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EXISTENCE AND UNIQUENESS RESULTS FOR CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper successfully applies the modified Adomian decomposition method to find the approximate solutions of the Caputo fractional integro-differential equations. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. Also, the behavior of the solution can be formally determined by analytical approximation. Moreover, we proved the existence and uniqueness results and convergence of the solution. Finally, an example is included to demonstrate the validity and applicability of the proposed technique.

1. INTRODUCTION

In this article, we consider Caputo fractional integro-differential equation of the form:

$$ {}^cD^\alpha u(x) = a(x)u(x) + \int_0^x K(x,t)F(u(t))dt + g(x), \tag{1.1} $$

with the initial condition

$$ u(0) = u_0, \tag{1.2} $$

where ${}^cD^\alpha$ is the Caputo's fractional derivative, $0 < \alpha \leq 1$ and $u : J \longrightarrow \mathbb{R}$, where $J = [0, 1]$ is the continuous function which has to be determined, $a, g : J \longrightarrow \mathbb{R}$ and $K : J \times J \longrightarrow \mathbb{R}$ are continuous functions. $F : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz continuous functions. The modified decomposition method was introduced by Wazwaz [16]. In recent years, many authors focus on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, we can remember the following works. Al-Samadi and Gumah [4] applied the homotopy analysis method for fractional SEIR epidemic model, Zurigat et al. [20] applied HAM for system of fractional integro-differential equations. Yang and Hou [17] applied

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the Laplace decomposition method to solve the fractional integro-differential equations, Mittal and Nigam [14] applied the Adomian decomposition method to approximate solutions for fractional integro-differential equations, and Ma and Huang [13] applied hybrid collocation method to study integro-differential equations of fractional order. Moreover, properties of the fractional integro-differential equations have been studied by several authors [4, 18, 20]. The fractional integro-differential equations have attracted much more interest of mathematicians and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory and viscoelasticity [3, 5, 6, 7, 8, 9, 10, 13, 14, 17].

The main objective of the present paper is to study the behavior of the solution that can be formally determined by analytical approximated method as the modified Adomian decomposition method. Moreover, we proved the existence, uniqueness results and convergence of the solution of the Caputo fractional integro-differential equation.

The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related to fractional calculus are recalled. In Section 3, modified Adomian decomposition method is constructed for solving Caputo fractional integro-differential equations. In Section 4, the existence and uniqueness results and convergence of the solution have been proved. In Section 5, the analytical example is presented to illustrate the accuracy of this method. Finally, we will give a report on our paper and a brief conclusion is given in Section 6.

2. Preliminaries

In this section, we present some required notations, definitions and some theorems which are used Throughout this paper [12, 15, 19].

Definition 1. (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f is defined as

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad x > 0, \quad \alpha \in \mathbb{R}^+, \\ J^0 f(x) &= f(x), \end{aligned}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Definition 2. (Caputo fractional derivative). The fractional derivative of $f(x)$ in the Caputo sense is defined by

$$\begin{aligned} {}^cD_x^{\alpha} f(x) &= J^{m-\alpha} D^m f(x) \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} \frac{d^m f(t)}{dt^m} dt, & m-1 < \alpha < m, \\ \frac{d^m f(x)}{dx^m}, & \alpha = m, \quad m \in N, \end{cases} \end{aligned}$$

where the parameter α is the order of the derivative and is allowed to be real or even complex. In this paper, only real and positive α will be considered.

Hence, we have the following properties:

(1) $J^\alpha J^v f = J^{\alpha+v} f, \quad \alpha, v > 0.$
(2) $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha},$
(3) $D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad \alpha > 0, \quad \beta > -1, \quad x > 0.$
(4) $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \quad x > 0, \quad m-1 < \alpha \leq m.$

Definition 3. (Riemann-Liouville fractional derivative). The Riemann-Liouville fractional derivative of order $\alpha > 0$ is normally defined as

$$D^\alpha f(x) = D^m J^{m-\alpha} f(x), \qquad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.1}$$

Theorem 2.1. [19] ***(Banach contraction principle).*** *Let (X, d) be a complete metric space, then each contraction mapping $T : X \longrightarrow X$ has a unique fixed point x of T in X i.e. $Tx = x$.*

Theorem 2.2. [11] ***(Schauder's fixed point theorem).*** *Let X be a Banach space and let A a convex, closed subset of X. If $T : A \longrightarrow A$ be the map such that the set $\{Tu : u \in A\}$ is relatively compact in X (or T is continuous and completely continouous). Then T has at least one fixed point $u^* \in A : Tu^* = u^*$.*

3. Modified Adomian Decomposition Method

Consider the equation (1.1) with the initial condition (1.2) where ${}^cD^\alpha$ is the operator defined as (2.1). Operating with J^α on both sides of the equation (1.1) we get

$$u(x) = u_0 + J^\alpha \left(a(x)u(x) + \int_0^x K(x,t)F(u(t))dt + g(x) \right)$$

Adomian's method defines the solution $u(x)$ by the series

$$u = \sum_{n=0}^{\infty} u_n, \tag{3.1}$$

and the nonlinear function F is decomposed as

$$F = \sum_{n=0}^{\infty} A_n, \tag{3.2}$$

where A_n are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\Big[\frac{d^n}{d\phi^n}(F\sum_{i=0}^{n} \phi^i u_i)\Big]_{\phi=0},$$

The Adomian polynomials were introduced in [1, 2, 5] as:

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F^{'}(u_0), \\ A_2 &= u_2 F^{'}(u_0) + \frac{1}{2} u_1^2 F^{''}(u_0), \\ A_3 &= u_3 F^{'}(u_0) + u_1 u_2 F^{''}(u_0) + \frac{1}{3} u_1^3 F^{'''}(u_0), \\ &\vdots \end{aligned}$$

The components $u_0, u_1, u_2, ...$ are determined recursively by

$$\begin{aligned} u_0(x) &= u_0 + J^{\alpha} g(x), \\ u_{k+1}(x) &= J^{\alpha}\left(a(x) u_k(x)\right) + J^{\alpha}\left(\int_0^x K(x,t) A_k dt\right). \end{aligned}$$

Having defined the components $u_0, u_1, u_2, \cdots$, the solution u in a series form defined by (3.1) follows immediately. It is important to note that the decomposition method suggests that the 0^{th} component u_0 be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $u_1, u_2, \cdots$, are derived recurrently.

The modified decomposition method was introduced by Wazwaz [16]. This method is based on the assumption that the function $J^{\alpha} g(x) = R(x)$ can be divided into two parts, namely $R_1(x)$ and $R_2(x)$. Under this assumption we set

$$R(x) = R_1(x) + R_2(x).$$

We apply this decomposition when the function $R(x)$ consists of several parts and can be decomposed into two different parts [1, 2, 9, 16]. In this case, $R(x)$ is usually a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part R_1 is important. For the method to be more efficient, we select R_1 as one term of $R(x)$ or at least a number of terms if possible and R_2 consists of the remaining terms of $R(x)$. In comparison with the standard decomposition method, the MADM minimizes the size of calculations and the cost of com putational operations in the algorithm. Both standard and modified decomposition methods are reliable for solving linear or nonlinear problems such as Volterra-Fredholm integro-differential equations, but in order to decrease the complexity of the algorithm and simplify the calculations we prefer to use the MADM. The MADM will accelerate the rapid convergence of the series solution in comparison with the standard Adomian decomposition method. The modified technique may give the exact solution for equations without the necessity to find the Adomian polynomials. We refer the reader to [16] for more details about the MADM. Accordingly, a slight variation was proposed only on the components u_0 and u_1. The suggestion was that only the part R_1 is assigned to the component u_0, whereas the remaining part R_2 is combined with the other terms to define u_1. Consequently, the following modified recursive relation was developed:

$$\begin{aligned} u_0(x) &= u_0 + R_1(x), \\ u_1(x) &= R_2(x) + J^{\alpha}\left(a(x)u_0(x)\right) + J^{\alpha}\left(\int_0^x K(x,t)A_0 dt\right). \\ &\vdots \\ u_{k+1}(x) &= J^{\alpha}\left(a(x)u_k(x)\right) + J^{\alpha}\left(\int_0^x K(x,t)A_k dt\right), k \geq 1. \end{aligned}$$

4. Main Results

In this section, we shall give an existence and uniqueness results of Eq. (1.1), with the initial condition (1.2) and prove it. Before starting and proving the main results, we introduce the following hypotheses:

(H1): The two functions $a, g : J \to \mathbb{R}$ are continuous.

(H2): There exists a function $K^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(x,t) \in \mathbb{R} \times \mathbb{R} : 0 \leq t \leq x \leq 1\}$ such that

$$K^* = \sup_{x,t\in[0,1]} \int_0^x |K(x,t)|\, dt < \infty,$$

(H3): There exists a constant $L_F > 0$ such that, for any $u_1, u_2 \in C(J, \mathbb{R})$

$$|F(u_1(x)) - F(u_2(x))| \leq L_F\, |u_1 - u_2|$$

Lemma 1. If $u_0(x) \in C(J, \mathbb{R})$, then $u(x) \in C(J, \mathbb{R}^+)$ is a solution of the problem $(1.1)-(1.2)$ iff u satisfying

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} \\ &\quad \times \left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right) ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s)ds, \end{aligned}$$

for $x \in J$.

Now, we will study the existence result by means of Schauder's fixed point theorem.

Theorem 4.1. *Assume that F is continuous functions and (H1), (H2) hold, If*

$$\frac{\|a\|_{\infty}}{\Gamma(\alpha+1)} < 1. \tag{4.1}$$

Then there exists at least a solution $u(x) \in C(J, \mathbb{R})$ to problem $(1.1) - (1.2)$.

Proof. Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$\begin{aligned}(Tu)(x) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s K(s,\tau)F(u(\tau))d\tau\right)ds,\end{aligned}$$

Firstly, we prove that the operator T is completely continuous.

(**1**) We show that T is continuous.

Let u_n be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $u_n, u \in C(J,\mathbb{R})$ and for any $x \in J$ we have

$$\begin{aligned}& |(Tu_n)(x) - (Tu)(x)| \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)|\,|u_n(s)-u(s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u_n(\tau)) - F(u(\tau))|\,d\tau\right)ds \\ \leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sup_{s\in J}|a(s)|\sup_{s\in J}|u_n(s)-u(s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\sup_{s,\tau\in J}\int_0^\tau |K(s,\tau)|\sup_{\tau\in J}|F(u_n(\tau)) - F(u(\tau))|\,d\tau\right)ds \\ \leq\ & \|a\|_\infty \|u_n(.) - u(.)\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds \\ & + K^* \|F(u_n(.)) - F(u(.))\|_\infty \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds.\end{aligned}$$

Since $\int_0^x (x-s)^{\alpha-1}ds$ is bounded, $\lim_{n\to\infty} u_n(x) = u(x)$ and F is continuous functions, we conclude that $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$, thus, T is continuous on $C(J,\mathbb{R})$.

(**2**) We verify that T maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, just we show that for any $\lambda > 0$ there exists a positive constant ℓ such that for each $u \in \mathbb{B}_\lambda = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \leq \lambda\}$, one has $\|Tu\|_\infty \leq \ell$. Let $\mu = \sup_{(u)\in J\times[0,\lambda]} F(u(x)) + 1$.

and for any $u \in \mathbb{B}_r$ and for each $x \in J$, we have

$$\begin{aligned}& |(Tu)(x)| \\ =\ & |u_0| + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)|\,|u(s)|\,ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|g(s)|\,ds \\ & + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds\end{aligned}$$

$$
\begin{aligned}
&\leq |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K^*\mu x^\alpha}{\Gamma(\alpha+1)} \\
&\leq \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K^*\mu}{\Gamma(\alpha+1)} \right) \\
&: \quad = \ell.
\end{aligned}
$$

Therefore, $\|Tu\| \leq \ell$ for every $u \in \mathbb{B}_r$, which implies that $T\mathbb{B}_r \subset \mathbb{B}_\ell$.
(3) We examine that T maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.
Let $\mathbb{B}_\lambda$ is defined as in (**2**) and for each $u \in \mathbb{B}_\lambda$, $x_1, x_2 \in [0, 1]$, with $x_1 < x_2$ we have

$$
\begin{aligned}
& |(Tu)(x_2) - (Tu)(x_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \right| \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \Big| \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \\
& - \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \Big| \\
=\ & \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds \\
& + \int_0^{x_1} (x_2 - s)^{\alpha-1} a(s)u(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} a(s)u(s)ds \Big| \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds \\
& + \int_0^{x_1} (x_2 - s)^{\alpha-1} g(s)ds - \int_0^{x_1} (x_1 - s)^{\alpha-1} g(s)ds \Big| \\
& + \frac{1}{\Gamma(\alpha)} \Big| \int_0^{x_2} (x_2 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \\
& - \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \\
& + \int_0^{x_1} (x_2 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \\
& - \int_0^{x_1} (x_1 - s)^{\alpha-1} \left(\int_0^s K(s,\tau)F(u(\tau))d\tau \right) ds \Big|.
\end{aligned}
$$

Consequently,

$$\begin{aligned}
& |(Tu)(x_2)-(Tu)(x_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\,|a(s)|\,|u(s)|\,ds \\
& +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\,|a(s)|\,|u(s)|\,ds\Big) \\
& +\frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\,|g(s)|\,ds \\
& +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\,|g(s)|\,ds\Big) \\
& +\frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\Big(\int_0^{s}|K(s,\tau)|\,|F(u(\tau))|\,d\tau\Big)ds \\
& +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\Big(\int_0^{s}|K(s,\tau)|\,|F(u(\tau))|\,d\tau\Big)ds\Big) \\
=\ & I_1+I_2+I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)}\Big(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\,|a(s)|\,|u(s)|\,ds \\
&\quad +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\,|a(s)|\,|u(s)|\,ds\Big) \\
&\leq \frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\,\|a\|_{\infty}\,\lambda+\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\,\|a\|_{\infty}\,\lambda+\frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\,\|a\|_{\infty}\,\lambda-\frac{x_2^{\alpha}}{\Gamma(\alpha+1)}\,\|a\|_{\infty}\,\lambda \\
&= \frac{\|a\|_{\infty}\,\lambda}{\Gamma(\alpha+1)}\left(2\,(x_2-x_1)^{\alpha}+(x_1^{\alpha}-x_2^{\alpha})\right) \\
&\leq \frac{\|a\|_{\infty}\,\lambda}{\Gamma(\alpha+1)}2\,(x_2-x_1)^{\alpha}, \qquad (4.2)
\end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\,|g(s)|\,ds+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\,|g(s)|\,ds\right) \\
&\leq \frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\,\|g\|_{\infty}+\frac{x_1^{\alpha}}{\Gamma(\alpha+1)}\,\|g\|_{\infty}+\frac{(x_2-x_1)^{\alpha}}{\Gamma(\alpha+1)}\,\|g\|_{\infty}-\frac{x_2^{\alpha}}{\Gamma(\alpha+1)}\,\|g\|_{\infty} \\
&= \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(2\,(x_2-x_1)^{\alpha}+(x_1^{\alpha}-x_2^{\alpha})\right) \\
&\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}2\,(x_2-x_1)^{\alpha}, \qquad (4.3)
\end{aligned}$$

and

$$\begin{aligned} I_3 &= \frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds \\ &\quad+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds \\ &\le \frac{(K^*\mu)}{\Gamma(\alpha+1)}\Big(2\,(x_2-x_1)^{\alpha}+(x_1^{\alpha}-x_2^{\alpha})\Big) \\ &\le \frac{(K^*\mu)}{\Gamma(\alpha+1)}2\,(x_2-x_1)^{\alpha}, \end{aligned} \tag{4.4}$$

we can conclude the right-hand side of (4.2), (4.3) and (4.4) is independently of $u \in \mathbb{B}_\lambda$ and tends to zero as $x_2 - x_1 \to 0$. This leads to $|(Tu)(x_2) - (Tu)(x_1)| \to 0$ as $x_2 \to x_1$. i.e. the set $\{T\mathbb{B}_\lambda\}$ is equicontinuous. From I_1 to I_3 together with the Arzela–Ascoli theorem, we can conclude that $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is completely continuous.

Finally, we need to investigate that there exists a closed convex bounded subset $\mathbb{B}_{\widetilde{\lambda}} = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \le \widetilde{\lambda}\}$ such that $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$. For each positive integer $\widetilde{\lambda}$, then $\mathbb{B}_{\widetilde{\lambda}}$ is clearly closed, convex and bounded of $C(J,\mathbb{R})$. We claim that there exists a positive integer ϵ such that $T\mathbb{B}_\epsilon \subseteq \mathbb{B}_\epsilon$. If this property is false, then for every positive integer $\widetilde{\lambda}$, there exists $u_{\widetilde{\lambda}} \in \mathbb{B}_{\widetilde{\lambda}}$ such that $(Tu_{\widetilde{\lambda}}) \notin T\mathbb{B}_{\widetilde{\lambda}}$, i.e. $\left\|Tu_{\widetilde{\lambda}}(t)\right\|_\infty > \widetilde{\lambda}$ for some $x_{\widetilde{\lambda}} \in J$ where $x_{\widetilde{\lambda}}$ denotes x depending on $\widetilde{\lambda}$. But by using the previous hypotheses we have

$$\begin{aligned} &\le |u_0| + \|u\|_\infty \|a\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{K^*\mu x^\alpha}{\Gamma(\alpha+1)} \\ &\le \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + K^*\mu}{\Gamma(\alpha+1)}\right) \\ \widetilde{\lambda} &< \left\|Tu_{\widetilde{\lambda}}\right\|_\infty \\ &= \sup_{x\in J}\left|(Tu_{\widetilde{\lambda}})(x)\right| \\ &\le \sup_{x\in J}\Big\{ |u_0| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)\,|u(s)|\,ds\right| + \left|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds\right| \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s |K(s,\tau)|\,|F(u(\tau))|\,d\tau\right)ds\Big\}ds \\ &\le \sup_{x\in J}\Big\{ |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{K^*\mu x^\alpha}{\Gamma(\alpha+1)}\Big\} \\ &\le \sup_{x\in J}\Big(|u_0| + \frac{\|a\|_\infty \widetilde{\lambda} + \|g\|_\infty + K^*\mu}{\Gamma(\alpha+1)}\Big). \end{aligned}$$

Dividing both sides by $\widetilde{\lambda}$ and taking the limit as $\widetilde{\lambda} \to +\infty$, we obtain

$$1 < \frac{\|a\|_\infty}{\Gamma(\alpha+1)},$$

which contradicts our assumption (4.1). Hence, for some positive integer $\widetilde{\lambda}$, we must have $T\mathbb{B}_{\widetilde{\lambda}} \subseteq \mathbb{B}_{\widetilde{\lambda}}$.

An application of Schauder's fixed point Theorem shows that there exists at least a fixed point u of T in $C(J, \mathbb{R})$. Then u is the solution to $(1.1)-(1.2)$ on J, and the proof is completed. □

Now, our result is based on the Banach contraction principle.

Theorem 4.2. *Assume that (H1)–(H3) hold. If*

$$\left(\frac{\|a\|_\infty + K^* L_F}{\Gamma(\alpha+1)} \right) < 1. \tag{4.5}$$

Then there exists a unique solution $u(x) \in C(J)$ *to* $(1.1) - (1.2)$.

Proof. By Lemma 1, we know that a function u is a solution to $(1.1) - (1.2)$ iff u satisfies

$$\begin{aligned} u(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s) u(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s,\tau) F(u(\tau)) d\tau \right) ds. \end{aligned}$$

Let the operator $T : C(J, \mathbb{R}) \to C(J, \mathbb{R})$, be defined as in Theorem 4.1. We can see that, if $u \in C(J, \mathbb{R})$ is a fixed point of T, then u is a solution of (1.1) – (1.2).

Now we prove T has a fixed point u in $C(J, \mathbb{R})$. For that, let $u_1, u_2 \in C(J, \mathbb{R})$ and for any $x \in [0, 1]$ such that

$$\begin{aligned} u_1(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s) u_1(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s,\tau) F(u_1(\tau)) d\tau \right) ds, \end{aligned}$$

and,

$$\begin{aligned} u_2(x) &= u_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} a(s) u_2(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s K(s,\tau) F(u_2(\tau)) d\tau \right) ds. \end{aligned}$$

Consequently, we get

$$\begin{aligned}
& |(Tu_1)(x) - (Tu_2)(x)| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)|\, |u_1(s) - u_2(s)|\, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |K(s,\tau)|\, |F(u_1(\tau)) - F(u_2(\tau))|\, d\tau \right) ds \\
\leq\ & \frac{\|a\|_\infty}{\Gamma(\alpha+1)} |u_1(x) - u_2(x)| + \frac{K^* L_F}{\Gamma(\alpha+1)} |u_1(x) - u_2(x)| \\
=\ & \left(\frac{\|a\|_\infty + K^* L_F}{\Gamma(\alpha+1)} \right) |u_1(x) - u_2(x)| .
\end{aligned}$$

From the inequality (4.5) we have

$$\|Tu_1 - Tu_2\|_\infty \leq \|u_1 - u_2\|_\infty .$$

This means that T is contraction map. By the Banach contraction principle, we can conclude that T has a unique fixed point u in $C(J, \mathbb{R})$. □

Theorem 4.3. *Suppose that (H1)–(H3), and* (4.5) *hold, if the series solution* $u(x) = \sum_{i=0}^{\infty} u_i(x)$ *and* $\|u_1\|_\infty < \infty$ *obtained by the* m*-order deformation is convergent, then it converges to the exact solution of the fractional integro-differential equation* $(1.1) - (1.2)$.

Proof. Denote as $(C[0,1], \|.\|)$ the Banach space of all continuous functions on J, with $|u_1(x)| \leq \infty$ for all x in J.

Frist we define the sequence of partial sums s_n, let s_n and s_m be arbitrary partial sums with $n \geq m$. We are going to prove that $s_n = \sum_{i=0}^{n} u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}
\Big\| s_n - s_m \Big\|_\infty &= \max_{\forall x \in J} \Big| s_n - s_m \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=0}^{n} u_i(x) - \sum_{i=0}^{m} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} u_i(x) \Big| \\
&= \max_{\forall x \in J} \Big| \sum_{i=m+1}^{n} \Big(\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [a(t) u_i(t) + \int_0^t K(t,s) A_i(s) ds] dt \Big) \Big| \\
&= \max_{\forall x \in J} \Big| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \Big[a(t) \sum_{i=m}^{n-1} u_i(t) + \int_0^t K(t,s) \sum_{i=m}^{n-1} A_i(s) ds \Big] dt \Big| .
\end{aligned}$$

From (3.1) and (3.2), we have

$$\sum_{i=m}^{n-1} A_i = F(s_{n-1}) - F(s_{m-1}),$$

$$\sum_{i=m}^{n-1} u_i = u(s_{n-1}) - u(s_{m-1}).$$

So,

$$\begin{aligned}
\|s_n - s_m\|_\infty &= \max_{\forall x\in J}\Big(\Big|\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\Big[a(t)(u(s_{n-1}) - u(s_{m-1}))\\
&\quad + \int_0^t K(t,s)(F(s_{n-1}) - F(s_{m-1}))ds\Big]dt\Big|\Big),\\
&\le \max_{\forall x\in J}\Big(\frac{1}{\Gamma(\alpha)}\int_0^x |x-t|^{\alpha-1}\Big[|a(t)||u(s_{n-1}) - u(s_{m-1})|\\
&\quad + \int_0^t |K(t,s)||(F(s_{n-1}) - F(s_{m-1}))|ds\Big]dt\Big),\\
&\le \frac{1}{\Gamma(\alpha+1)}\Big[\|a(t)\|_\infty\|s_{n-1} - s_{m-1}\|_\infty + K^* L_F\|s_{n-1} - s_{m-1}\|_\infty\Big],\\
&= \Big(\frac{\|a\|_\infty + K^* L_F}{\Gamma(\alpha+1)}\Big)\|s_{n-1} - s_{m-1}\|_\infty,\\
&= \delta\|s_{n-1} - s_{m-1}\|_\infty.
\end{aligned}$$

where

$$\delta = \left(\frac{\|a\|_\infty + K^* L_F}{\Gamma(\alpha+1)}\right)$$

Let $n = m+1$, then

$$\|s_n - s_m\|_\infty \le \delta\|s_m - s_{m-1}\|_\infty \le \delta^2\|s_{m-1} - s_{m-2}\|_\infty \le \cdots \le \delta^m\|s_1 - s_0\|_\infty,$$

so,

$$\begin{aligned}
\|s_n - s_m\|_\infty &\le \|s_{m+1} - s_m\|_\infty + \|s_{m+2} - s_{m+1}\|_\infty + \cdots + \|s_n - s_{n-1}\|_\infty\\
&\le [\delta^m + \delta^{m+1} + \cdots + \delta^{n-1}]\|s_1 - s_0\|_\infty\\
&\le \delta^m[1 + \delta + \delta^2 + \cdots + \delta^{n-m-1}]\|s_1 - s_0\|_\infty\\
&\le \delta^m(\frac{1-\delta^{n-m}}{1-\delta})\|u_1\|_\infty.
\end{aligned}$$

Since $0 < \delta < 1$, we have $(1 - \delta^{n-m}) < 1$, then

$$\|s_n - s_m\|_\infty \le \frac{\delta^m}{1-\delta}\|u_1\|_\infty.$$

But $|u_1(x)| < \infty$, so, as $m \longrightarrow \infty$, then $\|s_n - s_m\|_\infty \longrightarrow 0$.

We conclude that s_n is a Cauchy sequence in $C[0,1]$, therefore $u = \lim_{n\to\infty} u_n$. Then, the series is convergence and the proof is complete. □

5. Illustrative Example

In this section, we present the analytical technique based on MADM to solve Caputo fractional integro-differential equations.

Example 5.1 Consider the following Caputo fractional integro-differential equation:

$$ {}^cD^{0.5}[u(x)] = -\frac{x^2e^x}{3}u(x) + \int_0^x e^x s u(s)ds + \int_0^1 x^2u(s)ds + \frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}, \qquad (5.1)$$

with the initial condition

$$u(0) = 0,$$

and the the exact solution is $u(x) = x$. Applying the operator $J^{0.5}$ to both sides of Eq. (5.1)

$$u(x) = 0 + J^{0.5}\Big[-\frac{x^2e^x}{3}u(x)\Big] + J^{0.5}\Big[\int_0^x e^x s u(s)ds + \int_0^1 x^2u(s)ds\Big] + J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big].$$

Then,

$$u(x) = J^{0.5}\Big[-\frac{x^2e^x}{3}u(x)\Big] + J^{0.5}\Big[\int_0^x e^x s u(s)ds + \int_0^1 x^2u(s)ds\Big] + J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big]. (5.2)$$

From Eq. (5.1) we see $g(x) = \frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}$, suppose $R(x) = J^{0.5}g(x)$, from Eq. (5.2) we have

$$\begin{aligned}
R(x) &= J^{0.5}g(x) = J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big],\\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x (x-s)^{-0.5}s^{0.5}ds - \frac{1}{2\Gamma(0.5)}\int_0^x (x-s)^{-0.5}s^2ds,\\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x (1-\frac{s}{x})^{-0.5}x^{-0.5}s^{0.5}ds - \frac{1}{2\Gamma(0.5)}\int_0^x (1-\frac{s}{x})^{-0.5}x^{-0.5}s^2ds,\\
&= \frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^1 (1-\tau)^{-0.5}\tau^{0.5}xd\tau - \frac{1}{2\Gamma(0.5)}\int_0^1 (1-\tau)^{-0.5}x^{2.5}\tau^2d\tau,\\
&= \frac{x}{\Gamma(1.5)\Gamma(0.5)}\beta(0.5,1.5) - \frac{x^{2.5}}{2\Gamma(0.5)}\beta(0.5,3),\\
&= x - \frac{x^{2.5}}{\Gamma(3.5)}.
\end{aligned}$$

Now, we apply the modified Adomian decomposition method,

$$R(x) = R_1(x) + R_2(x) = J^{0.5}\Big[\frac{x^{0.5}}{\Gamma(1.5)} - \frac{x^2}{2}\Big] = x - \frac{x^{2.5}}{\Gamma(3.5)}.$$

The modified recursive relation

$$\begin{aligned}
u_0(x) &= R_1(x) = x. \\
u_1(x) &= R_2(x) + J^{0.5}\left(-\frac{x^2 e^x}{3} u_0(x)\right) + J^{0.5}\left(\int_0^x e^x s A_0(s) ds + \int_0^1 x^2 B_0(s) ds\right), \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2 e^x}{3} x\right) + J^{0.5}\left(\int_0^x e^x s^2 ds + \int_0^1 x^2 s ds\right), \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3 e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3} + \frac{x^2}{2}\right), \\
&= -\frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3 e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3}\right) + J^{0.5}\left(\frac{x^2}{2}\right), \\
&= 0. \\
u_2(x) &= 0. \\
&\vdots \\
u_n(x) &= 0.
\end{aligned}$$

Therefore, the obtained solution is

$$u(x) = \sum_{i=0}^{\infty} u_i(x) = x.$$

6. CONCLUSIONS

The modified Adomian decomposition method is successfully applied to find the approximate solution of Caputo fractional integro-differential equation. The reliability of the method and reduction in the size of the computational work give this method a wider applicability. The method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of linear and nonlinear fractional integro-differential equations. Moreover, we proved the existence and uniqueness of the solution. The convergence theorem and the illustrative example establish the precision and efficiency of the proposed technique.

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