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ON p-ADIC EULER L-FUNCTION OF TWO VARIABLES†

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Abstract. We consider a p-adic Euler L-function of two variables which interpolate the generalized Euler polynomials at nonpositive integers. We also show that the reflection formula and the functional equation for these functions.

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1. Introduction

Let p be an odd prime number. Let $\mathbb{Q}_p$ be the topological completion of $\mathbb{Q}$ with respect to the metric topology induced by $|\cdot|_p$. Let $\mathbb{C}_p$ be the field of p-adic completion of algebraic closure of $\mathbb{Q}_p$. Let v_p denote the p-adic exponential valuation on $\mathbb{C}_p$, normalized so that $v_p(p) = 1$.

For a primitive Dirichlet character χ with odd conductor f_χ, the generalized Euler polynomials $E_{n,\chi}(t)$ are defined by the generating function

$$2\sum_{a=1}^{f_\chi} \frac{(-1)^a\chi(a)e^{(a+t)x}}{e^{f_\chi x}+1} = \sum_{n=0}^{\infty} E_{n,\chi}(t)\frac{x^n}{n!} \tag{1}$$

(see [5, 9, 16, 19]). The corresponding generalized Euler numbers can be defined by $E_{n,\chi} = E_{n,\chi}(0)$. With this definition, the generalized Euler polynomials can also be expressed in terms of the expansion

$$E_{n,\chi}(t) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k,\chi}t^k, \tag{2}$$

which may be derived from (1). Let $\mathbb{Q}(\chi)$ denote the field generated over $\mathbb{Q}$ by all the values $\chi(a), a \in \mathbb{Z}$. Then it can be shown that $E_{n,\chi} \in \mathbb{Q}(\chi)$ for each

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$n \geq 0$, and $E_{n,\chi}(t) \in \mathbb{Q}(\chi, t)$. Recently, many authors have studied these and other related subject (see, e.g., [8, 14, 15]).

The p-adic analogue of Dirichlet L-functions were introduced and studied by Kubota and Leopoldt [11]. It becomes quite important in number theory after the works of Iwasawa [4], particularly in the theory of cyclotomic fields [12] and p-adic modular forms [22]. Recently, properties for several variations of Kubota-Leopoldt's p-adic L-functions have been studied by many authors (see [2, 4, 9, 13, 17, 18, 19, 23, 21, 24, 25]). And the p-adic functions which interpolate the Bernoulli and Euler polynomials have also been investigated by Tsumura [23], Kim [5, 6, 7], Cohn [1] and Young [20]. The constructions are based on p-adic gamma transforms, although Tsumura, Kim, Cohn and Young applied this technique to different areas of p-adic complex plane $\mathbb{C}_p$.

The two variable p-adic L-functions have been studied by Fox [2], Simsek [19] and Young [21]. These functions interpolate the generalized Bernoulli polynomials at nonpositive integers. By using these functions, Kummer's congruences for generalized Bernoulli polynomials are established.

In this paper, we construct the p-adic Euler L-functions $L_{p,E}(s,t;\chi)$ which interpolate the generalized Euler polynomials $E_{n,\chi}(t)$ at nonpositive integers, in analogue with Fox's construction of p-adic L-functions of two variable $L_p(s,t;\chi)$ in [2]. The methods follow from Iwasawa' s construction of p-adic L-functions in [4, Chapter 3] not involving the p-adic gamma transforms. We also prove several properties of $L_{p,E}(s,t;\chi)$, such as the reflection formula and the functional equation.

2. Construction of the p-adic Euler L-function $L_{p,E}(s,t;\chi)$

Throughout this paper, let p be an odd rational prime number.

In this section, by applying the method of Fox [2, Theorem 3.13] on the existence of a specific two-variable p-adic L-function, we construct the p-adic function $L_{p,E}(s,t;\chi)$ and we also express them in an explicit form.

Note that there exist $\varphi(p)$ distinct solutions, modulo p, to the equation $x^{\varphi(p)} - 1 = 0$, and each solution must be congruent to one of the values $a \in \mathbb{Z}$, where $1 \leq a \leq p, (a,p) = 1$. Thus, by Hensel's Lemma, given $a \in \mathbb{Z}$ with $(a,p) = 1$, there exists a unique $\omega(a) \in \mathbb{Z}_p$, where $\omega(a)^{\varphi(p)} = 1$, such that $\omega(a) \equiv a \pmod{p\mathbb{Z}_p}$. Letting $\omega(a) = 0$ for $a \in \mathbb{Z}$ such that $(a,p) \neq 1$, it can be seen that ω is actually a Dirichlet character having conductor $f_\omega = p$, called the Teichmüller character. Let $\langle a \rangle = \omega^{-1}(a)a$. Then $\langle a \rangle \equiv 1 \pmod{p\mathbb{Z}_p}$.

If $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbb{Z}, a + pt \equiv a \pmod{p\mathbb{Z}_p[t]}$. Thus, we define $\omega(a+pt) = \omega(a)$ for these values of t. We also define $\langle a+pt \rangle = \omega^{-1}(a)(a+pt)$ for such t. Therefore, $\langle a+pt \rangle = \langle a \rangle + p\omega^{-1}(a)t$, so that $\langle a+pt \rangle \equiv 1 \pmod{p\mathbb{Z}_p[t]}$.

We also define a particular subring of $\mathbb{C}_p$ by

$$D = \left\{ s \in \mathbb{C}_p \,:\, v_p(s) > -1 + \frac{1}{p-1} \right\}.$$

Since $1 \in D$ and any point of a p-adic disc is its center, D is the same as the set $D = \{s \in \mathbb{C}_p : v_p(1-s) > -1 + \frac{1}{p-1}\}$.

To our purpose, we shall need to make a slight extension of the definition of p-adic Euler L-functions. Additional informations about these functions can be found in [10].

Let χ be the Dirichlet character with odd conductor $f = f_\chi$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Q}_p(\chi)$ denote the field generated over $\mathbb{Q}_p$ by $\chi(a), a \in \mathbb{Z}$ (in an algebraic closure of $\mathbb{Q}_p$). $\mathbb{Q}_p(\chi)$ is a locally compact topological field containing $\mathbb{Q}(\chi)$ as a dense subfield. Let $t \in \mathbb{C}_p, |t|_p \leq 1$, and let $\mathbb{Q}_p(\chi, t)$, the field generated over $\mathbb{Q}_p$ by adjoining t and the values $\chi(a), a \in \mathbb{Z}$. For $n \in \mathbb{N}$, we define χ_n to be the primitive character associated with the character $\chi_n : (\mathbb{Z}/\text{l.c.m.}(f,p)\mathbb{Z})^\times \to \mathbb{C}^\times$ defined by $\chi_n(a) = \chi(a)\omega^{-n}(a)$. We define a sequence of elements $\epsilon_{n,\chi}(t), n \geq 0$, in $\mathbb{Q}_p(\chi, t)$ by

$$\epsilon_{n,\chi}(t) = E_{n,\chi_n}(pt) - \chi_n(p)p^n E_{n,\chi_n}(t), \tag{3}$$

where $E_{n,\chi_n}(t)$ is the generalized Euler polynomial and $n \in \mathbb{N}_0$. Note that $\chi_n(a)$ is in $\mathbb{Q}_p(\chi)$ for any $n \in \mathbb{N}_0$ and $a \in \mathbb{Z}$.

In what follows, we construct a p-adic L-function of two variables which interpolates the generalized Euler polynomials at nonpositive integers. First, we need the following two lemmas.

Lemma 2.1 ([10, Proposition 5.4(2)])**.** *If $n \in \mathbb{Z}$ and $n \geq 0$, then there exists a Witt's formula of $E_{n,\chi}(t)$ in the p-adic field $\mathbb{Q}_p(\chi, t)$ such that*

$$E_{n,\chi}(t) = \lim_{N\to\infty} \sum_{a=1}^{fp^N} (-1)^a \chi(a)(a+t)^n,$$

where we set $\chi(a) = 0$ if a is not prime to the conductor f.

Lemma 2.2. *In the p-adic field $\mathbb{Q}_p(\chi, t)$, for $n \in \mathbb{Z}, n \geq 0$, we have*

$$E_{n,\chi}(t) - \chi(p)p^n E_{n,\chi}\left(\frac{t}{p}\right) = \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a)(a+t)^n.$$

Proof. From Lemma 2.1, we see that

$$\begin{aligned}
&E_{n,\chi}(t) - \chi_n(p)p^n E_{n,\chi}\left(\frac{t}{p}\right) \\
&= \lim_{N\to\infty} \left(\sum_{a=1}^{fp^N} (-1)^a \chi(a)(a+t)^n - \sum_{a=1}^{fp^N-1} (-1)^{ap} \chi(ap)(ap+t)^n \right) \\
&= \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a)(a+t)^n,
\end{aligned}$$

since p is an odd prime number. □

Now we consider a p-adic Euler L-function of two variables and we also compute its value at nonpositive integers.

Define

$$L_{p,E}(s,t;\chi)=\lim_{N\to\infty}\sum_{\substack{a=1\\(a,p)=1}}^{fp^N}(-1)^a\chi(a)\langle a+pt\rangle^{1-s}, \tag{4}$$

which is analytic for $s\in D$ and $t\in\mathbb{C}_p$ such that $|t|_p\leq 1$.

Theorem 2.3. *For $n\in\mathbb{N}$ and $t\in\mathbb{C}_p$ such that $|t|_p\leq 1$, we have*

$$L_{p,E}(1-n,t;\chi)=\epsilon_{n,\chi}(t).$$

Proof. Let $n\in\mathbb{N}$ and $t\in\mathbb{C}_p$ such that $|t|_p\leq 1$. From (4), we obtain

$$\begin{aligned}L_{p,E}(1-n,t;\chi)&=\lim_{N\to\infty}\sum_{\substack{a=1\\(a,p)=1}}^{fp^N}(-1)^a\chi(a)\langle a+pt\rangle^{n}\\&=\lim_{N\to\infty}\sum_{\substack{a=1\\(a,p)=1}}^{fp^N}(-1)^a\chi(a)\omega^{-n}(a)(a+pt)^{n}.\end{aligned} \tag{5}$$

Therefore by Lemma 2.2, we have

$$\begin{aligned}&L_{p,E}(1-n,t;\chi)\\&=\lim_{N\to\infty}\left(\sum_{a=1}^{fp^N}(-1)^a\chi_n(a)(a+pt)^n-\sum_{a=1}^{fp^{N-1}}(-1)^{ap}\chi_n(ap)(ap+pt)^n\right)\\&=E_{n,\chi_n}(pt)-\chi_n(p)p^nE_{n,\chi_n}(t)\\&=\epsilon_{n,\chi}(t).\end{aligned} \tag{6}$$

This is the desired result. □

Define the forward difference operator Δ_c by the equation

$$\Delta_c a_n=a_{n+c}-a_n, \tag{7}$$

where a_n denotes a function of n, that is, $a_n=a(n)$. The power Δ_c^k of Δ_c are defined by Δ_c^0 =identity and $\Delta_c^k=\Delta_c\circ\Delta_c^{k-1}$ for positive integers k. The following is a fundamental result for the kth difference of the function a_n, which can be found in [3, p. 196].

Lemma 2.4. *Let Δ be forward difference operator, which assigns to every function $a_n\in A^{\mathbb{R}}$, defined on the real numbers, and with values in a ring A. Then*

$$\Delta_c^k a_n=\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}a_{n+jc}$$

for some nonnegative integer $k\in\mathbb{Z}$.

In particular, Fox [3, Lemma 5] derives the following congruence property of $\langle a+pt\rangle^m$ by elementary means:

Lemma 2.5. *Let $a, m \in \mathbb{Z}$, with $(a,p)=1$ and $m \geq 2$, and let $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. Then*

$$\langle a+pt\rangle^m - 1 \equiv m(\langle a\rangle - 1 + p\omega^{-1}(a)t) \pmod{m(m-1)p^2\mathbb{Z}_p[t]}.$$

Theorem 2.6. *If $c, k, n \geq 1$ and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. Then*

$$c^{-k}p^{-k}\Delta_c^k \epsilon_{n,\chi}(t) \in \mathbb{Z}_p[\chi, t].$$

Proof. Let $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. An application of Theorem 2.3 and Lemma 2.4 to the sequence $\{\Delta_c^k \epsilon_{n,\chi}(t)\}$ yields

$$\Delta_c^k \epsilon_{n,\chi}(t) = \Delta_c^k L_{p,E}(1-n, t, \chi) = \lim_{N\to\infty} \sum_{\substack{a=1 \\ (a,p)=1}}^{fp^N} (-1)^a \chi(a) \langle a+pt\rangle^n,$$

where $c, n, k \geq 1$. Note that

$$\begin{aligned}
\Delta_c^k \langle a+pt\rangle^n &= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \langle a+pt\rangle^{n+jc} \\
&= \langle a+pt\rangle^n \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \langle a+pt\rangle^{jc} \\
&= \langle a+pt\rangle^n (\langle a+pt\rangle^c - 1)^k,
\end{aligned}$$

so that,

$$\begin{aligned}
\Delta_c^k \epsilon_{n,\chi}(t) &= \Delta_c^k L_{p,E}(1-n, t, \chi) \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \lim_{N\to\infty} \sum_{\substack{a=1 \\ (a,p)=1}}^{fp^N} (-1)^a \chi(a) \langle a+pt\rangle^{n+jc} \\
&= \lim_{N\to\infty} \sum_{\substack{a=1 \\ (a,p)=1}}^{fp^N} (-1)^a \chi(a) \langle a+pt\rangle^n (\langle a+pt\rangle^c - 1)^k.
\end{aligned} \tag{8}$$

By Lemma 2.5, we obtain $(\langle a+pt\rangle^c - 1)^k \equiv 0 \pmod{c^k p^k \mathbb{Z}_p[t]}$ and $\Delta_c^k \epsilon_{n,\chi}(t) \equiv 0 \pmod{c^k p^k \mathbb{Z}_p[\chi, t]}$. □

Corollary 2.7. *Let $n' \in \mathbb{Z}$ such that $n' > n \geq 1$ and let $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. Then*

$$\Delta_c^k \epsilon_{n,\chi}(t) \equiv \Delta_c^k \epsilon_{n',\chi}(t) \pmod{(n'-n)c^k p^{k+1}\mathbb{Z}_p[\chi, t]}.$$

Proof. From (8), we have

$$\Delta_c^k \epsilon_{n',\chi}(t) - \Delta_c^k \epsilon_{n,\chi}(t) = \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a) \langle a+pt\rangle^n \times ((\langle a+pt\rangle^c - 1)^k (\langle a+pt\rangle^{n'-n} - 1).$$

By Lemma 2.5, we obtain $(\langle a+pt\rangle^c - 1)^k(\langle a+pt\rangle^{n'-n} - 1) \equiv 0 \pmod{(n'-n)c^k p^{k+1}\mathbb{Z}_p[t]}$, and the result follows. □

We denote two particular subrings of $\mathbb{C}_p$ in the following manner

$$\mathfrak{o} = \{t \in \mathbb{C}_p : |t|_p \leq 1\}, \quad \mathfrak{p} = \{t \in \mathbb{C}_p : |t|_p < 1\}.$$

If $t \in \mathbb{C}_p$ such that $|t|_p \leq |p|_p^s$, then $t \in p^s\mathfrak{o}$ for $s \in \mathbb{Q}$, and we also write this as $t \equiv 0 \pmod{p^s\mathfrak{o}}$ as usual. Let

$$c_{n,\chi}(t) = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \epsilon_{i,\chi}(t)$$

for all $t \in \mathbb{C}_p$ with $|t|_p \leq 1$. We now derive our bound on the magnitude of $c_{n,\chi}(t)$.

Lemma 2.8.

$$|c_{n,\chi}(t)|_p \leq |p^n|_p, \quad n \geq 0.$$

Proof. From the definition of $\epsilon_{i,\chi}$ and (5), we have

$$\begin{aligned} c_{n,\chi}(t) &= \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a) \langle a+pt\rangle^i \\ &= \lim_{N\to\infty} \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a) (\langle a+pt\rangle - 1)^n. \end{aligned}$$

Since $\langle a+pt\rangle \equiv 1 \pmod{p\mathfrak{o}}$ then $(\langle a+pt\rangle - 1)^n \equiv 0 \pmod{p^n\mathfrak{o}}$, we conclude that

$$\begin{aligned} |c_{n,\chi}(t)|_p &= \lim_{N\to\infty} \left| \sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a) (\langle a+pt\rangle - 1)^n \right|_p \\ &= \lim_{N\to\infty} |p^n \theta_n(N,t)|_p \leq |p^n|_p, \end{aligned}$$

because

$$\sum_{\substack{a=1\\(a,p)=1}}^{fp^N} (-1)^a \chi(a) (\langle a+pt\rangle - 1)^n = p^n \theta_n(N,t)$$

for some $\theta_n(N,t)$ with $|\theta_n(N,t)|_p \leq 1$. This is the desired result. □

Now we apply Theorem 1 in [4, p. 22] to the above sequences $\epsilon_{n,\chi}(t)$ and $c_{n,\chi}(t)$, $n \geq 0$, in $\mathbb{Q}_p(\chi, t)$ and for

$$r = |p|_p < |p|_p^{1/(p-1)}.$$

This theorem shows that there exists a unique power series $A_{\chi,t}(x) \in \mathbb{Q}_p(\chi, t)[[x]]$ convergent for $|\xi|_p < |p|_p^{1/(p-1)}|p|_p^{-1} = |p|_p^{-(p-2)/(p-1)}$ which takes the prescribed values at the nonnegative integers, that is,

$$A_{\chi,t}(n) = \epsilon_{n,\chi}(t).$$

Denote by

$$L_{p,E}(s, t; \chi) = A_{\chi,t}(1 - s).$$

We have the following theorem.

Theorem 2.9. *Let χ be a Dirichlet character with odd conductor f. For each $t \in \mathbb{C}_p$, with $|t|_p \leq 1$, there exists a unique p-adic analytic function*

$$L_{p,E}(s, t; \chi) = \sum_{n=0}^{\infty} (-1)^n a_n(t)(s - 1)^n, \quad a_n(t) \in \mathbb{Q}_p(\chi, t)$$

on $\{s \in \mathbb{C}_p : |s - 1|_p < p^{-(p-2)/(p-1)}\}$ such that

$$L_{p,E}(1 - n, t; \chi) = \epsilon_{n,\chi}(t), \quad n \geq 1.$$

3. Properties of $L_{p,E}(s, t; \chi)$

This section will show some properties of the p-adic function $L_p(s, t; \chi)$ constructed above, including the reflection formula (see Theorem 3.3) and the functional equation (see Theorem 3.4). And in what follows, let p be an odd prime and χ a Dirichlet character with odd conductor $f = f_\chi$.

The polynomials $E_{n,\chi}(t)$ has the property that, for all $n \geq 0$,

$$E_{n,\chi}(-t) = (-1)^{n-1}\chi(-1)E_{n,\chi}(t) \tag{9}$$

if $\chi \neq 1$. From (1), if $\chi = 1$, we recover the original Euler polynomials, that is, $E_{n,1}(x) = E_n(x)$.

Lemma 3.1. *For all $n \in \mathbb{Z}, n \geq 0$, we have*

$$E_{n,1}(-t) = (-1)^{n-1}E_{n,1}(t) - (-1)^{n-1}2t^n.$$

Proof. The formula holds for $n = 0$ because $E_{0,1}(t) = 1$. If $n \geq 1$, then since $E_{n,1}(0) = 0$ for even $n \geq 2$, we may write (2) in the form

$$\begin{aligned} E_{n,1}(t) &= \sum_{k=0}^{n} \binom{n}{k} E_{n-k,1}(0)t^k \\ &= \sum_{\substack{k=0 \\ n-k \text{ odd}}}^{n} \binom{n}{k} E_{n-k,1}(0)t^k + E_{0,1}(0)t^n. \end{aligned}$$

Any k such that $n-k$ is odd must have the different parity as n. Hence

$$\begin{aligned}E_{n,1}(-t) &= (-1)^{n-1}\sum_{\substack{k=0\\ n-k \text{ odd}}}^{n}\binom{n}{k}E_{n-k,1}(0)t^k + (-1)^n E_{0,1}(0)t^n\\ &= (-1)^{n-1}E_{n,1}(t) - (-1)^{n-1}2E_{0,1}(0)t^n\\ &= (-1)^{n-1}E_{n,1}(t) - (-1)^{n-1}2t^n,\end{aligned}$$

the lemma then follows. □

Lemma 3.2. *For all $n \in \mathbb{Z}, n \geq 0$, we have*

$$\epsilon_{n,\chi}(-t) = -\chi(-1)\epsilon_{n,\chi}(t).$$

Proof. This is true for $n = 0$ since $\epsilon_{0,\chi}(-t) = -\chi(-1)\epsilon_{0,\chi}(t)$ and $\epsilon_{0,1}(-t) = -\epsilon_{0,1}(t)$. So we assume that $n \geq 1$.

First we consider the cases of $\chi_n = 1$. From Lemma 3.1, we obtain

$$\begin{aligned}\epsilon_{n,1}(-t) &= E_{n,1}(-pt) - p^n E_{n,1}(-t)\\ &= (-1)^{n-1}E_{n,1}(pt) - (-1)^{n-1}2(pt)^n\\ &\quad - p^n((-1)^{n-1}E_{n,1}(t) - (-1)^{n-1}2t^n)\\ &= (-1)^{n-1}(E_{n,1}(pt) - p^n E_{n,1}(t))\\ &= (-1)^{n-1}\epsilon_{n,1}(t).\end{aligned}$$

Therefore the lemma holds for $\chi_n = 1$.

Next we assume that $\chi_n \neq 1$. Then, from (9), we have

$$\begin{aligned}\epsilon_{n,\chi}(-t) &= E_{n,\chi_n}(-pt) - \chi_n(p)p^n E_{n,\chi_n}(-t)\\ &= (-1)^{n-1}\chi_n(-1)E_{n,\chi_n}(pt) - \chi_n(p)p^n(-1)^{n-1}\chi_n(-1)E_{n,\chi_n}(t)\\ &= (-1)^{n-1}\chi_n(-1)(E_{n,\chi_n}(pt) - \chi_n(p)p^n E_{n,\chi_n}(t))\\ &= (-1)^{n-1}\chi_n(-1)\epsilon_{n,\chi}(t)\\ &= -\chi(-1)\epsilon_{n,\chi}(t),\end{aligned}$$

since $\chi_n(-1) = (-1)^n\chi(-1)$. Thus the lemma also holds for $\chi_n \neq 1$. This completes the proof of our assertion. □

Theorem 3.3 (Reflection formula)**.** *Let $s \in D$ and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. Then*

$$L_{p,E}(s,-t;\chi) = -\chi(-1)L_{p,E}(s,t;\chi).$$

Proof. From Lemma 3.2, we have $\epsilon_{n,\chi}(-t) = -\chi(-1)\epsilon_{n,\chi}(t)$, then using Theorem 2.9, we see this implies $L_{p,E}(s,-t;\chi) = -\chi(-1)L_{p,E}(s,t;\chi)$. □

Now, by (1) we obtain

$$(-1)^{m-1}E_{n,\chi}(t+mf) + E_{n,\chi}(t) = 2\sum_{a=1}^{mf}(-1)^a\chi(a)(t+a)^n, \qquad (10)$$

where χ is the Dirichlet character with odd conductor $f = f_\chi$ and $m \geq 1$. For the character χ, let $F_0 = \text{l.c.m}(f, p)$. Then we have $\chi_n \mid F_0$ for each $n \in \mathbb{Z}$. We also denote F a positive multiple of F_0.

Theorem 3.4 (Functional equation). *Let $s \in D$ and $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$. Then we have*

$$L_{p,E}(s, t+F; \chi) + L_{p,E}(s, t; \chi) = 2 \sum_{\substack{a=1 \\ (p,a)=1}}^{pF} (-1)^a \chi(a) \langle a + pt \rangle^{1-s}. \tag{11}$$

Proof. Let $t \in \mathbb{C}_p$ such that $|t|_p \leq 1$, and let $n \geq 1$. Then from Theorem 2.3, we have

$$L_{p,E}(1-n, t+F; \chi) + L_{p,E}(1-n, t; \chi) = \epsilon_{n,\chi}(t+F) + \epsilon_{n,\chi}(t). \tag{12}$$

By using (3) and (10), we have

$$\begin{aligned} \epsilon_{n,\chi}(t+F) + \epsilon_{n,\chi}(t) &= E_{n,\chi_n}(p(t+F)) - \chi_n(p)p^n E_{n,\chi_n}(t+F) \\ &\quad + E_{n,\chi_n}(pt) - \chi_n(p)p^n E_{n,\chi_n}(t) \\ &= E_{n,\chi_n}(p(t+F)) + E_{n,\chi_n}(pt) \\ &\quad - \chi_n(p)p^n (E_{n,\chi_n}(t+F) + E_{n,\chi_n}(t)) \\ &= 2\sum_{a=1}^{pF} (-1)^a \chi_n(a)(pt+a)^n \\ &\quad - \chi_n(p)p^n 2\sum_{a=1}^{F} (-1)^a \chi_n(a)(t+a)^n. \end{aligned} \tag{13}$$

We also note that

$$\begin{aligned} \chi_n(p)p^n 2\sum_{a=1}^{F} (-1)^a \chi_n(a)(t+a)^n &= 2\sum_{a=1}^{F} (-1)^{pa} \chi_n(pa)(pt+pa)^n \\ &= 2\sum_{\substack{a=1 \\ p \mid a}}^{pF} (-1)^a \chi_n(a)(pt+a)^n. \end{aligned} \tag{14}$$

Substitution the above equality into (13), we have

$$\begin{aligned} \epsilon_{n,\chi}(t+F) + \epsilon_{n,\chi}(t) &= 2\sum_{a=1}^{pF} (-1)^a \chi_n(a)(pt+a)^n - 2\sum_{\substack{a=1 \\ p \mid a}}^{pF} (-1)^a \chi_n(a)(pt+a)^n \\ &= 2 \sum_{\substack{a=1 \\ (p,a)=1}}^{pF} (-1)^a \chi_n(a)(pt+a)^n. \end{aligned}$$

From which, by using (12), we conclude that for $n \geq 1$,

$$L_{p,E}(1-n, t+F; \chi) + L_{p,E}(1-n, t; \chi) = 2 \sum_{\substack{a=1 \\ (p,a)=1}}^{pF} (-1)^a \chi_n(a)(pt+a)^n$$
$$= 2 \sum_{\substack{a=1 \\ (p,a)=1}}^{pF} (-1)^a \chi(a) \langle a+pt \rangle^n,$$

since $\chi_n = \chi\omega^{-n}$. Thus Theorem 3.4 holds for all $s = 1-n$, where $n \geq 1$. Since the negative integers have 0 as a limit point, using the uniqueness property for power series in [4, p. 19, Lemma 1], we have shown that Theorem 3.4 holds for all s in any neighborhood about 0 common to the domain of the functions on either side of (11).

It is obvious that the definition domains in the variable s of the functions on the left hand-side of (11) contain D. For $t \in \mathbb{C}_p, |t|_p \leq 1$, the same argument as the proof of Theorem 4.8 of [2] shows that

$$2 \sum_{\substack{a=1 \\ (p,a)=1}}^{pF} (-1)^a \chi(a) \langle a+pt \rangle^{1-s}$$

is convergence as a power series of s, whenever $s \in D$. Therefore, the definition domains of the functions on either side of (11) contain D, and our theorem holds. □

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