}\}$.

Then we find that

$$c = \begin{cases} b, & \text{if } \alpha < 1, \\ b - [\alpha] - 1, & \text{if } 1 \leq \alpha < b - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, we deduce that

$$\frac{-|\alpha| + |\alpha - b| + b}{2} - 1 \leq c \leq \frac{-|\alpha| + |\alpha - b| + b}{2} + 1. \tag{2}$$

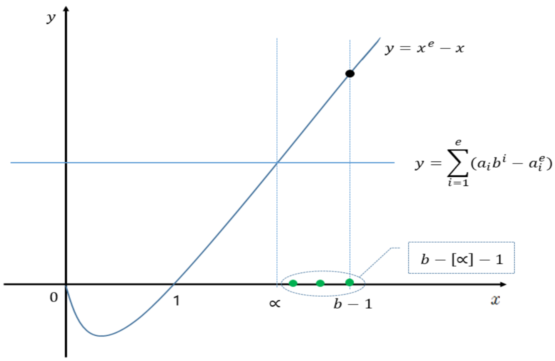

Figure 3. The cardinality of a_0, which satisfies $a_0^e - a_0 > \sum_{i=1}^{e} \left(a_i b^i - a_i^e\right)$

Now, we put

$$\beta = \begin{cases} \left(\sum_{i=1}^{e} \left(a_i b^i - a_i^e\right)\right)^{\frac{1}{e}}, & \text{if } \sum_{i=1}^{e} \left(a_i b^i - a_i^e\right) \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

i.e.,

$$\beta = \left(\frac{|\sum_{i=1}^{e}(a_i b^i - a_i^e)| + \sum_{i=1}^{e}(a_i b^i - a_i^e)}{2} \right)^{\frac{1}{e}}. \tag{3}$$

Therefore, we have $\alpha^e - \alpha = \beta^e$.
Since $\beta \geq 0$, we put $\alpha = \beta + h$ with $h > 0$. Then we have

$$(\beta + h)^e - (\beta + h) = \beta^e \tag{4}$$

and $h < 1$. If $h \geq 1$, the equation (4) does not make sense.
On the other hand, if $\beta = 0$, that is, $\sum_{i=1}^{e} \left(a_i b^i - a_i^e\right) \leq 0$, then $\alpha < 1$.
Thus, we have $\alpha - 1 < \beta \leq \alpha$ and

$$\lim_{b \to \infty} \frac{\beta}{b} = \lim_{b \to \infty} \frac{\alpha}{b}. \tag{5}$$

By Lemma 2.1, we get $x < (e-1)\, b^e$.

If $b > e-1$ then we obtain $0 \leq a_e \leq e-2$ *and* $0 \leq a_i \leq b-1$ $(1 \leq i \leq e-1)$. (6)

Hence, for sufficiently large b (that is, $b > e-1$), we have

$$N(e,b) = \sum_{a_e=0}^{e-2} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} c \right).$$

Thus,

$$\begin{aligned} F(b) &:= \lim_{b \to \infty} \frac{N(e,b)}{b^e} \\ &= \lim_{b \to \infty} \frac{\sum_{a_e=0}^{e-2} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} c \right)}{b^e} \\ &= \lim_{b \to \infty} \sum_{a_e=0}^{e-2} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{c}{b} \right) \\ &= \sum_{a_e=0}^{e-2} \lim_{b \to \infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{-|\alpha| + |\alpha - b| + b}{2b} \right) \end{aligned} \tag{7}$$

by (2).
By (5) and (7), we obtain

$$F(b) = \sum_{a_e=0}^{e-2} \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{-|\beta| + |\beta - b| + b}{2b} \right).$$

Since $\beta \geq 0$, we deduce that

$$F(b) = \sum_{a_e=0}^{e-2} \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{|\beta - b| - \beta + b}{2b} \right).$$

On the other hand,

$$\begin{aligned} G(b) &:= \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{(\sum_{i=1}^{e}(a_i b^i - a_i^e))^{\frac{1}{e}}}{b} \right) \\ &= \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} \frac{((a_e b^e - a_e^e) + \sum_{i=1}^{e-1}(a_i b^i - a_i^e))^{\frac{1}{e}}}{b} \right) \\ &= \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} ((a_e - \frac{a_e^e}{b^e}) + \sum_{i=1}^{e-1}(a_i b^{i-e} - \frac{a_i^e}{b^e}))^{\frac{1}{e}} \right). \end{aligned} \tag{8}$$

From (6), we easily check that

$$\lim_{b\to\infty} \frac{a_e^e}{b^e} = 0 \text{ and} \lim_{b\to\infty} \frac{a_t}{b^{e-t}} = 0 \ (1 \leq t \leq e-2). \tag{9}$$

By (8) and (9), we get

$$\begin{aligned} G(b) &= \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} ((a_e + \sum_{i=1}^{e-1}(a_i b^{i-e} - \frac{a_i^e}{b^e}))^{\frac{1}{e}} \right) \\ &= \lim_{b\to\infty} \frac{1}{b^{e-1}} \left(\sum_{a_{e-1}=0}^{b-1} \sum_{a_{e-2}=0}^{b-1} \cdots \sum_{a_1=0}^{b-1} ((a_e + \frac{a_{e-1}}{b} - \sum_{i=1}^{e-1}(\frac{a_i}{b})^e)^{\frac{1}{e}} \right) \\ &= \lim_{b\to\infty} \left(\frac{1}{b} \sum_{a_{e-1}=1}^{b} \frac{1}{b} \sum_{a_{e-2}=1}^{b} \cdots \frac{1}{b} \sum_{a_1=1}^{b} ((a_e + \frac{a_{e-1}}{b} - \sum_{i=1}^{e-1}(\frac{a_i}{b})^e)^{\frac{1}{e}} \right). \end{aligned}$$

Accordingly, if

$$y = \left(\frac{|a_e + x_{e-1} - \sum_{i=1}^{e-1} x_i^e| + a_e + x_{e-1} - \sum_{i=1}^{e-1} x_i^e}{2} \right)^{\frac{1}{e}}$$

then

$$\lim_{b\to\infty}\frac{N(e,b)}{b^e}=\sum_{a_e=0}^{e-2}\lim_{b\to\infty}\frac{1}{b^{e-1}}\left(\sum_{a_{e-1}=0}^{b-1}\sum_{a_{e-2}=0}^{b-1}\cdots\sum_{a_1=0}^{b-1}\frac{|\beta-b|-\beta+b}{2b}\right)$$
$$=\frac{1}{2}\sum_{a_e=0}^{e-2}\int_0^1\int_0^1\cdots\int_0^1(|y-1|-y+1)dx_1dx_2\cdots dx_{e-1}.$$

This is completed the proof of Theorem 1.1. □

Remark 3.1. According to Theorem 1.1, if $z=a_e+x_{e-1}-\sum_{i=1}^{e-1}x_i^e$, then

$$y=\begin{cases} z^{\frac{1}{e}}, & \text{if } z\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

And, if $f(y)=\frac{|y-1|-y+1}{2}$, then

$$f(y)=\begin{cases} 0, & \text{if } y\geq 1,\\ -y+1, & \text{if } 0\leq y<1.\end{cases}$$

Therefore, we have

$$f(y)=\begin{cases} 1, & \text{if } z<0,\\ 1-z^{\frac{1}{e}}, & \text{if } 0\leq z\leq 1,\\ 0, & \text{if } z>1.\end{cases}$$

Although the case of when $e=2$ in Theorem 1.1 is already explained in Theorem 2.2, let us solve it again using the Theorem 1.1. Putting $y=\left(\frac{|x_1-x_1^2|+x_1-x_1^2}{2}\right)^{\frac{1}{2}}$, we have an example as follow.

Example 3.1. Let $e=2$ in Theorem 1.1. We only consider $0\leq x_1\leq 1$, $x_1-x_1^2\geq 0$. Then $y=\left(x_1-x_1^2\right)^{\frac{1}{2}}$.

Now, we obtain $0\leq y\leq 1$, $|y-1|-y+1=1-y-y+1=2-2y$.

Thus,

$$\begin{aligned}
\lim_{b\to\infty}\frac{N(2,b)}{b^2} &= \frac{1}{2}\int_0^1 (2-2y)\ dx_1 \\
&= \int_0^1 \left(1-\sqrt{x_1-x_1^2}\right)\ dx_1 \\
&= 1-\int_0^1 \sqrt{\frac{1}{4}-\left(x_1-\frac{1}{2}\right)^2}\ dx_1 \\
&= 1-\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\cdot\frac{1}{2}\cos\theta d\theta \\
&= 1-\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} d\theta \\
&= 1-\frac{\pi}{8}.
\end{aligned}$$

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Kyung Ho Jeong
Unnam high school, 17, Imbangul-daero, 165beon-gil, Gwangsan-gu, Gwangju, 62325, Republic of Korea.
e-mail: jkh0702@empas.com

Jisuk So
Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, 567, Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.
e-mail: goleta961@jbnu.ac.kr

Soeun Kim
Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, 567, Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.
e-mail: cybersoeun@jbnu.ac.kr

Daeyeoul Kim

Department of Mathematics and Institute of Pure and Applied Mathematics, Chonbuk National University, 567, Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896, Republic of Korea.
e-mail: kdaeyeoul@jbnu.ac.kr