*J. Appl. Math. & Informatics Vol.* **36***(2018), No. 5 - 6, pp. 411 - 418*
https://doi.org/10.14317/jami.2018.411

# DISCRETE COMPACTNESS PROPERTY FOR KIM-KWAK FINITE ELEMENTS†

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ABSTRACT. In this paper, we prove the discrete compactness property for Kim-Kwak finite element spaces of any order under a weak quasi-uniformity assumption.

AMS Mathematics Subject Classification : 65N30, 65N25.
*Key words and phrases* : Discrete compactness property, edge finite elements, Maxwell's eigenvalue problem.

## 1. Introduction

The edge finite elements are widely used in the appoximation of the eigenvalue problem which arise from electromagnetics. In the theoretical analysis, the important question is to determine the convergence rate of the resulting approximation. For the lowest order edge element space of Nedelec[9] on a tetrahedral mesh, convergence was proved by Kikuchi[2], [3] and [4] using the discrete compactness property. Under the quasi-uniformity assumption, P. Monk and L. Demkowicz[8] observed the convergence for the Nedelec edge elements of all order based on a tetrahedral or hexahedral mesh using the theory of collectively compact operators. The key property in establishing the applicability of this theory is the discrete compactness property of Kikuchi. In a recent article [5] , Kim and Kwak introduced a new family of edge elements on hexahedral grid which, contrary to the classical edge elements, has fewer degrees of freedom and still provides an optimal order approximation. The aim of this paper is to prove the discrete compactness property for Kim-Kwak spaces of any order.

An outline of the article is as follows. In section 2, we present the model problem and the discretization of the problem. Section 3 contains Kim-Kwak edge finite element spaces and the associated scalar spaces. In section 4, we

Received March 10, 2018. Revised July 12, 2018. Accepted August 10, 2018.
†This work was supported by 2017 Hannam University Research Fund.

prove the main result of this article concerning the discrete compactness property under some assumption.

## 2. Setting of the model problem

Let $\Omega \in \mathbb{R}^3$ be a bounded, Lipschitz, polyhedral domain with simply connected boundary $\partial\Omega$. The eigenvalue problem is to find an electric field $\mathbf{E} \neq 0$ and an electric eigenvalue $\lambda$ such that

$$\begin{aligned} \nabla \times \nabla \times \mathbf{E} &= \lambda \mathbf{E}, \quad \text{in } \Omega, && (1)\\ \nabla \cdot \mathbf{E} &= 0, \quad \text{in } \Omega, && (2)\\ \mathbf{n} \times \mathbf{E} &= 0, \quad \text{on } \partial\Omega, && (3) \end{aligned}$$

where $\mathbf{n}$ is an unit outward normal. Since boundary $\partial\Omega$ is simply connected, $\lambda = 0$ is not an eigenvalue for this problem. Also, there is a discrete set of real eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ with $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$. The eigenspaces $W(\lambda_n)$ associated with $\lambda_n$, $n = 1, \cdots$, is finite dimensional.

For the finite element approximation of this problem, we let

$$\mathbf{H}_0^{curl}(\Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 \mid \nabla \times \mathbf{u} \in (L^2(\Omega))^3,\ \mathbf{n} \times \mathbf{u} = 0\}.$$

We denote by $\|\cdot\|$ the $L^2(\Omega)$ or $(L^2(\Omega))^3$ norm, and for other Hilbert space $H^k$ we denote by $\|\cdot\|_k$ the norm on that space. Also we denote

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} dV.$$

Since the solution $\mathbf{E}$ of Maxwell's equations satisfies $\nabla \cdot \mathbf{E} = 0$ in $\Omega$, we need to define

$$\mathbf{X} = \{\mathbf{u} \in \mathbf{H}_0^{curl}(\Omega) \mid \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\}.$$

Then the weak form of the eigenvalue problem $(1) - (3)$ is to find $\mathbf{E} \in \mathbf{X},\ \mathbf{E} \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$(\nabla \times \mathbf{E},\ \nabla \times \boldsymbol{\phi}) = \lambda(\mathbf{E},\ \boldsymbol{\phi}),\ \forall \boldsymbol{\phi} \in \mathbf{X}. \tag{4}$$

Now suppose that we discretize $\mathbf{H}_0^{curl}(\Omega)$ using edge finite elements $\mathbf{V}_h$ parametrized by the mesh size $h > 0$. Then for the edge spaces, we can define a scalar space $S_h \subset H_0^1(\Omega)$ such that $\nabla S_h \subset \mathbf{V}_h$. Since the divergence constraint, we cannot easily approximate $\mathbf{X}$ by an interior approximation. Let

$$\mathbf{X}_h = \{\mathbf{u}_h \in \mathbf{V}_h \mid (\mathbf{u}_h,\ \nabla p_h) = 0,\ \forall p_h \in S_h\}.$$

Then $\mathbf{X}_h \nsubseteq \mathbf{X}$ and we can write the following orthogonal decomposition:

$$\mathbf{V}_h = \mathbf{X}_h \oplus S_h. \tag{5}$$

Using $\mathbf{X}_h$, the discrete eigenvalue problem corresponding to (4) is to find $\mathbf{E}_h \in \mathbf{X}_h,\ \mathbf{E}_h \neq 0$ and $\lambda_h \in \mathbb{R}$ such that

$$(\nabla \times \mathbf{E}_h,\ \nabla \times \boldsymbol{\phi}_h) = \lambda_h(\mathbf{E}_h,\ \boldsymbol{\phi}_h),\ \forall \boldsymbol{\phi}_h \in \mathbf{X}_h. \tag{6}$$

In practical calculation, we usually use $\mathbf{V}_h$ in place of $\mathbf{X}_h$. Thus we would compute $\mathbf{E}_h \in \mathbf{V}_h$, $\mathbf{E}_h \neq 0$ and $\lambda_h \in \mathbb{R}$ such that

$$(\nabla \times \mathbf{E}_h,\ \nabla \times \boldsymbol{\phi}_h) = \lambda_h(\mathbf{E}_h,\ \boldsymbol{\phi}_h),\ \forall \boldsymbol{\phi}_h \in \mathbf{V}_h.$$

Using the decomposition (5), we choose $\boldsymbol{\phi}_h = \nabla p_h$ for $p_h \in S_h$. Then

$$\lambda_h(\mathbf{E}_h,\ \nabla p_h) = 0,\ \forall p_h \in S_h.$$

Since physical eigenvalues are nonzero, it suffices to analyze (6).

## 3. Kim-Kwak edge finite element spaces

In this section, we shall consider the edge finite element spaces due to Kim-Kwak[5] and the scalar spaces due to Kim[6] . We start by covering $\Omega$ by a regular mesh using hexahedra with each edge parallel to one of the coordinate axes. Let us denote the mesh by $\tau_h$ where $h$ is the maximum diameter of the elements in $\tau_h$. For the proof of the discrete compactness property, we need a weak quasi-uniformity restriction. Let $h_K$ denote the diameter of the smallest sphere containing the element $K \in \tau_h$ and let $h_{min} = \min_{K\in\tau_h} h_K$. If there is a constant $C$ with $0 < C < 1$ such that $hh_{min}^{-C} \to 0$ as $h \to 0$, then the mesh is weakly quasi-uniform. Let $Q_{\ell,m,n}$ be the space of polynomials of maximum degree $\ell$ in $x$, $m$ in $y$ and $n$ in z.

For given $k \geq 1$, we define the edge finite element space $\mathbf{U}(K)$ is the subspace of $Q_{k,k+1,k+1}(K) \times Q_{k+1,k,k+1}(K) \times Q_{k+1,k+1,k}(K)$, where the elements in the set $\{\alpha_{i,j}\}_{j=1,2}$ are replaced by the elements $\beta_i$, and the three elements $\gamma_i$ are replaced by the single element $\delta$ for $i = 1, 2, 3$ as follows:

$$\begin{aligned}
&\begin{matrix}\alpha_{11} = \{(0,\ x^{k+1}y^k z^\ell,\ 0)\}_{\ell=0}^k \\ \alpha_{12} = \{(x^k y^{k+1} z^\ell,\ 0,\ 0)\}_{\ell=0}^k\end{matrix} \Rightarrow \beta_1 = \{(x^k y^{k+1} z^\ell,\ x^{k+1} y^k z^\ell,\ 0)\}_{\ell=0}^k,\\
&\begin{matrix}\alpha_{21} = \{(0,\ 0,\ x^\ell y^{k+1} z^k)\}_{\ell=0}^k \\ \alpha_{22} = \{(0,\ x^\ell y^k z^{k+1},\ 0)\}_{\ell=0}^k\end{matrix} \Rightarrow \beta_2 = \{(0,\ x^\ell y^k z^{k+1},\ x^\ell y^{k+1} z^k)\}_{\ell=0}^k,\\
&\begin{matrix}\alpha_{31} = \{(0,\ 0,\ x^{k+1} y^\ell z^k)\}_{\ell=0}^k \\ \alpha_{32} = \{(x^k y^\ell z^{k+1},\ 0,\ 0)\}_{\ell=0}^k\end{matrix} \Rightarrow \beta_3 = \{(x^k y^\ell z^{k+1},\ 0,\ x^{k+1} y^\ell z^k)\}_{\ell=0}^k,\\
&\begin{matrix}\gamma_1 = \{(x^k y^{k+1} z^{k+1},\ 0,\ 0)\} \\ \gamma_2 = \{(0,\ x^{k+1} y^k z^{k+1},\ 0)\} \\ \gamma_3 = \{(0,\ 0,\ x^{k+1} y^{k+1} z^k)\}\end{matrix} \Rightarrow \delta = \{(x^k y^{k+1} z^{k+1},\ x^{k+1} y^k z^{k+1},\ x^{k+1} y^{k+1} z^k)\}.
\end{aligned}$$

For the degrees of freedom, we need to define two auxiliary spaces. First, we define $\mathbf{\Phi}_k^{curl}(x, y)$ to be the subspace of $Q_{k-1,\ k}(x, y) \times Q_{k,\ k-1}(x, y)$ where the two elements $(x^{k-1}y^k, 0)$ and $(0, x^k y^{k-1})$ are replaced by the single element $(x^{k-1}y^k, x^k y^{k-1})$. To define the second space, we use a replacement rule similar to the definition of $\mathbf{U}(K)$. We define $\mathbf{\Psi}_k^{curl}(K)$ to be the subspace of $Q_{k,\ k-1,\ k-1}(K) \times Q_{k-1,\ k,\ k-1}(K) \times Q_{k-1,\ k-1,\ k}(K)$, where the elements

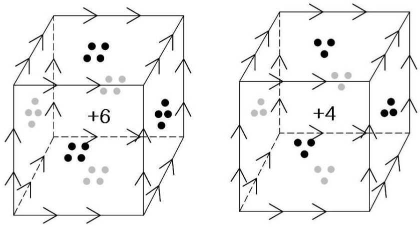


FIGURE 1. For simplicity, we only show the degrees of freedom for $k = 1$. Left one is Nedelec element: there are two tangential component degrees of freedom per edge, four per face and six interior degrees of freedom. Right one is new element: there are two tangential component degrees of freedom per edge, three per face and four interior degrees of freedom

$\{\phi_{ij}\}_{j=1,2}$ are replaced by the elements $\psi_i$ and the three elements $\xi_i$ are replaced by the single element $\zeta$ for $i = 1, 2, 3$ as follows:

$$\begin{array}{l} \phi_{11} = \{(0,\ x^{k-1}y^k z^\ell,\ 0)\}_{\ell=0}^{k-2} \\ \phi_{12} = \{(x^k y^{k-1} z^\ell,\ 0,\ 0)\}_{\ell=0}^{k-2} \end{array} \Rightarrow \psi_1 = \{(x^k y^{k-1} z^\ell,\ x^{k-1} y^k z^\ell,\ 0)\}_{\ell=0}^{k-2},$$

$$\begin{array}{l} \phi_{21} = \{(0,\ 0,\ x^\ell y^{k-1} z^k)\}_{\ell=0}^{k-2} \\ \phi_{22} = \{(0,\ x^\ell y^k z^{k-1},\ 0)\}_{\ell=0}^{k-2} \end{array} \Rightarrow \psi_2 = \{(0,\ x^\ell y^k z^{k-1},\ x^\ell y^{k-1} z^k)\}_{\ell=0}^{k-2},$$

$$\begin{array}{l} \phi_{31} = \{(0,\ 0,\ x^{k-1} y^\ell z^k)\}_{\ell=0}^{k-2} \\ \phi_{32} = \{(x^k y^\ell z^{k-1},\ 0,\ 0)\}_{\ell=0}^{k-2} \end{array} \Rightarrow \psi_3 = \{(x^k y^\ell z^{k-1},\ 0,\ x^{k-1} y^\ell z^k)\}_{\ell=0}^{k-2},$$

$$\begin{array}{l} \xi_1 = \{(x^k y^{k-1} z^{k-1},\ 0,\ 0)\} \\ \xi_2 = \{(0,\ x^{k-1} y^k z^{k-1},\ 0)\} \\ \xi_3 = \{(0,\ 0,\ x^{k-1} y^{k-1} z^k)\} \end{array} \Rightarrow \zeta = \{(x^k y^{k-1} z^{k-1},\ x^{k-1} y^k z^{k-1},\ x^{k-1} y^{k-1} z^k)\}.$$

If $K$ is a cube with general edge $e$ and face $f$, and if $\mathbf{t}$ is a unit tangent vector along $e$, we define the degrees of freedom as follows:

$$\int_e \mathbf{u} \cdot \mathbf{t}\, q\ ds, \qquad \text{for each edges } e \text{ of } K, \quad \forall q \in P_k(e), \tag{7}$$

$$\int_f (\mathbf{u} \times \mathbf{n}) \cdot\ \mathbf{q}\ dA, \qquad \text{for each faces } f \text{ of } K, \quad \forall \mathbf{q} \in \mathbf{\Phi}_k^{curl}(f), \tag{8}$$

$$\int_K \mathbf{u} \cdot \mathbf{q}\ d\mathbf{x}, \qquad \forall \mathbf{q} \in \mathbf{\Psi}_k^{curl}(K). \tag{9}$$

Then a vector function in $\mathbf{U}(K)$ is uniquely determined by the degrees of freedom (7) – (9). And the finite element space $\mathbf{U}(K)$ is curl conforming. Our new element has smaller number of degrees of freedom than the well known Nedelec finite elements on parallelepiped(see Figure 1). Hence it is more efficient.

Then we have the following space:

$$\mathbf{U}_h = \{\mathbf{u}_h \in \mathbf{H}_0^{curl}(\Omega)|\ \mathbf{u}_h|_K \in \mathbf{U}(K),\ \forall K \in \tau_h\}.$$

Now we discretize $\mathbf{H}_0^{curl}(\Omega)$ using edge finite elements. We take subspace $\mathbf{V}_h \subset \mathbf{H}_0^{curl}(\Omega)$ to be

$$\mathbf{V}_h = \{\mathbf{u}_h \in \mathbf{U}_h|\ \mathbf{n} \times \mathbf{u}_h = 0,\ \text{on } \partial\Omega\}. \tag{10}$$

Using the degrees of freedom (7) – (9), we can define an interpolation operator

$$\mathbf{r}_K : \mathbf{H}^{k+1}(K) \to \mathbf{U}(K)$$

for an arbitrary element $K$. Then the global interpolation operator $\mathbf{r}_h$ is defined piecewise by

$$(\mathbf{r}_h\mathbf{u})|_K = \mathbf{r}_K(\mathbf{u}|_K)$$

for all $K \in \tau_h$.

To define the associated scalar space $S_h$, we first let $S(K)$ is the subspace of $Q_{k+1,\ k+1,\ k+1}(K)$ except constant multiple of the term $x^{k+1}y^{k+1}z^{\ell},\ x^{k+1}y^{\ell}z^{k+1}$, $x^{\ell}y^{k+1}z^{k+1}$ and $x^{k+1}y^{k+1}z^{k+1}$ for $\ell = 0 \ldots k$. For any scalar function $p \in S(K)$, we define the following degrees of freedom:

$$p(\mathbf{a}), \qquad \text{for the eight vertices } \mathbf{a} \text{ of } K, \tag{11}$$

$$\int_e pq\ ds, \qquad \text{for each edges } e \text{ of } K, \quad \forall q \in P_{k-1}(e), \tag{12}$$

$$\int_f pq\ dA, \qquad \text{for each faces } f \text{ of } K, \quad \forall q \in Q^*_{k-1,\ k-1}(f), \tag{13}$$

$$\int_K pq\ dA, \qquad \forall q \in Q^*_{k-1,\ k-1,\ k-1}(K), \tag{14}$$

where $Q^*_{k-1,\ k-1}(f)$ is the subspace of $Q_{k-1,\ k-1}(f)$ except constant multiple of the term $x^{k-1}y^{k-1},\ y^{k-1}z^{k-1},\ z^{k-1}x^{k-1}$ and $Q^*_{k-1,\ k-1,\ k-1}(K)$ is the subspace of $Q_{k-1,\ k-1,\ k-1}(K)$ except constant multiple of the term $x^{k-1}y^{k-1}z^{\ell}$, $x^{k-1}y^{\ell}z^{k-1}$, $x^{\ell}y^{k-1}z^{k-1}$ and $x^{k-1}y^{k-1}z^{k-1}$ for $\ell = 0 \ldots k-2$. Then a scalar function in $S(K)$ is uniquely determined by the degrees of freedom (11) – (14). And $S(K)$ is the gradient conforming element space. Now we define the scalar space $S_h$ as follows:

$$S_h = \{p_h \in H_0^1(\Omega)|\ p_h|_K \in S(K),\ \forall K \in \tau_h\}. \tag{15}$$

Then $\nabla S_h \subset \mathbf{V}_h$. Using the degrees of freedom (11) – (14), we can defined an interpolation operator

$$\pi_K : H^{\frac{3}{2}+\delta}(K) \to S(K)$$

by requiring the degrees of freedom of $\pi_K p - p$ vanish. Then the global interpolation operator $\pi_h$ is defined element-wise by

$$(\pi_h p)|_K = \pi_K(p|_K)$$

for all $K \in \tau_h$.

Using the finite element spaces and the interpolation operators presented above, we have the following result.

**Theorem 3.1** (Kim [6]). *If $p$ is sufficiently smooth such that $\mathbf{r}_h\nabla p$ and $\pi_h p$ are defined, then we have*

$$\nabla\pi_h p = \mathbf{r}_h\nabla p.$$

*And there is a constant $C$ independent of $h$ such that*

$$\begin{aligned} \| p - \pi_h p \|_1 &\leq Ch^k \| p \|_{k+1}, \\ \| \mathbf{u} - \mathbf{r}_h\mathbf{u} \| + \| \nabla\times(\mathbf{u} - \mathbf{r}_h\mathbf{u}) \| &\leq Ch^{k+1}(\| \mathbf{u} \|_{k+1} + \| \nabla\times\mathbf{u} \|_{k+1}). \end{aligned}$$

## 4. Discrete compactness property for the Kim-Kwak elements

In this section, we will show that Kim-Kwak edge element spaces satisfy the discrete compactness property under the weak quasi-uniformity assumption on the mesh $\tau_h$. Let $\{h_n\}_{n=0}^{\infty}$ denote a refinement path so that $h_0 > h_1 > h_2 > \cdots > 0$ and $h_n \to 0$ as $n\to\infty$. The numbers $h_n$ index a sequence of progressively finer meshes used to approximate the problem.

First, we consider a regularity result due to Costabel and Dauge [1]. For given $\mathbf{f}\in X'$($X'$ is the $(L^2(\Omega))^3$ dual space of $X$) with $\nabla\cdot\mathbf{f} = 0$ in $\Omega$, let $\mathbf{u}\in X$ satisfy

$$\nabla\times\nabla\times\mathbf{u} = \mathbf{f}$$

in $\Omega$. For a given domain $\Omega$, there is a constant $\varepsilon_0 > 0$ such that for all $\varepsilon$ with $0\leq\varepsilon<\varepsilon_0$ and $\mathbf{f}\in(H^{\varepsilon-1}(\Omega))^3$ we can write

$$\mathbf{u} = \mathbf{w} + \nabla\chi,$$

where $\mathbf{w}\in(H^{\varepsilon+1}(\Omega))^3$ and $\chi\in H_0^1(\Omega)$ with $\Delta\chi\in H^{\varepsilon}(\Omega)$. In addition, we have

$$\| \mathbf{w} \|_{\varepsilon+1} + \| \chi \|_1 \leq C \| \mathbf{f} \|_{\varepsilon-1}, \tag{16}$$

$$\| \Delta\chi \| \leq C \| \mathbf{f} \|_{-1} . \tag{17}$$

We also assume that there is an $\delta > 0$ such that $\chi\in H^{\frac{3}{2}+\delta}(\Omega)$ and

$$\| \chi \|_{\frac{3}{2}+\delta} \leq C \| \Delta\chi \| . \tag{18}$$

**Theorem 4.1.** *Let the mesh $\tau_h$ is regular and weakly quasi-uniform. Assume that the regularity results* (16) − (18) *hold. Suppose that the sequence $\{\mathbf{u}_n\}_{n=1}^{\infty}$ has the following properties:*
*(i) $\{\mathbf{u}_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbf{H}_0^{curl}(\Omega)$.*
*(ii) $\mathbf{u}_n\in X_{h_n}$ for each $n$ and $h_n\to 0$ as $n\to\infty$.*
*Then we can always choose a subsequence, still denoted by $\{\mathbf{u}_n\}_{n=1}^{\infty}$ for simplicity such that*

$$\mathbf{u}_n\to\mathbf{u}$$

*strongly in $(L^2(\Omega))^3$ and weakly in $X$, where $\mathbf{u}$ is a certain element of $X$.*

*Proof.* Using the Helmholtz decomposition, we can write

$$\mathbf{u}_n = \mathbf{u}^n + \nabla p^n, \tag{19}$$

where $p^n \in H_0^1(\Omega)$ satisfies $(\mathbf{u}_n, \nabla q) = (\nabla p^n, \nabla q)$ for all $q \in H_0^1(\Omega)$, and $\mathbf{u}^n \in X$ satisfies $\nabla \times \nabla \times \mathbf{u}^n = \nabla \times \nabla \times \mathbf{u}_n$ in $\Omega$. Since $\nabla \times \mathbf{u}_n$ is a piecewise polynomial vector, $\nabla \times \nabla \times \mathbf{u}_n \in (H^{\varepsilon-1}(\Omega))^3$ for any $\varepsilon$. By the regularity assumption, $\mathbf{u}^n = \mathbf{w}^n + \nabla \chi^n$ for $\mathbf{w}^n \in (H^{\varepsilon+1}(\Omega))^3$ for all $\varepsilon$ with $0 \le \varepsilon < \varepsilon_0$. Hence $\mathbf{r}_{h_n}\mathbf{w}^n$ is well-defined. Since $\chi \in H^{\frac{3}{2}+\delta}(\Omega)$ for some $\delta > 0$, $\mathbf{r}_{h_n}\nabla\chi^n$ is well-defined. By Theorem 1, $\nabla \pi_{h_n}\chi^n = \mathbf{r}_{h_n}\nabla\chi^n$. Since $\mathbf{r}_{h_n}\mathbf{u}^n = \mathbf{u}^n$, we have

$$\begin{aligned} \mathbf{u}_n &= \mathbf{r}_{h_n}\mathbf{u}^n + \nabla \pi_{h_n} p^n & (20)\\ &= (\mathbf{r}_{h_n}\mathbf{u}^n - \mathbf{u}^n) + \mathbf{u}^n + \nabla \pi_{h_n} p^n & (21) \end{aligned}$$

Using the continuous Friedrichs inequality and Weber's compactness property[7], the sequence $\{\mathbf{u}^n\}_{n=1}^{\infty} \subset X$, there is a subsequence, again denote by $\{\mathbf{u}^n\}_{n=1}^{\infty}$, such that

$$\mathbf{u}^n \to \mathbf{u} \tag{22}$$

strongly in $(L^2(\Omega))^3$ and weakly in $X$, for some function $\mathbf{u} \in X$. Since $\mathbf{u}_n \in X_{h_n}$ and $\mathbf{u}^n \in X$,

$$\begin{aligned} (\nabla \pi_{h_n} p^n, \nabla \pi_{h_n} p^n) &= (\mathbf{u}_n - \mathbf{r}_{h_n}\mathbf{u}^n, \nabla \pi_{h_n} p^n)\\ &= (\mathbf{u}^n - \mathbf{r}_{h_n}\mathbf{u}^n, \nabla \pi_{h_n} p^n). \end{aligned}$$

Therefore, we have

$$\| \nabla \pi_{h_n} p^n \| \le \| \mathbf{u}^n - \mathbf{r}_{h_n}\mathbf{u}^n \| . \tag{23}$$

Now, we will prove that $\| \mathbf{r}_{h_n}\mathbf{u}^n - \mathbf{u}^n \| \to 0$ as $n \to \infty$. By the definition of $\mathbf{u}^n$,

$$\mathbf{r}_{h_n}\mathbf{u}^n - \mathbf{u}^n = (\mathbf{r}_{h_n}\mathbf{w}^n - \mathbf{w}^n) + \nabla(\pi_{h_n}\chi^n - \chi^n). \tag{24}$$

Using the regularity of $\mathbf{w}^n$ and the error estimate of Theorem 1, we have

$$\| \mathbf{r}_{h_n}\mathbf{w}^n - \mathbf{w}^n \| \le \; C h_n^{\varepsilon+1} \| \mathbf{w}^n \|_{\varepsilon+1} .$$

From (16) and the weak quasi-uniformity assumption,

$$\| \mathbf{r}_{h_n}\mathbf{w}^n - \mathbf{w}^n \| \le \; C h_n^{\varepsilon+1} \| \nabla \times \mathbf{u}_n \|_{\varepsilon} \le \; C h_n^{\varepsilon+1} h_{min,n}^{-\varepsilon} \| \nabla \times \mathbf{u}_n \| \to 0 \tag{25}$$

as $h \to 0$ and $n \to \infty$. By the assumed regularity of $\chi^n$,

$$\| \nabla(\pi_{h_n}\chi^n - \chi^n) \| \le \; C h_n^{\frac{1}{2}+\delta} \| \nabla \chi^n \|_{\frac{1}{2}+\delta} \le \; C h_n^{\frac{1}{2}+\delta} \| \Delta \chi^n \| .$$

However, $\| \Delta\chi^n \| \le \; C$ for each $n$. So we have

$$\| \pi_{h_n}\chi^n - \chi^n \|_1 \to 0 \tag{26}$$

as $n \to \infty$. From (25) – (26), we show that

$$\| \mathbf{r}_{h_n}\mathbf{u}^n - \mathbf{u}^n \| \to 0 \tag{27}$$

as $n \to \infty$. Using (22), (23) and (27) in (21) completes the proof.

□

## 5. Conculusions

Discretization of Maxwell eigenvalue problems with Kim-Kwak edge finite elements involves a simultaneous use of two discrete subspaces (10) and (15) of $H_0^1(\Omega)$ and $\mathbf{H}_0^{curl}(\Omega)$, reproducing the exact sequence condition. Kim-Kwak edge finite element spaces and the scalar spaces of Kim have fewer degrees of freedom than well-known Nedelec finite element spaces. Especially, stiffness matrix resulting from Kim-Kwak element has a similar data structure as those of standard Nedelec space. However, the resulting stiffness matrix is smaller than the original system and the elementary operation involved does not increase the condition numbers. Hence, the system can be solved by more efficiently than original one.

Kikuchi's discrete compactness property, along with appropriate approximability conditions, guarantees the convergence of discrete Maxwell eigenvalues to the exact ones. Relying on our main result Theorem 4.1, we can prove the convergence of eigenvalue approximation.

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