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# CONVERGENCE ANALYSIS OF PARALLEL $S$-ITERATION PROCESS FOR A SYSTEM OF VARIATIONAL INEQUALITIES USING ALTERING POINTS

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Abstract. In this paper we have considered a system of mixed generalized variational inequality problems defined on two different domains in a Hilbert space. It has been shown that the solution of a system of mixed generalized variational inequality problems is equivalent to altering point formulation of some mappings. A new parallel $S$-iteration type process has been considered which converges strongly to the solution of a system of mixed generalized variational inequality problems.

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## 1. Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let $C$ be a nonempty subset of $H$ and $T : C \to H$ an operator. The variational inequality problem $VI(C, T)$ is to find $x^* \in C$ such that

$$\langle Tx^*, x - x^* \rangle \geq 0 \quad \text{for all } x \in C. \tag{1}$$

The set of solutions of variational inequality $VI(C, T)$ is denoted by $\Omega[VI(C, T)]$, i.e.,

$$\Omega[VI(C, T)] := \{x^* \in C : \langle Tx^*, x - x^* \rangle \geq 0 \text{ for all } x \in C\}.$$

It is well known that the variational inequality problem (1) is equivalent to the following fixed point problem:

$$\text{to find } x^* \in C \text{ such that } x^* = P_C(I - \lambda T)x^*,$$

where $\lambda > 0$ is a constant and $P_C$ is a projection mapping from $H$ onto $C$.

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The classical variational inequality problem was initially introduced by Stampacchia [19, 34] in 1964. The variational inequality problem is one of the very useful and interesting problem in the literature. Many of the problems of pure and applied sciences can be formulated in form of variational inequality problem. Several existence results, iterative algorithms, extensions and generalizations for the variational inequality problems has been studied by many authors in past years (see [3–15, 17, 18, 22–24, 28, 36–41]). One of the important generalization of classical variational inequality problem is a system of variational inequality problems which has been studied by many authors in various frameworks (see [3–6, 9, 17, 24, 36]).

In 2001, Verma [37] introduced and studied a new system of monotone variational inequalities and developed some iterative algorithms for approximation of solutions of considered problems in Hilbert spaces. Since then the system of monotone variational inequalities has been generalized and studied by many authors in different ways (see, [7, 8, 12, 14, 18, 22, 38–40]).

In 2012, Wan and Zhan [41] considered a new system of generalized mixed variational inequality problems (GMVIP) in Hilbert spaces. By using concept of $\eta$-subdifferential and $\eta$-proximal mapping they demonstrated that GMVIP is equivalent to a fixed point problem. They suggested some iterative technique to solve the system of generalized mixed variational inequalities. In 2013, Guo et al. [13] introduced a system of generalized nonlinear mixed variational inequalities and obtained the approximate solution by using the resolvent parallel technique.

In 2014, Sahu [28] introduced the notion of altering points and studied existence and approximation results for altering points. It is remarkable that many problems of nonlinear analysis such as best proximity pairs, a system of nonlinear variational inequalities and a system of hierarchical variational inequalities are equivalent to altering point formulation of some mappings (see [28]).

It is well known that $S$-iteration process introduced by Agarwal et al. [1] is a faster method to find the fixed point of contraction operator than the Picard [26], Mann [21], and Ishikawa [16] iteration processes (see [2, 20, 30]). The $S$-iteration process is more applicable than the Picard, Mann, and Ishikawa iteration processes because it is faster for contraction mappings and also works for nonexpansive type mappings (see [25, 35]). Because of its super convergence, the $S$-iteration process attracted many researchers as an alternate iteration process for solving various nonlinear problems (see [25, 29, 31–33, 35]). In 2011, Sahu [27] introduced the notion of $S$-operator as follows:

Let $C$ be a nonempty convex subset of a vector space $X$ and $T : C \to C$ an operator. Then, an operator $G_{\alpha,\beta,T} : C \to C$ is said to be an $S$-operator generated by $\alpha \in (0, 1]$, $\beta \in (0, 1)$ and $T$ if

$$G_{\alpha,\beta,T} = (1 - \alpha)T + \alpha T((1 - \beta)I + \beta T),$$

and an operator $G_{\beta,T} : C \to C$ is said to be an $S$-operator generated by $\beta \in (0, 1)$ and $T$ if

$$G_{\beta,T} = T((1 - \beta)I + \beta T).$$

It is easy to see that $G_{\alpha,\beta,T}$ is contraction with contractivity factor $k(1-\alpha\beta(1-k))$ if $T$ is a contraction with contractivity factor $k$ and $G_{\alpha,\beta,T}$ is nonexpansive if $T$ is a nonexpansive.

Motivated by $S$-operator, Sahu [27] introduced normal $S$-iteration process as follows:

Let $C$ be a nonempty convex subset of a normed space $X$ and $T : C \to C$ an operator. Then, for arbitrary $x_1 \in C$, the normal $S$-iteration process [27] is defined by

$$x_{n+1} = T[(1-\alpha_n)x_n + \alpha_n T x_n], \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

Using the idea of normal $S$-iteration process, Sahu [28] introduced a parallel $S$-iteration process for finding altering points of mappings $T_1$ and $T_2$ as follows:

Let $C_1$ and $C_2$ be two nonempty closed convex subsets of a Banach space $X$. Let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two mappings. For $\alpha \in (0, 1)$ and arbitrary $(x_1, y_1) \in C_1 \times C_2$, parallel $S$-iteration process is defined by

$$\begin{cases} x_{n+1} = T_2[(1-\alpha)y_n + \alpha T_1 x_n]; \\ y_{n+1} = T_1[(1-\alpha)x_n + \alpha T_2 y_n], \quad n \in \mathbb{N}. \end{cases} \tag{2}$$

In [41], Wan and Zhan considered the following generalized mixed variational inequality problems in Hilbert spaces:

Let $C$ be a closed and convex set in a Hilbert space $H$. Let $T_i, \eta_i : H \times H \to H$ and $g_i : H \to H$ be single-valued mappings and let $\psi_i : H \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous, $\eta_i$-subdifferentiable and proper function on $H$ $(i = 1, 2)$. Find $x^*, y^* \in H$ such that, for all $x \in H$

$$\begin{cases} \langle \rho T_1(y^*, x^*) + x^* - g_1(y^*), \eta_1(x, x^*)\rangle + \rho'\psi_1(x) - \rho'\psi_1(x^*) \geq 0; \\ \langle \sigma T_2(x^*, y^*) + y^* - g_2(x^*), \eta_2(x, y^*)\rangle + \sigma'\psi_2(x) - \sigma'\psi_2(y^*) \geq 0, \end{cases} \tag{3}$$

where the parameters $\rho, \rho', \sigma, \sigma' > 0$ are constants. Under suitable conditions on mappings and parameters, they proved that the sequences $\{x_n\}, \{y_n\}$ generated by following Mann type iteration process

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_{\rho'}^{\Delta\psi_1}[g_1(y_n) - \rho T_1(y_n, x_n)]; \\ y_n = J_{\sigma'}^{\Delta\psi_2}[g_2(x_n) - \sigma T_2(x_n, y_n)], \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, converges strongly to $x^*$ and $y^*$, respectively.

Recently, Sahu et al. [29] defined a new system of generalized variational inequalities on two closed convex subsets of a real Hilbert space and established a strong convergence result using altering points technique.

Motivated and inspired by works of Wan and Zhan [41], Guo et al. [13], Sahu [28] and Sahu et al. [29], the main purpose of this paper is to introduce a new system of mixed generalized variational inequality problems (8) in Hilbert space and to show its equivalence altering point formulation. We introduce a parallel $S$-iteration process to approximate the solution of considered system of mixed generalized variational inequalities. Our result significantly extends the corresponding result of Wan and Zhan [41] for parallel $S$-iteration process and generalizes the result of Sahu [28].

## 2. Preliminaries

Throughout this paper, $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote by $I$ the identity operator of $H$. Also, we denote by $\to$ the strong convergence. The symbol $\mathbb{N}$ stands for the set of all natural numbers.

Let $C$ be a nonempty subset of $H$. A mapping $T : C \to C$ is said to be

(1) $\beta$-*strongly monotone* if there exists a constant $\beta > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \beta \|x - y\|^2 \quad \text{for all } x, y \in C,$$

(2) $\mu$-*cocoercive* if there exists $\mu > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq \mu \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in C,$$

(3) *relaxed* $\gamma$-*cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq (-\gamma) \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in C,$$

(4) *relaxed* $(\gamma, r)$-*cocoercive* if there exist constants $\gamma \geq 0$ and $r > 0$ such that

$$\langle T(x) - T(y), x - y \rangle \geq (-\gamma) \|T(x) - T(y)\|^2 + r\|x - y\|^2 \quad \text{for all } x, y \in C.$$

It is clear that every $\beta$-strongly monotone mapping is $\beta$-expansive and when $\beta = 1$, it is expansive. Every $\mu$-cocoercive mapping is $\frac{1}{\mu}$-Lipschitz continuous mapping. If $\gamma = 0$, then relaxed $(\gamma, r)$-cocoercive mapping is $r$-strongly monotone. Thus, the class of relaxed $(\gamma, r)$-cocoercive mappings is more general than that of the class of strongly monotone mappings.

**Definition 2.1.** [28] Let $C_1, C_2, ..., C_k$ be nonempty subsets of a metric space $X$ and $T_1 : C_1 \to C_2, T_2 : C_2 \to C_3, ..., T_k : C_k \to C_1$ be mappings. Then $x_1 \in C_1, x_2 \in C_2, ..., x_k \in C_k$ are said to be altering points of mappings $T_1, T_2, ..., T_k$ if $T_1 x_1 = x_2, T_2 x_2 = x_3, ..., T_k x_k = x_1$.

In particular for $k = 2$, the point $(x^*, y^*) \in C_1 \times C_2$ is altering point of mappings $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ if

$$\begin{cases} T_1(x^*) = y^*, \\ T_2(y^*) = x^*. \end{cases} \tag{5}$$

Thus $x^*$ and $y^*$ are altering points of $T_1$ and $T_2$ if (5) holds. The set of altering points of mappings $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ is denoted by $Alt(T_1, T_2)$ i.e.,

$$Alt(T_1, T_2) = \{(x^*, y^*) \in C_1 \times C_2 : T_1(x^*) = y^* \text{ and } T_2(y^*) = x^*\}.$$

**Example 2.2.** [28] Let $X = C_1 = C_2 = [0, 1]$ and define $T_1$, $T_2 : X \to X$ by $T_1(x) = 1 - x$ and $T_2(x) = x^2$, $x \in X$. Note $T_2T_1(x) = T_2(1 - x) = (1 - x)^2$ and $T_1T_2(x) = T_1(x^2) = 1 - x^2$ for all $x \in X$. Then $x^* = \frac{\sqrt{5}-1}{2}$ and $y^* = \frac{3-\sqrt{5}}{2}$ are altering points of $T_1$ and $T_2$. The graphical representation of altering points of mappings $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ is given in Figure 1.

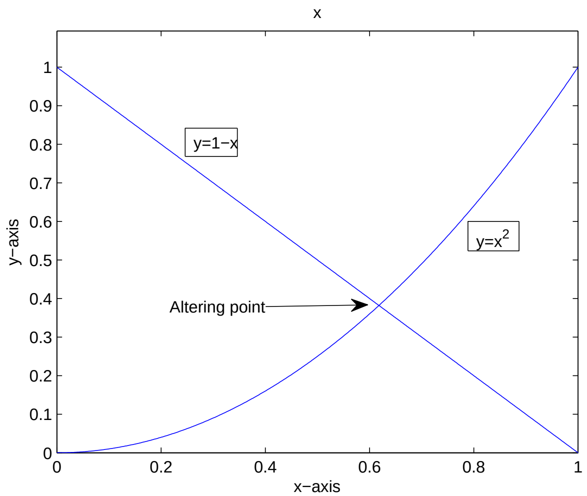


Figure 1. Graphical representation of altering points

**Example 2.3.** Let $X = \ell_2$, $C_1 = \{(x_1, x_2, ..., x_n, ...) \in \ell_1 : |x_n| \leq \frac{1}{2}, \forall n \in \mathbb{N}\}$ and $C_2 = \{(x_1, x_2, ..., x_n, ...) \in \ell_1 : |x_n| \leq 1, \forall n \in \mathbb{N}\}$. Define $T_1 : C_1 \to C_2$ by $T_1(x_1, x_2, ..., x_n, ...) = (0, x_1, x_2, ..., x_{n-1}, ...)$ for all $(x_1, x_2, ..., x_n, ...) \in C_1$ and $T_2 : C_2 \to C_1$ by $T_2(x_1, x_2, ..., x_n, ...) = (\frac{x_1^2}{2}, \frac{x_2^2}{2}, ..., \frac{x_n^2}{2}, ...)$ for all $(x_1, x_2, ..., x_n, ...) \in C_2$. Note that the mapping $T_2T_1 : C_1 \to C_1$ defined by $T_2T_1(x_1, x_2, ..., x_n, ...) = (0, \frac{x_1^2}{2}, \frac{x_2^2}{2}, ..., \frac{x_{n-1}^2}{2}, ...)$ for all $(x_1, x_2, ..., x_n, ...) \in C_1$ is a contraction mapping and the points $x^* = (0, 0, ..., 0, ...) \in C_1$ and $y^* = (0, 0, ..., 0, ...) \in C_2$ are altering points of mappings $T_1$ and $T_2$. The point $x^* = (0, 0, ..., 0, ...) \in C_1$ is also a fixed point of mapping $T_2T_1 : C_1 \to C_1$.

The following existence and approximation results for altering points are given in Sahu [28].

**Theorem 2.4.** [28, Theorem 3.1] *Let $C_1$ and $C_2$ be nonempty closed subsets of a complete metric space $X$ and let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two Lipschitz continuous mappings with Lipschitz constants $k_1$ and $k_2$, respectively such that $k_1k_2 < 1$. Then we have the following:*

*(a) There exists a unique point $(x^*, y^*) \in C_1 \times C_2$ such that $x^*$ and $y^*$ are altering points of mappings $T_1$ and $T_2$.*

*(b) For arbitrary $x_0 \in C_1$, a sequence $\{(x_n, y_n)\}$ in $C_1 \times C_2$ generated by*

$$\begin{cases} y_n = T_1 x_n, \\ x_{n+1} = T_2 y_n \quad \textit{for all } n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \end{cases}$$

*converges to* $(x^*, y^*)$.

**Theorem 2.5.** [28, Theorem 3.6] *Let* $C_1$ *and* $C_2$ *be two nonempty closed convex subsets of a Banach space* $X$. *Let* $T_1 : C_1 \to C_2$ *and* $T_2 : C_2 \to C_1$ *be two Lipschitz continuous mappings with Lipschitz constants* $k_1$ *and* $k_2$ *such that* $k_1 k_2 < 1$. *Then the sequence* $\{(x_n, y_n)\}$ *in* $C_1 \times C_2$ *generated by parallel S-iteration process* (2) *converges strongly to a unique point* $(x^*, y^*) \in C_1 \times C_2$ *such that* $x^*$ *and* $y^*$ *are altering points of mappings* $T_1$ *and* $T_2$.

**Definition 2.6.** [10, 11] Let $\eta : H \times H \to H$ be a single-valued mapping. A proper function $\psi : H \to \mathbb{R} \cup \{+\infty\}$ is said to be *$\eta$-subdifferentiable* at a point $x \in H$ if there exists a point $x^* \in H$ such that

$$\psi(y) - \psi(x) \geq \langle x^*, \eta(y, x)\rangle \quad \text{for all } y \in H,$$

where $x^*$ is called an *$\eta$-subgradient* of $\psi$ at $x$. The set of all $\eta$-subgradients of $\psi$ at $x$ is denoted by $\Delta\psi(x)$. The mapping $\Delta\psi : H \to 2^H$ defined by

$$\Delta\psi(x) = \{x^* \in H : \psi(y) - \psi(x) \geq \langle x^*, \eta(y, x)\rangle \text{ for all } y \in H\} \tag{6}$$

is said to be *$\eta$-subdifferential* of $\psi$ at $x$.

**Remark 2.1.** If $\eta(y, x) = y - x$ for all $y, x \in H$, then Definition 2.6 reduces to the usual definition of subdifferential of a functional $\psi$. If $\psi$ is defferentiable at $x \in H$ and satisfies

$$\psi(x + \lambda\eta(y, x)) \leq \lambda\psi(y) + (1 - \lambda)\psi(x) \quad \text{for all } y \in H,\ \lambda \in [0, 1],$$

then $\psi$ is $\eta$-subdifferentiable at $x \in H$.

**Definition 2.7.** [10, 11] Let $\psi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper functional. For any given $x \in H$ and any $\rho > 0$, if there exists a mapping $\eta : H \times H \to H$ and a unique point $u \in H$ such that

$$\langle u - x, \eta(y, u)\rangle \geq \rho\psi(u) - \rho\psi(y) \quad \text{for all } y \in H, \tag{7}$$

then the mapping $x \mapsto u$, denoted by $J_\rho^{\triangle\psi}(x)$, is said to be an *$\eta$-proximal mapping* of $\psi$.

**Definition 2.8.** [7, 15] A two-variable mapping $T : C \times C \to H$ is said to be *strongly relaxed $(\gamma, r)$-cocoercive in the first variable* if there exist constants $\gamma, r > 0$ such that, for all $x, y \in C$

$$\langle T(x, u) - T(y, v), x - y\rangle \geq (-\gamma)\|T(x, u) - T(y, v)\|^2 + r\|x - y\|^2 \quad \text{for all } u, v \in C.$$

**Definition 2.9.** [41] A two-variable mapping $T : C \times C \to H$ is said to be *relaxed $(\gamma, r)$-cocoercive in the first variable* if there exist constants $\gamma, r > 0$ such that, for all $x, y \in C$

$$\langle T(x, u) - T(y, u), x - y\rangle \geq (-\gamma)\|T(x, u) - T(y, u)\|^2 + r\|x - y\|^2 \quad \text{for all } u \in C.$$

If $T$ is the univariate operator, then the relaxed $(\gamma, r)$-cocoercive in the first variable of two-variable mapping $T(\cdot, \cdot)$ reduces to the relaxed $(\gamma, r)$-cocoercive of univariate operator $T$.

**Definition 2.10.** [23] A mapping $\eta : H \times H \to H$ is said to be

(1) *$\delta$-strongly monotone* if there exists a constant $\delta > 0$ such that

$$\langle \eta(x,y), x-y\rangle \geq \delta\|x-y\|^2 \quad \text{for all } x, y \in H,$$

(2) *$\tau$-Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(x,y)\| \leq \tau\|x-y\| \quad \text{for all } x, y \in H.$$

**Definition 2.11.** [10, 11] A function $f : H \times H \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (in short, 0-DQCV) in $x$ if for any finite set $\{x_1, x_2, \cdots, x_n\} \subset H$ and for any $y = \sum_{i=1}^n \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$,

$$\min_{1 \leq i \leq n} f(x_i, y) \leq 0.$$

**Lemma 2.12.** [7] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n + c_n \quad \textit{for all } n \geq n_0,$$

*where $n_0$ is some nonnegative integer, $\lambda_n \in (0,1)$ with $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ and $\sum_{n=0}^{\infty} c_n \leq \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Let $H$ be a real Hilbert space and let $C_1, C_2$ be two nonempty closed convex subsets of $H$. Let $T_1 : C_1 \to C_2$ and $T_2 : C_2 \to C_1$ be two mappings, $g_i : H \to H$ be single valued mappings and $\psi_i : H \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and $\eta$-subdifferentiable function ($i = 1, 2$). Consider the following system of mixed generalized variational inequality problems (SMGVIP):

Find $(x^*, y^*) \in C_1 \times C_2$ such that, for all $y \in C_2$ and $x \in C_1$

$$\begin{cases} \langle \rho T_1(x^*) + y^* - g_1(x^*), \eta_1(y, y^*)\rangle + \rho'\psi_1(y) - \rho'\psi_1(y^*) \geq 0; \\ \langle \sigma T_2(y^*) + x^* - g_2(y^*), \eta_2(x, x^*)\rangle + \sigma'\psi_2(x) - \sigma'\psi_2(x^*) \geq 0, \end{cases} \tag{8}$$

where $\sigma > 0$, $\sigma' > 0$, $\rho > 0$ and $\rho' > 0$ are constants.

Define

$$I_{C_1}(u) = \begin{cases} 0 & \text{if } u \in C_1, \\ +\infty, & \text{otherwise,} \end{cases} \quad I_{C_2}(u) = \begin{cases} 0 & \text{if } u \in C_2, \\ +\infty, & \text{otherwise.} \end{cases}$$

Now consider the following particular cases of the problem (8):

(I) If $\eta_1(u, v) = \eta_2(u, v) = u - v$, then the SMGVIP (8) is equivalent to the following system of generalized mixed variational inequalities:

Find $(x^*, y^*) \in C_1 \times C_2$ such that, for all $y \in C_2$ and $x \in C_1$

$$\begin{cases} \langle \rho T_1(x^*) + y^* - g_1(x^*), y - y^*\rangle + \rho'\psi_1(y) - \rho'\psi_1(y^*) \geq 0; \\ \langle \sigma T_2(y^*) + x^* - g_2(y^*), x - x^*\rangle + \sigma'\psi_2(x) - \sigma'\psi_2(x^*) \geq 0. \end{cases} \tag{9}$$

(II) If $\eta_1(u, v) = \eta_2(u, v) = u - v$, $\psi_1(u) = I_{C_1}(u)$ and $\psi_2(u) = I_{C_2}(u)$, then SMGVIP (8) reduces to the following system of generalized variational inequalities:

Find $(x^*, y^*) \in C_1 \times C_2$ such that

$$\begin{cases} \langle \rho T_1(x^*) + y^* - g_1(x^*), y - y^* \rangle \geq 0 & \text{for all } y \in C_2; \\ \langle \sigma T_2(y^*) + x^* - g_2(y^*), x - x^* \rangle \geq 0 & \text{for all } x \in C_1. \end{cases} \tag{10}$$

(III) If $g_1 = g_2 = I$, $\eta_1(u,v) = \eta_2(u,v) = u - v$, $\psi_1(u) = I_{C_1}(u)$ and $\psi_2(u) = I_{C_2}(u)$, then the SMGVIP (8) reduces to the following system of variational inequalities considered by Sahu [28]:

Find $(x^*, y^*) \in C_1 \times C_2$ such that

$$\begin{cases} \langle \rho T_1(x^*) + y^* - x^*, y - y^* \rangle \geq 0 & \text{for all } y \in C_2; \\ \langle \sigma T_2(y^*) + x^* - y^*, x - x^* \rangle \geq 0 & \text{for all } x \in C_1. \end{cases} \tag{11}$$

The following lemma will be useful in equivalence formulation between system of variational inequalities and altering point problem:

**Lemma 2.13.** [10, 11] *Let $\eta : H \times H \to H$ be $\tau$-Lipschitz continuous and $\delta$-strongly monotone such that $\eta(x,y) + \eta(y,x) = 0$ for all $x, y \in H$ and for any given $x \in H$, the function $h(y,u) = \langle x - u, \eta(y,u) \rangle$ is 0-DQCV in $y$. Let $\psi : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and $\eta$- subdifferentiable proper functional. Then, for any given $\rho > 0$ and $x \in H$, there exists a unique $u \in H$ such that*

$$\langle u - x, \eta(y,u) \rangle \geq \rho\psi(u) - \rho\psi(y) \quad \textit{for all } y \in H,$$

*that is, $u = J_\rho^{\Delta\psi}(x)$ and $\eta$-proximal mapping $J_\rho^{\Delta\psi}$ of $\psi$ is $\frac{\tau}{\delta}$-Lipschitzian mapping.*

By using Lemma 2.13, one can easily observe that the system of mixed generalized variational inequality problems (8) is equivalent to following altering point problem:

$$\text{to find } (x^*, y^*) \in C_1 \times C_2 \text{ such that } \begin{cases} x^* = J_{\sigma'}^{\Delta\psi_2}[g_2 - \sigma T_2](y^*); \\ y^* = J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1](x^*), \end{cases} \tag{12}$$

that is, $x^* \in C_1$ and $y^* \in C_2$ are altering points of the mappings $S_1 := J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1]$ and $S_2 := J_{\sigma'}^{\Delta\psi_2}[g_2 - \sigma T_2]$.

Following the idea of Sahu [28], we will consider the following parallel $S$-iteration process for the problem (8).

**Algorithm 2.14.** *For any given $(x_1, y_1) \in C_1 \times C_2$, the iterative sequence $\{(x_n, y_n)\}$ is defined by*

$$\begin{cases} x_{n+1} = S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)], \\ y_{n+1} = S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] & \textit{for all } n \in \mathbb{N}, \end{cases} \tag{13}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$, $S_1 := J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1]$ and $S_2 := J_{\sigma'}^{\Delta\psi_2}[g_2 - \sigma T_2]$.*

Note that if $\eta_1(u,v)=\eta_2(u,v)=u-v$, then the $\eta$-proximal mapping $J_{\rho'}^{\Delta\psi_1}$ is just the resolvent operator $J_{\psi_1}=(I+\rho'\partial\psi_1)^{-1}$, the $\eta$-proximal mapping $J_{\sigma'}^{\Delta\psi_2}$ is just the resolvent operator $J_{\psi_2}=(I+\sigma'\partial\psi_2)^{-1}$. Therefore, we have the following particular parallel $S$-iterative algorithm for the problem (9):

**Algorithm 2.15.** *For any given $(x_1,y_1)\in C_1\times C_2$, the iterative sequence $\{(x_n,y_n)\}$ is defined by*

$$\begin{cases} x_{n+1}=U_2[(1-\alpha_n)y_n+\alpha_n U_1(x_n)], \\ y_{n+1}=U_1[(1-\alpha_n)x_n+\alpha_n U_2(y_n)] \quad \text{for all } n\in\mathbb{N}, \end{cases} \tag{14}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$, $U_1=J_{\psi_1}[g_1-\rho T_1]$ and $U_2=J_{\psi_2}[g_2-\sigma T_2]$.*

**Algorithm 2.16.** *If $\eta_1(u,v)=\eta_2(u,v)=u-v$, $\psi_1(u)=I_{C_1}(u)$ and $\psi_2(u)=I_{C_2}(u)$, then the resolvent operator $J_{\psi_1}$ is just the projection operator $P_{C_1}$ and $J_{\psi_2}$ is just the projection operator $P_{C_2}$. Consequently, we have the following algorithm for problem (10): For any given $(x_1,y_1)\in C_1\times C_2$, the iterative sequence $\{(x_n,y_n)\}$ is defined by*

$$\begin{cases} x_{n+1}=V_2[(1-\alpha_n)y_n+\alpha_n V_1(x_n)], \\ y_{n+1}=V_1[(1-\alpha_n)x_n+\alpha_n V_2(y_n)] \quad \text{for all } n\in\mathbb{N}, \end{cases} \tag{15}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$, $V_1=P_{C_1}[g_1-\rho T_1]$ and $V_2=P_{C_2}[g_2-\sigma T_2]$.*

**Algorithm 2.17.** *If $g_1=g_2=I$, then Algorithm 2.16 reduces to the following iterative Algorithm for the problem (11): For any given $(x_1,y_1)\in C_1\times C_2$, the iterative sequence $\{(x_n,y_n)\}$ is defined by*

$$\begin{cases} x_{n+1}=W_2[(1-\alpha_n)y_n+\alpha_n W_1(x_n)], \\ y_{n+1}=W_1[(1-\alpha_n)x_n+\alpha_n W_2(y_n)], \quad \text{for all } n\in\mathbb{N}, \end{cases} \tag{16}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$, $W_1=P_{C_1}[I-\rho T_1]$ and $W_2=P_{C_2}[I-\sigma T_2]$.*

## 3. Main results

First we study the convergence analysis of Mann iteration process for solving the SMGVIP (8).

**Theorem 3.1.** *Let $C_1$ and $C_2$ be nonempty closed convex subsets of a real Hilbert space $H$. Let $T_1:C_1\to C_2$ and $T_2:C_2\to C_1$ be relaxed $(\gamma_i,r_i)$-cocoercive and $\mu_i$-Lipschitz continuous and let $g_i:H\to H$ be relaxed $(l_i,p_i)$-cocoercive and $\xi_i$-Lipschitz continuous $(i=1,2)$. Let $\eta_i:H\times H\to H$ be $\tau_i$-Lipschitz continuous and $\delta_i$ strongly monotone such that $\eta_i(x,y)+\eta_i(y,x)=0$ for all $x,y\in H$ and for any $x\in H$, the function $h_i(y,u)=\langle x-u,\eta_i(y,u)\rangle$ is 0-DQCV in $y$ $(i=1,2)$. Let $\psi_i$ be a lower semicontinuous $\eta_i$-subdifferentiable proper function $(i=1,2)$. Define $S_1=J_{\rho'}^{\Delta\psi_1}[g_1-\rho T_1]$ and $S_2=J_{\sigma'}^{\Delta\psi_2}[g_2-\sigma T_2]$. Let $\{x_n\}$ and $\{y_n\}$*

*be sequences in $C_1$ and $C_2$, respectively, generated by the following Mann type algorithm:*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S_2(y_n); \\ y_n = S_1(x_n), \ n \in \mathbb{N}, \end{cases} \tag{17}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1]$. Then, we have the followings:*

*(a) The mappings $S_1$ and $S_2$ are Lipschitz continuous with Lipschitz constants $\frac{\tau_1}{\delta_1}(\theta_1 + \kappa_1)$ and $\frac{\tau_1}{\delta_2}(\theta_2 + \kappa_2)$, respectively, where*

$$\theta_i = \sqrt{1 + 2l_i\xi_i^2 - 2p_i + \xi_i^2} \quad \text{and}$$
$$\kappa_i = \sqrt{1 + 2\rho\gamma_i\mu_i^2 - 2\rho r_i + \rho^2\mu_i^2} \quad (i = 1, 2).$$

*(b) If $\tau_i(\theta_i + \kappa_i) < \delta_i$ $(i = 1, 2)$, then there exists a unique point $(x^*, y^*) \in C_1 \times C_2$ which solves the SMGVIP* (8).

*(c) In addition, if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\tau_i(\theta_i + \kappa_i) < \delta_i$ $(i = 1, 2)$, then the sequences $\{x_n\}$ and $\{y_n\}$ converges strongly to $x^*$ and $y^*$, respectively.*

*Proof.* $(a)$ Let $x, y \in C_1$. By Lemma 2.13, we have

$$\begin{aligned} &\|S_1(x) - S_1(y)\| \\ &= \|J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1](x) - J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1](y)\| \\ &\le \frac{\tau_1}{\delta_1}\|[g_1 - \rho T_1](x) - [g_1 - \rho T_1](y)\| \\ &\le \frac{\tau_1}{\delta_1}\|x - y - (g_1(x) - g_1(y))\| + \frac{\tau_1}{\delta_1}\ \|x - y - \rho(T_1(x) - T_1(y))\|. \end{aligned} \tag{18}$$

Observe that

$$\begin{aligned} &\|x - y - (g_1(x) - g_1(y))\|^2 \\ &= \|x-y\|^2 - 2\langle x - y, g_1(x) - g_1(y)\rangle + \|g_1(x) - g_1(y)\|^2 \\ &\le \|x-y\|^2 - 2(-l_1\|g_1(x) - g_1(y)\|^2 + p_1\|x-y\|^2) + \|g_1(x) - g_1(y)\|^2 \\ &\le \|x-y\|^2 + 2l_1\xi_1^2\|x-y\|^2 - 2p_1\|x-y\|^2 + \xi_1^2\|x-y\|^2 \\ &= (1 + 2l_1\xi_1^2 - 2p_1 + \xi_1^2)\|x-y\|^2 \\ &= \theta_1^2\|x-y\|^2 \end{aligned} \tag{19}$$

and

$$\begin{aligned} &\|x - y - \rho(T_1(x) - T_1(y))\|^2 \\ &= \|x-y\|^2 - 2\rho\langle x - y, T_1(x) - T_1(y)\rangle + \|T_1(x) - T_1(y)\|^2 \\ &\le \|x-y\|^2 - 2\rho(-\gamma_1\|T_1(x) - T_1(y)\|^2 + r_1\|x-y\|^2) + \|T_1(x) - T_1(y)\|^2 \\ &\le \|x-y\|^2 + 2\rho\gamma_1\mu_1^2\|x-y\|^2 - 2\rho r_1\|x-y\|^2 + \mu_1^2\|x-y\|^2 \\ &= (1 + 2\rho\gamma_1\mu_1^2 - 2\rho r_1 + \mu_1^2)\|x-y\|^2 \\ &= \kappa_1^2\|x-y\|^2. \end{aligned} \tag{20}$$

Using (19) and (20) in (18), we get

$$\|S_1(x) - S_1(y)\| \leq \frac{\tau_1}{\delta_1}(\theta_1 + \kappa_1)\|x - y\|.$$

Similarly, we can show that $S_2$ is $\frac{\tau_2}{\delta_2}(\theta_2 + \kappa_2)$-Lipschitz continuous.

($b$) Suppose that $\tau_i(\theta_i + \kappa_i) < \delta_i$ $(i = 1, 2)$. It is clear from part ($a$) that mappings $S_1$ and $S_2$ are contraction mappings. Therefore, from Theorem 2.4, there exists a unique point $(x^*, y^*) \in C_1 \times C_2$ such that $x^*$ and $y^*$ are altering points of mappings $S_1$ and $S_2$. Thus, $(x^*, y^*)$ is the unique solution of the SMGVIP (8).

($c$) Suppose that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\tau_i(\theta_i + k_i) < \delta_i$ $(i = 1, 2)$. From (17), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n S_2 y_n - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\|S_2 y_n - S_1 y^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \frac{\tau_2}{\delta_2}(\theta_2 + \kappa_2)\|y_n - y^*\| \end{aligned} \tag{21}$$

and

$$\|y_n - y^*\| = \|S_1 x_n - S_1 x^*\| \leq \frac{\tau_1}{\delta_1}(\theta_1 + \kappa_1)\|x_n - x^*\|. \tag{22}$$

Using (22) in (21), we get

$$\begin{aligned} &\|x_{n+1} - x^*\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \frac{\tau_1}{\delta_1}(\theta_1 + \kappa_1)\frac{\tau_2}{\delta_2}(\theta_2 + \kappa_2)\|x_n - x^*\| \\ &= \left[1 - \alpha_n\left(1 - \frac{\tau_1}{\delta_1}\frac{\tau_2}{\delta_2}(\theta_1 + \kappa_1)(\theta_2 + \kappa_2)\right)\right]\|x_n - x^*\|. \end{aligned} \tag{23}$$

Note that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\frac{\tau_1}{\delta_1}\frac{\tau_2}{\delta_2}(\theta_1 + \kappa_1)(\theta_2 + \kappa_2) < 1$. Therefore, from Lemma 2.12, we have $\lim_{n\to\infty} x_n = x^*$. Hence, from (22) we obtain that $\lim_{n\to\infty} y_n = y^*$. □

Taking $C_1 = C_2 = C$ in Theorem 3.1, we have the following which can be also derived from [41]:

**Corollary 3.2.** *Let $C$ be a closed convex subset of a real Hilbert space $H$ and let $(x^*, y^*)$ be the solution of the problem* (8). *Let $T_i : C \to C$ is relaxed $(\gamma_i, r_i)$-cocoercive and $\mu_i$-Lipschitz continuous and let $g_i : H \to H$ be relaxed $(l_i, p_i)$-cocoercive and $\xi_i$-Lipschitz continuous $(i = 1, 2)$. Let $\eta_i : H \times H \to H$ be $\tau_i$-Lipschitz continuous and $\delta_i$ strongly monotone such that $\eta_i(x, y) + \eta_i(y, x) = 0$ for all $x, y \in H$ and for any $x \in H$, the function $h_i(y, u) = \langle x - u, \eta_i(y, u)\rangle$ is 0-DQCV in $y$ $(i = 1, 2)$. Let $\psi_i$ be a lower semicontinuous $\eta_i$-subdifferentiable proper function $(i = 1, 2)$. Define $S_1 = J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1]$ and $S_2 = J_{\sigma'}^{\Delta\psi_2}[g_2 - \sigma T_2]$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by iterative algorithm* (17). *Then, we have the followings:*

(*a*) *The mappings* $S_1$ *and* $S_2$ *are Lipschitz continuous with Lipschitz constant* $\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)$ *and* $\frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)$, *respectively, where*

$$\theta_i = \sqrt{1+2l_i\xi_i^2 - 2p_i + \xi_i^2} \text{ and } \kappa_i = \sqrt{1+2\rho\gamma_i\mu_i^2 - 2\rho r_i + \rho^2\mu_i^2} \quad (i=1,2).$$

(*b*) *If* $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\tau_i(\theta_i+\kappa_i) < \delta_i$ $(i=1,2)$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converges strongly to* $x^*$ *and* $y^*$, *respectively.*

Now we study convergence analysis of parallel $S$-iteration process defined by (13).

**Theorem 3.3.** *Let* $C_1$ *and* $C_2$ *be nonempty closed convex subsets of* $H$. *Let* $T_1 : C_1 \to C_2$ *be relaxed* $(\gamma_1, r_1)$*-cocoercive,* $\mu_1$*-Lipschitz continuous and let* $T_2 : C_2 \to C_1$ *be relaxed* $(\gamma_2, r_2)$*-cocoercive,* $\mu_2$*-Lipschitz continuous. Let* $g_i : H \to H$ *be single valued relaxed* $(l_i, p_i)$*-cocoercive,* $\xi_i$*-Lipschitz continuous and let* $\eta_i : H \times H \to H$ *be* $\tau_i$*-Lipschitz continuous and* $\delta_i$*-strongly monotone such that* $\eta_i(x,y)+\eta_i(y,x) = 0$ *for all* $x, y \in H$ *and for any given* $x \in H$, *the function* $h_i(y,u) = \langle x-u, \eta_i(y,u)\rangle$ *is* 0*-DQCV in* $y$ $(i=1,2)$. *Let* $\psi_i$ *be a lower semicontinuous* $\eta_i$*-subdifferentiable proper function* $(i=1,2)$.. *Define* $S_1 := J_{\rho'}^{\Delta\psi_1}[g_1-\rho T_1]$ *and* $S_2 := J_{\sigma'}^{\Delta\psi_2}[g_2-\sigma T_2]$. *Then we have the following:*

(*a*) *The mappings* $S_1$ *and* $S_2$ *are* $\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)$ *and* $\frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)$*-Lipschitzian, respectively, where*

$$\theta_i = \sqrt{1+2l_i\xi_i^2 - 2p_i + \xi_i^2} \text{ and } \kappa_i = \sqrt{1+2\rho\gamma_i\mu_i^2 - 2\rho r_i + \rho^2\mu_i^2} \quad (i=1,2).$$

(*b*) *If* $\tau_i(\theta_i+\kappa_i) < \delta_i$ $(i=1,2)$, *then there exists a unique point* $(x^*, y^*) \in C_1 \times C_2$, *which solves the SMGVIP* (8).

(*c*) *In addition, if* $\max\{\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1), \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\} \le k < 1$, *then the sequence* $\{(x_n, y_n)\}$ *generated by iterative process* (13) *converges strongly to the point* $(x^*, y^*)$.

*Proof.* Parts (*a*) and (*b*) follows from Theorem 3.1..

(*c*) Suppose that $\max\{\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1), \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\} \le k < 1$. From (13) and part (a), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\| \\
&= \|S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] - x^*\| \\
&= \|S_2[(1-\alpha_n)y_n + \alpha_n S_1(x_n)] - S_2(y^*)\| \\
&\le \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\|(1-\alpha_n)y_n + \alpha_n S_1(x_n) - y^*\| \qquad (24)\\
&\le \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)(1-\alpha_n)\|y_n-y^*\| + \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\alpha_n\|S_1(x_n)-S_1(x^*)\| \\
&\le \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)(1-\alpha_n)\|y_n-y^*\| + \frac{\tau_1\tau_2}{\delta_1\delta_2}(\theta_1+\kappa_1)(\theta_2+\kappa_2)\alpha_n\|x_n-x^*\|.
\end{aligned}$$

Again from (13) and part $(b)$, we get

$$
\begin{aligned}
&\|y_{n+1} - y^*\| \\
&= \|S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] - y^*\| \\
&= \|S_1[(1-\alpha_n)x_n + \alpha_n S_2(y_n)] - S_1(x^*)\| \\
&\leq \frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\|(1-\alpha_n)x_n + \alpha_n S_2(y_n) - x^*\| \qquad (25)\\
&\leq \frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)(1-\alpha_n)\|x_n - x^*\| + \frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\alpha_n\|S_2 y_n - S_2 y^*\| \\
&\leq \frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)(1-\alpha_n)\|x_n - x^*\| + \frac{\tau_1\tau_2}{\delta_1\delta_2}(\theta_1+\kappa_1)(\theta_2+\kappa_2)\alpha_n\|y_n - y^*\|.
\end{aligned}
$$

Adding (24) and (25), we get

$$
\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\leq \frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\left(1-\alpha_n\left(1-\frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\right)\right)\|x_n - x^*\| \\
&\quad + \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\left(1-\alpha_n\left(1-\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\right)\right)\|y_n - y^*\|.
\end{aligned} \tag{26}
$$

Note that $\max\{\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1), \frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\} \leq k < 1$ and $\alpha_n \in (0,1)$ for all $n \in \mathbb{N}$. Hence

$$
\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\left(1-\alpha_n\left(1-\frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\right)\right) \leq k[1-(1-k)\alpha_n] \leq k.
$$

Similarly, we get

$$
\frac{\tau_2}{\delta_2}(\theta_2+\kappa_2)\left(1-\alpha_n\left(1-\frac{\tau_1}{\delta_1}(\theta_1+\kappa_1)\right)\right) \leq k[1-(1-k)\alpha_n] \leq k.
$$

Then, (26) reduces to

$$
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \leq k[1-(1-k)\alpha_n](\|x_n - x^*\| + \|y_n - y^*\|). \tag{27}
$$

Define the norm $\|\cdot\|_1$ on $H \times H$ by $\|(x,y)\|_1 = \|x\| + \|y\|$ for all $(x,y) \in H \times H$. Note that $(H \times H, \|\cdot\|_1)$ is a Banach space. From (27), we get

$$
\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_1 \leq k[1-(1-k)\alpha_n]\|(x_n, y_n) - (x^*, y^*)\|_1.
$$

Since $k[1-(1-k)\alpha_n] \leq k < 1$, we obtain that $\lim_{n\to\infty} \|(x_n, y_n) - (x^*, y^*)\|_1 = 0$. Hence, we get that

$$
\lim_{n\to\infty} \|x_n - x^*\| = \lim_{n\to\infty} \|y_n - y^*\| = 0.
$$

Therefore, $\{x_n\}$ and $\{y_n\}$ converges to $x^*$ and $y^*$, respectively. □

**Remark 3.1.** For convergence of Mann iteration process defined by (17) to unique solution of the SMGVIP (8), the condition $\sum_{n=0}^{\infty} \alpha_n = \infty$ is required. But by the parallel $S$-iteration process defined by (13) such condition does not required.

**Corollary 3.4.** *Let $C_1$ and $C_2$ be nonempty closed convex subsets of $H$.. Let $T_1 : C_1 \to C_2$ be $r_1$-strongly monotone, $\mu_1$-Lipschitz continuous and let $T_2 : C_2 \to C_1$ be $r_2$-strongly monotone, $\mu_2$-Lipschitz continuous. Let $g_i : H \to H$ be single valued $p_i$-strongly monotone, $\xi_i$-Lipschitz continuous and let $\eta_i : H \times H \to H$ be $\tau_i$-Lipschitz continuous and $\delta_i$-strongly monotone such that $\eta_i(x,y)+\eta_i(y,x) = 0$ for all $x, y \in H$ and for any given $x \in H$, the function $h_i(y,u) = \langle x-u, \eta_i(y,u)\rangle$ is 0-DQCV in $y$ $(i = 1,2)$. Let $\psi_i$ be a lower semicontinuous $\eta_i$-subdifferentiable proper function $(i = 1,2)$. Let $\{(x_n, y_n)\}$ be the sequence generated by Algorithm 2.14 and $(x^*, y^*) \in C_1 \times C_2$ be the solution of (8). Define $S_1 := J_{\rho'}^{\Delta\psi_1}[g_1 - \rho T_1]$ and $S_2 := J_{\sigma'}^{\Delta\psi_2}[g_2 - \sigma T_2]$. Then we have the following:*

*(a) Mapping $S_1$ and $S_2$ are $\frac{\tau_1}{\delta_1}(\theta_1' + \kappa_1')$ and $\frac{\tau_2}{\delta_2}(\theta_2' + \kappa_2')$- Lipschitzian, respectively, where*

$$\theta_i' = \sqrt{1 - 2p_i + \xi_i^2} \text{ and } \kappa_i' = \sqrt{1 - 2\rho r_i + \rho^2\mu_i^2} \quad (i = 1,2).$$

*(b) If $\tau_i(\theta_i' + \kappa_i') < \delta_i$ $(i = 1,2)$, then there exists a unique point $(x^*, y^*) \in C_1 \times C_2$ such that $x^*$ and $y^*$ are altering points of mappings $S_1$ and $S_2$.*

*(c) In addition, if $\max\{\frac{\tau_1}{\delta_1}(\theta_1' + \kappa_1'), \frac{\tau_2}{\delta_2}(\theta_2' + \kappa_2')\} \le k < 1$, then the sequence $\{(x_n, y_n)\}$ generated by iterative process (13) converges strongly to the points $(x^*, y^*)$.*

*Proof.* If $\gamma = 0$, then relaxed $(\gamma, r)$-cocoercive mapping is $r$-strongly monotone mapping. Therefore proof follows from Theorem 3.3. □

Taking $g_1 = g_2 = I$, $\eta_1(u,v) = \eta_2(u,v) = u - v$, $\psi_1(u) = I_{C_1}(u)$ and $\psi_2(u) = I_{C_2}(u)$ in Theorem 3.3, we get the following:

**Corollary 3.5.** [28, Theorem 4.4] *Let $C_i$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T_i : C_i \to H$ be a $\mu_i$-Lipschitzian and $r_i$-strongly monotone operator with $0 < \rho$ and $\sigma < \frac{2r_i}{\mu_i^2}$ for $i = 1,2$. Then, the system of variational inequalities (11) has a unique solution $(x^*, y^*) \in C_1 \times C_2$ and for $\alpha_n = \alpha \in (0,1)$ for all $n \in \mathbb{N}$ and arbitrary $(x_1, y_1) \in C_1 \times C_2$, the sequence $\{(x_n, y_n)\}$ generated by iteration process (16) converges strongly to $(x^*, y^*)$.*

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