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HELICOIDAL KILLING FIELDS, HELICOIDS AND RULED MINIMAL SURFACES IN HOMOGENEOUS THREE-MANIFOLDS

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Abstract. We provide definitions for the helicoidal Killing field and the helicoid in arbitrary three-manifolds, and investigate helicoids and ruled minimal surfaces in homogeneous three-manifolds, mainly in $\mathrm{SL}_2\mathbb{R}$ and Sol(3). In so doing we finish our classification of ruled minimal surfaces in homogeneous three-manifolds with the isometry group of dimension 4.

1. Introduction

This work is a continuation of our efforts in classifying the ruled minimal surfaces of homogeneous three-manifolds [5, 11, 12]. But we slightly change our emphasis from ruled minimal surfaces to helicoids. A big difference between the two surfaces is that, in our opinion, there does not seem to be a universal definition for helicoids in arbitrary manifolds, unlike the one for ruled minimal surfaces. The most commonly accepted definition for helicoids relies on screw motions, which is valid in many spaces but still cannot be applied in arbitrary three-manifolds. While investigating ruled minimal surfaces in homogeneous three-manifolds we came up with a definition of helicoids in arbitrary three-manifolds, which we believe is quite general. See Definition 3.4.

Ruled minimal surfaces and helicoids are closely related. It is a classical fact that any nontrivial ruled minimal surface in $\mathbb{E}^3$ is a part of the helicoid, which is obtained by moving a geodesic with a screw motion whose Killing field is orthogonal to that geodesic. The same fact holds for other 3-dimensional space forms [8]. We have shown that the same is true for $\mathbb{S}^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$, the 3-dimensional Riemannian Heisenberg group Nil_3 and the 3-dimensional

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Berger sphere [5,11,12]. Plehnert provides examples of ruled minimal surfaces from a perspective different from ours [10]. Bekkar, Bouziani, Boukhatem and Inoguchi deal in [1] with helicoids of $E(\kappa,\tau)$ but their definition of helicoids is different from ours. See §3.

Our results are spread in §3, §4, and §5. Roughly speaking, in §3, we first define helicoidal Killing fields and helicoids and show that in a generic sense helicoidal Killing fields induce ruled minimal surfaces (cf. Proposition 3.2), that there are abundant examples of helicoidal Killing fields (cf. Theorem 3.3), and that a helicoid is a ruled a minimal surface (cf. Theorem 3.5). In §4, we classify all ruled minimal surfaces in $\mathrm{SL}_2\mathbb{R}$ (cf. Theorem 4.7), and show that all of them are helicoids in the sense of §3 (cf. Theorem 4.8). In §5, we show that in Sol(3) there are Killing fields which are not helicoidal (cf. Theorem 5.1), and that there is a ruled minimal surface which is not a helicoid in the sense of §3 (cf. Theorem 5.1).

2. Preliminaries

2.1. Definitions

For each pair of real numbers κ and τ, $E(\kappa,\tau)$ is $\mathbb{R}^2\times\mathbb{R}$ if $\kappa\geq 0$, or $D(\frac{2}{\sqrt{-\kappa}})\times\mathbb{R}$ if $\kappa<0$, equipped with the metric

$$(1)\quad ds^2=\lambda^2(dx_1^2+dx_2^2)+[\tau\lambda(x_2dx_1-x_1dx_2)+dx_3]^2,\quad \lambda:=\frac{1}{1+\frac{\kappa}{4}(x_1^2+x_2^2)}.$$

The following are an orthonormal frame for $TE(\kappa,\tau)$:

(2)

$$\begin{aligned}
e_1&=\tfrac{1}{\lambda}\left(\cos(\sigma x_3)\tfrac{\partial}{\partial x_1}+\sin(\sigma x_3)\tfrac{\partial}{\partial x_2}\right)+\tau\left(x_1\sin(\sigma x_3)-x_2\cos(\sigma x_3)\right)\tfrac{\partial}{\partial x_3},\\
e_2&=\tfrac{1}{\lambda}\left(-\sin(\sigma x_3)\tfrac{\partial}{\partial x_1}+\cos(\sigma x_3)\tfrac{\partial}{\partial x_2}\right)+\tau\left(x_1\cos(\sigma x_3)+x_2\sin(\sigma x_3)\right)\tfrac{\partial}{\partial x_3},\\
e_3&=\tfrac{\partial}{\partial x_3},
\end{aligned}$$

where $\sigma:=\frac{\kappa}{2\tau}$. We also consider the following subsets of the set of all 2×2 complex matrices $M(2,\mathbb{C})$

$$\begin{aligned}
\mathrm{SL}_2\mathbb{R}&:=\{M\in M(2,\mathbb{C}):\bar{M}=M,\det M=1\},\\
\mathrm{SU}(1,1)&:=\{M\in M(2,\mathbb{C}):M\begin{pmatrix}1&0\\0&-1\end{pmatrix}M^*=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\det M=1\},
\end{aligned}$$

where M^* denotes the conjugate transpose of M. They can be rewritten as

$$\begin{aligned}
\mathrm{SL}_2\mathbb{R}&:=\left\{\begin{pmatrix}x+u&y-v\\-y-v&x-u\end{pmatrix}:x,y,u,v\in\mathbb{R},x^2+y^2-u^2-v^2=1\right\},\\
\mathrm{SU}(1,1)&:=\left\{\begin{pmatrix}z&w\\\bar{w}&\bar{z}\end{pmatrix}:z,w\in\mathbb{C},z\bar{z}-w\bar{w}=1\right\}.
\end{aligned}$$

2.2. Transformations between $E(\kappa,\tau)$, $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{SU}(1,1)$

From now on, we restrict κ to be negative, and define

$$\mu := \sqrt{-\kappa}.$$

The following map $T : E(\kappa,\tau) \to \mathrm{SL}_2\mathbb{R}$

$$T(x_1,x_2,x_3) := \frac{1}{\sqrt{4+\kappa(x_1^2+x_2^2)}}\left[\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} - \mu\begin{pmatrix}-x_1 & x_2\\ x_2 & x_1\end{pmatrix}\right]\begin{pmatrix}\cos\left(\frac{\kappa x_3}{4\tau}\right) & \sin\left(\frac{\kappa x_3}{4\tau}\right)\\ -\sin\left(\frac{\kappa x_3}{4\tau}\right) & \cos\left(\frac{\kappa x_3}{4\tau}\right)\end{pmatrix}$$

and the map $S : \mathrm{SL}_2\mathbb{R} \to E(\kappa,\tau)$

$$S\begin{pmatrix}x+u & y-v\\ -y-v & x-u\end{pmatrix} := \left(\frac{2}{\mu}\frac{xu-yv}{x^2+y^2}, \frac{2}{\mu}\frac{xv+yu}{x^2+y^2}, \frac{4\tau}{\kappa}\arg(x+iy)\right)$$

are local inverses of each other. We can identify $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{SU}(1,1)$ via

$$\rho : \mathrm{SL}_2\mathbb{R} \to \mathrm{SU}(1,1), \qquad \rho(X) := R^{-1}XR, \qquad R := \frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix},$$

which is a group isomorphism (cf. [4]). It is easy to see that

$$\rho\begin{pmatrix}x+u & y-v\\ -y-v & x-u\end{pmatrix} = \begin{pmatrix}x+iy & u+iv\\ u-iv & x-iy\end{pmatrix}.$$

With this, we identify $E(\kappa,\tau)$ with $\kappa<0$, $\tau\neq 0$ with the universal cover of $\mathrm{SU}(1,1)$. The map

$$\Xi : \mathbb{R}^+\times\mathbb{R}/2\pi\times\mathbb{R}/2\pi \to \mathrm{SU}(1,1),\ \Xi(s,\theta,\phi) := \begin{pmatrix}e^{i\phi}\cosh s & e^{i\theta}\sinh s\\ e^{-i\theta}\sinh s & e^{-i\phi}\cosh s\end{pmatrix} \tag{3}$$

can be conveniently used to describe the geometry of $\mathrm{SU}(1,1)$, where $\mathbb{R}^+$ is the set of all nonnegative real numbers. Combined with

$$f := (\rho\circ S)^{-1} : \mathrm{SU}(1,1) \to E(\kappa,\tau) \cong D(\tfrac{2}{\mu})\times\mathbb{R},$$

it yields

$$f\circ\Xi(s,\theta,\phi) = \left(\frac{2}{\mu}\tanh s\cos(\theta+\phi), \frac{2}{\mu}\tanh s\sin(\theta+\phi), 4\frac{\tau}{\kappa}\phi\right).$$

In this coordinate system of $E(\kappa,\tau)$, the metric (1) takes the form

$$\begin{aligned} ds^2 = &-\tfrac{4}{\kappa}(ds)^2 - \tfrac{2\sinh^2(s)\left(\kappa+4\tau^2+\left(\kappa-4\tau^2\right)\cosh(2s)\right)}{\kappa^2}(d\theta)^2 \\ &- \tfrac{2\left(\kappa-4\tau^2\right)\sinh^2(2s)}{\kappa^2}d\theta d\phi + \tfrac{2\cosh^2(s)\left(\kappa+4\tau^2-\left(\kappa-4\tau^2\right)\cosh(2s)\right)}{\kappa^2}(d\phi)^2. \end{aligned}$$

We also observe that

$$f\circ\Xi(s,\theta,0) = \left(\frac{2}{\mu}\tanh s\cos\theta, \frac{2}{\mu}\tanh s\cos\theta, 0\right), \tag{4}$$

$$f\circ\Xi(s,0,0) = \left(\frac{2}{\mu}\tanh s, 0, 0\right), \tag{5}$$

$$f\circ\Xi(s,\frac{\pi}{2},0) = \left(0, \frac{2}{\mu}\tanh s, 0\right), \tag{6}$$

$$f \circ \Xi(0,\theta,\phi) = \left(0, 0, 4\frac{\tau}{\kappa}\phi\right). \tag{7}$$

From (4), (7) we see that the x_1x_2 plane and the x_3 axis in the $E(\kappa,\tau)$ correspond to $s\theta$-plane and the ϕ axis, respectively.

2.3. One parameter subgroups of isometries of SU(1,1)

Consider (cf. [4])

$$\Lambda_e(s) := \begin{pmatrix} e^{is} & 0 \\ 0 & e^{-is} \end{pmatrix},\ \Lambda_p(s) := \begin{pmatrix} 1+is & -is \\ is & 1-is \end{pmatrix},\ \Lambda_h(s) := \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}.$$

We also introduce

$$\Lambda_{p1}(s) := \Lambda_e(\frac{\pi}{2})\Lambda_p(s)\Lambda_e(\frac{-\pi}{2}) \text{ and } \Lambda_{h1}(s) := \Lambda_e(\frac{\pi}{4})\Lambda_h(s)\Lambda_e(\frac{-\pi}{4}),$$

i.e.,

$$\Lambda_{p1}(s) = \begin{pmatrix} 1+is & is \\ -is & 1-is \end{pmatrix}, \quad \Lambda_{h1}(s) = \begin{pmatrix} \cosh s & i\sinh s \\ -i\sinh s & \cosh s \end{pmatrix}.$$

Since left translation is an isometry, $M \mapsto \Lambda_\star(s)M$ for any of $\star = e, p, p1, h, h1$ is a one-parameter group of isometries. Furthermore,

$$M \mapsto M\Lambda_e(s) \tag{8}$$

is also a one parameter group of isometries. Consequently, the following maps are also one-parameter groups of isometries:

$$\begin{aligned}
f_1(M(s,\theta,\phi),t) &:= \Lambda_h(t)M(s,\theta,\phi) = M(s+t,\theta,\phi),\\
f_2(M(s,\theta,\phi),t) &:= \Lambda_{h1}(t)M(s,\theta,\phi) = \Lambda_e(\tfrac{\pi}{4})\Lambda_h(t)\Lambda_e(\tfrac{-\pi}{4})M(s,\theta,\phi),\\
f_3(M(s,\theta,\phi),t) &:= \Lambda_e(t/2)M(s,\theta,\phi)\Lambda_e(t/2) = M(s,\theta,\phi+t),\\
f_R(M(s,\theta,\phi),t) &:= \Lambda_e(t/2)M(s,\theta,\phi)\Lambda_e(-t/2) = M(s,\theta+t,\phi).
\end{aligned}$$

Note that $\Lambda_e(\mp\frac{\pi}{4})M(s,\theta,\phi) = M(s,\phi\mp\frac{\pi}{4},\theta\mp\frac{\pi}{4})$. Then it is easy to see that any of these maps sends a fiber to a fiber (cf. Subsection 2.6).

2.4. Orthogonal geodesics

Define $\gamma_i : \mathbb{R} \to \mathrm{SU}(1,1)$ for $i = 1, 2, 3$ by

$$\gamma_1(s) := \Lambda_h(s), \qquad \gamma_2(s) := \Lambda_{h1}(s), \qquad \gamma_3(s) := \Lambda_e(s).$$

They are geodesics through the identity, and are orthogonal. The isometries $f_i(\cdot,t)$ for $i = 1, 2, 3$ is the translation along the geodesic γ_i, respectively, and $f_R(\cdot,t)$ is the rotation around γ_3.

2.5. Fibration

The map (8) induces the foliation of SU(1,1) by circles. We call these circles as fibers. We may think of it as a fibration over the hyperbolic plane, as we consider the Hopf fibration as the circle fibers over $\mathbb{S}^2$. Note that each fiber corresponds to vertical lines in $E(\kappa,\tau)$.

2.6. Orthonormal basis

(5), (6), (7) imply that

$$f_*(e_i) = \frac{\partial}{\partial x_i}\Big|_{(0,0,0)}, \qquad i = 1,2,3$$

where

$$e_1 := \begin{pmatrix} 0 & \frac{\mu}{2} \\ \frac{\mu}{2} & 0 \end{pmatrix}, \qquad e_2 := \begin{pmatrix} 0 & \frac{i\mu}{2} \\ -\frac{i\mu}{2} & 0 \end{pmatrix}, \qquad e_3 := \begin{pmatrix} \frac{i\kappa}{4\tau} & 0 \\ 0 & -\frac{i\kappa}{4\tau} \end{pmatrix}.$$

The pull back by $\rho \circ T$ of the left invariant metric of $\mathrm{SU}(1,1)$ which makes these orthonormal is exactly the metric (1) of $E(\kappa,\tau)$. Hence the left invariant vector fields $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ generated by e_1, e_2, e_3 can be identified with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in (2). We will follow this notational convention if there seems to be no worry of confusion. Note that $\mathbf{e}_3$ is tangent to the fibers. We identify $E(\kappa,\tau)$ with $\kappa < 0$, $\tau \neq 0$ as the universal cover of $\mathrm{SU}(1,1)$, or equivalently of $\mathrm{SL}_2\mathbb{R}$.

2.7. Covariant derivatives

The orientation is given by $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$. We have (cf. [3])

$$[\mathbf{e}_1, \mathbf{e}_2] = 2\tau\mathbf{e}_3, \qquad [\mathbf{e}_2, \mathbf{e}_3] = \frac{\kappa}{2\tau}\mathbf{e}_1, \qquad [\mathbf{e}_3, \mathbf{e}_1] = \frac{\kappa}{2\tau}\mathbf{e}_2,$$

and

$$\begin{aligned}
&\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, && \nabla_{\mathbf{e}_2}\mathbf{e}_1 = -\tau\mathbf{e}_3, && \nabla_{\mathbf{e}_3}\mathbf{e}_1 = \frac{\kappa - 2\tau^2}{2\tau}\mathbf{e}_2, \\
&\nabla_{\mathbf{e}_1}\mathbf{e}_2 = \tau\mathbf{e}_3, && \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0, && \nabla_{\mathbf{e}_3}\mathbf{e}_2 = -\frac{\kappa - 2\tau^2}{2\tau}\mathbf{e}_1, \\
&\nabla_{\mathbf{e}_1}\mathbf{e}_3 = -\tau\mathbf{e}_2, && \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \tau\mathbf{e}_1, && \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0.
\end{aligned}$$

3. Helicoidal killing fields and helicoids in three-manifolds

In this section, we consider arbitrary three-manifolds. Recall that K is a Killing field if and only if

$$\langle \nabla_X K, Y\rangle + \nabla_Y K, X\rangle = 0$$

for all vector fields X and Y.

For a motivation we observe the helicoids in the Euclidean three-space. The screw motion in $\mathbb{E}^3$

$$\varphi_t(x,y,z) := (x\cos at - y\sin at, x\sin at + y\cos at, z + bt) \tag{9}$$

induces the Killing field $K(x,y,z) := -ay\partial_x + ax\partial_y + b\partial_z$. We observe that

$$V := \nabla_K K = -a^2(x\partial_x + y\partial_y)$$

has integral curves

$$\gamma(t) := (c_1 e^{-a^2 t}, c_2 e^{-a^2 t}, c_3) \tag{10}$$

for some constants c_1, c_2, c_3. If $a \neq 0$, then γ is a horizontal ray emanating from the z-axis; if $a = 0$ it is just a point on the z-axis. Interestingly we see that γ is a part of the ruling geodesic, i.e., the straight line through the z-axis, of the helicoids, hence that V satisfies the pre-geodesic equation

$$\nabla_V V \parallel V.$$

In fact, we have $\nabla_V V = a^2 V$ in this case. Note that if (9) is rotational, i.e., if $b = 0$, then K vanishes at the z-axis, and that if (9) is translational, i.e., if $a = 0$, then $\nabla_K K$ vanishes identically. Motivated by these observations we introduce the following definition

Definition 3.1. A nonzero Killing field K in a 3 dimensional manifold is called *helicoidal* if

$$\nabla_V V \times V = \vec{0} \qquad \text{where} \quad V := \nabla_K K. \tag{11}$$

A helicoidal Killing field which vanishes somewhere is called *rotationally helicoidal*, or *rotational* in short. A helicoidal Killing field K with identically vanishing $\nabla_K K$ is called *translationally helicoidal*, or *translational* in short. A helicoidal Killing field which is neither translational nor rotational is called *nontrivially helicoidal*.

Our definition is justified by the following:

Proposition 3.2. *Suppose that in an arbitrary three-manifold*

(1) *K is a helicoidal Killing field, and*
(2) *γ is a regular integral curve of $\nabla_K K$.*

Then the sweepout of γ by the one parameter group of isometries which induces K is regular, ruled and minimal.

Proof. Note that if K vanishes at a point, then $\nabla_K K$ also vanishes at that point, contradicting the second condition. So K never vanishes on the integral curve. Since K and $\nabla_K K$ are perpendicular to each other, it is clear that the sweepout is a regular surface.

By reparametrizing the integral curve if necessary, we may assume that it has unit speed everywhere. Then it is a geodesic, say $t \mapsto \gamma(t)$. (Given a regular curve $\alpha(u)$, let $\gamma(t) = \alpha(u(t))$ be its unit speed parametrization and v be the speed of α. Then $\nabla_{\gamma'}\gamma' = -v^{-3}\dot{v}\dot{\alpha} + v^{-2}\nabla_{\dot{\alpha}}\dot{\alpha}$, where $\gamma' = d\gamma/dt$, $\dot{v} = dv/du$, $\dot{\alpha} = d\alpha/du$. If in addition $\nabla_{\dot{\alpha}}\dot{\alpha} \times \dot{\alpha} = 0$, this implies that $\nabla_{\gamma'}\gamma' \times \gamma' = 0$, which together with $\nabla_{\gamma'}\gamma' \perp \gamma'$ implies again that $\nabla_{\gamma'}\gamma' = 0$.) Let $X(s,t) := \Psi_s \circ \gamma(t)$, where Ψ_s is the one parameter group of isometries associated with K. It is clear from the definition that X is a ruled surface. To show that it is minimal, we first recall that if $X(s,t)$ is a surface in M such that $t \mapsto X(s,t)$ are geodesics and that $\langle X_s, X_t \rangle = 0$, then $X(s,t)$ is minimal if and only if (cf. [11,12])

$$\langle \nabla_{X_s} X_s, X_s \times X_t \rangle = 0. \tag{12}$$

In our situation we have $X_s = K$ and $\nabla_K K = hX_t$ for some function $h = h(s,t)$. Because of condition (2), h is never 0. Since K is a Killing field, we have $\langle K, \nabla_K K\rangle = 0$, hence $\langle X_s, X_t\rangle = 0$. Furthermore, $\nabla_{X_s} X_s = \nabla_K K$. Therefore

$$\langle \nabla_{X_s} X_s, X_s \times X_t\rangle = \frac{1}{h}\langle \nabla_K K, K \times \nabla_K K\rangle,$$

which vanishes identically. Then the conclusion follows. □

Now, we show that there are abundant examples of helicoidal Killing fields.

Theorem 3.3. *In $E(\kappa,\tau)$, every nonzero Killing field is helicoidal.*

Proof. Recall from [2] that

$$\begin{aligned}
S_1 &:= \Big(1 + \frac{\kappa}{4}(x_1^2 - x_2^2)\Big)\frac{\partial}{\partial x_1} + \frac{\kappa}{2}x_1x_2\frac{\partial}{\partial x_2} + \tau x_2\frac{\partial}{\partial x_3},\\
S_2 &:= \frac{\kappa}{2}x_1x_2\frac{\partial}{\partial x_1} + \Big(1 + \frac{\kappa}{4}(x_2^2 - x_1^2)\Big)\frac{\partial}{\partial x_2} - \tau x_1\frac{\partial}{\partial x_3},\\
S_3 &:= \frac{\partial}{\partial x_3},\\
S_R &:= -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}
\end{aligned}$$

are a basis of the Killing fields of $E(\kappa,\tau)$. They can be written as

$$\begin{aligned}
S_1 = {} & \left(\frac{1}{2}\kappa\lambda x_1x_2 \sin(\sigma x_3) + \left(\frac{1}{4}\kappa\left(x_1^2 - x_2^2\right) + 1\right)\lambda\cos(\sigma x_3)\right) e_1\\
& + \left(\frac{1}{2}\kappa\lambda x_1x_2 \cos(\sigma x_3) - \left(\frac{1}{4}\kappa\left(x_1^2 - x_2^2\right) + 1\right)\lambda\sin(\sigma x_3)\right) e_2\\
& + \left(-\frac{1}{2}\kappa\lambda\tau x_2x_1^2 + \left(\frac{1}{4}\kappa\left(x_1^2 - x_2^2\right) + 1\right)\lambda\tau x_2 + \tau x_2\right) e_3,\\
S_2 = {} & \left(\left(1 - \frac{1}{4}\kappa\left(x_1^2 - x_2^2\right)\right)\lambda\sin(\sigma x_3) + \frac{1}{2}\kappa\lambda x_1x_2\cos(\sigma x_3)\right) e_1\\
& + \left(\left(1 - \frac{1}{4}\kappa\left(x_1^2 - x_2^2\right)\right)\lambda\cos(\sigma x_3) - \frac{1}{2}\kappa\lambda x_1x_2\sin(\sigma x_3)\right) e_2\\
& + \left(\frac{1}{2}\kappa\lambda\tau x_1x_2^2 - \left(1 - \frac{1}{4}\kappa\left(x_1^2 - x_2^2\right)\right)\lambda\tau x_1 - \tau x_1\right) e_3,\\
S_3 = {} & e_3,\\
S_R = {} & \left(\lambda x_1\sin(\sigma x_3) - \lambda x_2\cos(\sigma x_3)\right) e_1 + \left(\lambda x_2\sin(\sigma x_3) + \lambda x_1\cos(\sigma x_3)\right) e_2\\
& + \left(-\lambda\tau x_1^2 - \lambda\tau x_2^2\right) e_3.
\end{aligned}$$

Any Killing field K can be written as a linear combination of these four fields, that is,

$$K = a_1S_1 + a_2S_2 + a_3S_3 + a_RS_R$$

for some constants a_1, a_2, a_3 and a_R. By direct calculations, we see that

$$\nabla_K K = h\tilde{V},$$

where

$$\begin{aligned}
h &:= -\frac{a_R\left(x_1^2\left(\kappa-8\tau^2\right)+x_2^2\left(\kappa-8\tau^2\right)-4\right)+4\left(a_2x_1-a_1x_2\right)\left(\kappa-4\tau^2\right)+2a_3\tau\left(\kappa x_1^2+\kappa x_2^2+4\right)}{\left(\kappa x_1^2+\kappa x_2^2+4\right)^2},\\
\tilde{V} &:= \Bigg(-4a_R\left(x_2\sin\left(\frac{\kappa x_3}{2\tau}\right)+x_1\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\\
&\quad + a_1\left(\kappa x_1^2\sin\left(\frac{\kappa x_3}{2\tau}\right)+\left(4-\kappa x_2^2\right)\sin\left(\frac{\kappa x_3}{2\tau}\right)-2\kappa x_1x_2\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\\
&\quad + a_2\left(2\kappa x_1x_2\sin\left(\frac{\kappa x_3}{2\tau}\right)+\kappa x_1^2\cos\left(\frac{\kappa x_3}{2\tau}\right)-\left(\kappa x_2^2+4\right)\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\Bigg)e_1\\
&\quad + \Bigg(4a_R\left(x_1\sin\left(\frac{\kappa x_3}{2\tau}\right)-x_2\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\\
&\quad + a_1\left(2\kappa x_1x_2\sin\left(\frac{\kappa x_3}{2\tau}\right)+\kappa x_1^2\cos\left(\frac{\kappa x_3}{2\tau}\right)+\left(4-\kappa x_2^2\right)\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\\
&\quad + a_2\left(-\kappa x_1^2\sin\left(\frac{\kappa x_3}{2\tau}\right)+\left(\kappa x_2^2+4\right)\sin\left(\frac{\kappa x_3}{2\tau}\right)+2\kappa x_1x_2\cos\left(\frac{\kappa x_3}{2\tau}\right)\right)\Bigg)e_2,
\end{aligned}$$

which implies that the integral curves of $\nabla_K K$ and $\tilde{V}$ are the same. Note that $\nabla_K K$ is horizontal. By direct calculations again, we can see that

$$\nabla_{\tilde{V}}\tilde{V}\times\tilde{V}=0,$$

from which the proof follows. □

It would be great if *any* integral curve of $\nabla_K K$ for any helicoidal Killing field K induces a ruled minimal surface, and if any ruled minimal surface arises this way. But, unfortunately, both of them are not true. For example, the images of the integral curves of (10) with $a=0$ is just a single point, hence the integral curves do not induce a regular surface. Furthermore even if $a\neq 0$ and $c_1^2+c_2^2\neq 0$ the integral curves are only rays, hence the sweepout of any of these integral curves is just the half of the usual helicoid we know.

For these reasons, our definition of the helicoid becomes a little technical as follows:

Definition 3.4. We call a regular surface S in an arbitrary three manifold M a *helicoid* if there is a unit-speed geodesic γ and a helicoidal Killing field K such that

(i) $\gamma'(t)$ and $\nabla_K K$ are parallel to each other on $\gamma(t)$,
(ii) $\gamma'(t)$ and $K(\gamma(t))$ are an orthogonal basis of $T_{\gamma(t)}S$ for any t, and
(iii) S is (a part of) the image of the map $X(s,t):=\Psi_s\circ\gamma(t)$, where Ψ_s is the one parameter group of isometries associated with K.

We call a helicoid S nontrivial if the associated K is nontrivially helicoidal.

Note that (ii) prohibits K from vanishing on γ. So the pair

$$\gamma:(-\infty,\infty)\to\mathbb{E}^3,\ \gamma(t):=(t,0,0),\quad K(x,y,z):=-y\partial_x+x\partial_y$$

does not produce a helicoid. But the sweepout of γ by K is anyway a helicoid with the help of the pair $\gamma(t):=(t,0,0)$ and $K(x,y,z):=\partial_y$. Note that

$\nabla_K K = 0$ in this case, but this does not contradict (i). By definition, if S is a helicoid, then any surface contained in S is again a helicoid.

Of course we have the following:

Theorem 3.5. *A helicoid is a ruled minimal surface.*

Proof. The proof is almost identical to that of Proposition 3.2, and is omitted. □

One can ask immediately if the converse of this Theorem is true. In the next two sections, we provide a partial answer to this question.

4. Ruled minimal surface in $\mathbf{SL_2}\mathbb{R}$

4.1. Introduction

One may take a look at [7] by Kokubu for an account of the theory of minimal surfaces in $\mathrm{SL}_2\mathbb{R}$. All Killing fields of $\mathrm{SL}_2\mathbb{R}$ are helicoidal (cf. Theorem 3.3), hence we may expect that there are abundant examples of helicoids, hence ruled minimal surfaces, in $\mathrm{SL}_2\mathbb{R}$. But it is still to be determined if all ruled minimal surfaces are helicoids. Note that in Nil(3) and in Berger sphere, all ruled minimal surfaces are classified.

In this section, we classify ruled minimal surfaces in $E(\kappa,\tau)$ with $\kappa < 0$, $\tau > 0$, or equivalently in $\mathrm{SL}_2\mathbb{R}$, and show that they are helicoids in the sense of the previous section. We carry out the computations in $\mathrm{SU}(1,1)$ rather than in $\mathrm{SL}_2\mathbb{R}$. The involved computations are basically the same as the ones in [12], but we include them here for the convenience of the reader.

Let $X : U \subset \mathbb{R}^2 \to \mathrm{SU}(1,1)$ be a ruled minimal surface. We may assume without loss of generality that the t-curve $t \mapsto X(s,t)$ is a ruling geodesic with unit speed and the s-curve $s \mapsto X(s,t)$ is orthogonal to the ruling geodesics everywhere:

$$\nabla_{X_t} X_t = 0, \qquad \|X_t(s,t)\| = 1, \qquad \langle X_t(s,t), X_s(s,t)\rangle = 0 \qquad \text{for all } s,t.$$

The parametrization X satisfies the integrability condition:

$$\nabla_{X_s} X_t = \nabla_{X_t} X_s. \tag{13}$$

X is minimal if and only if

$$\langle \nabla_{X_s} X_s, X_s \times X_t\rangle = 0, \tag{14}$$

since $t \mapsto X(s,t)$ are geodesics for all s (cf. [11,12]).

Let $X_t(s,t) = X_t^1(s,t)\mathbf{e}_1 + X_t^2(s,t)\mathbf{e}_2 + X_t^3(s,t)\mathbf{e}_3$. Since the t-curve is a geodesic, we have (from the geodesic equations)

$$0 = \frac{\partial X_t^1}{\partial t} - \left(\frac{\kappa}{2\tau} - 2\tau\right) X_t^2 X_t^3,\ 0 = \frac{\partial X_t^2}{\partial t} + \left(\frac{\kappa}{2\tau} - 2\tau\right) X_t^1 X_t^3,\ 0 = \frac{\partial X_t^3}{\partial t}. \tag{15}$$

The following is a consequence of the geodesic equation whose proof can be found in [12]:

Lemma 4.1. *If γ is a geodesic in* SU(1,1)*, then the angle between geodesic γ and the fiber is constant along the geodesic, that is, $\langle \dot{\gamma}, \mathbf{e}_3 \rangle$ is constant along γ.*

Hence we call a geodesic *horizontal* if it is orthogonal to the fibers everywhere and *vertical* if it is tangent to the fibers everywhere.

Solving the geodesic equation (15) with the initial conditions

$$X_t^1(s,0) = a(s)\cos\theta(s), \qquad X_t^2(s,0) = a(s)\sin\theta(s), \qquad X_t^3(s,0) = b(s)$$

for some functions $a(s), b(s)$ and $\theta(s)$ with $a(s)^2 + b(s)^2 \equiv 1$, we have

$$X_t(s,t) = a(s)\cos\phi(s,t)\,\mathbf{e}_1 - a(s)\sin\phi(s,t)\,\mathbf{e}_2 + b(s)\,\mathbf{e}_3, \tag{16}$$

where

$$\phi(s,t) := -\left(2\tau - \frac{\kappa}{2\tau}\right) b(s)t - \theta(s). \tag{17}$$

Define

$$\begin{aligned} V(s,t) &:= \sin\phi(s,t)\mathbf{e}_1 + \cos\phi(s,t)\mathbf{e}_2, \\ W(s,t) &:= X_t(s,t) \times V(s,t) \\ &= -b(s)\cos\phi(s,t)\mathbf{e}_1 + b(s)\sin\phi(s,t)\mathbf{e}_2 + a(s)\mathbf{e}_3. \end{aligned}$$

Since $\langle X_t, X_s \rangle = 0$, we have

$$X_s(s,t) = f(s,t)V(s,t) + g(s,t)W(s,t) \tag{18}$$

for some functions $f(s,t)$ and $g(s,t)$. Then one can rewrite the integrability equation (13) and the ruled minimal surface equation (14) as follows (see [12] for detailed computation): (In the following, κ, τ are constants, a, b, θ are functions of s only, f, g are functions of both s and t, and $'$ is the differentiation with respect to s.)

Lemma 4.2. *The functions f and g satisfy*

$$a(g_t + 2\tau f) = b', \qquad 2\tau f_t - (\kappa a^2 + 4\tau^2 b^2)g = (4\tau^2 - \kappa)ab't + 2\tau a\theta'. \tag{19}$$

Lemma 4.3. *The ruled surface X is minimal if and only if $MC = 0$, where*

$$MC := f_s g - f g_s - (f^2 + g^2)\left(\left(2\tau - \frac{\kappa}{2\tau}\right) b't - 4ag + \theta'\right)b. \tag{20}$$

4.2. Character of the rulings of the ruled minimal surfaces in $\mathbf{SL_2}\mathbb{R}$

We will show in this section that all the geodesics $t \mapsto X(s,t)$ are either all horizontal or all vertical. The arguments are very similar to the ones as in [5, 6, 11, 12], but we present anyhow the details for the benefit of the reader.

Suppose a geodesic $t \mapsto X(s_0, t)$ is neither horizontal nor vertical. By reparametrizating the surface if necessary, we may assume that $s_0 = 0$. Then $a(0) \neq 0$ and $b(0) \neq 0$. Three cases arise when we try to solve (19):

$$\begin{array}{ll} \text{Case (i)} & \kappa a^2(0) + 4\tau^2 b^2(0) > 0, \\ \text{Case (ii)} & \kappa a^2(0) + 4\tau^2 b^2(0) < 0, \\ \text{Case (iii)} & \kappa a^2(0) + 4\tau^2 b^2(0) = 0. \end{array}$$

We consider each case separately:

Case (i): By continuity, we may assume that $\kappa a^2(s) + 4\tau^2 b^2(s) > 0$ around $s = 0$. Set

$$\beta(s) := \sqrt{\kappa a^2(s) + 4\tau^2 b^2(s)}.$$

Since $a(s)^2 + b(s)^2 = 1$, the general solutions f and g of the integrability equation (19) are

$$f(s,t) = A(s)\sin(B(s) + t\beta(s)) + \frac{2\tau b'(s)}{a(s)\beta(s)^2},$$

$$g(s,t) = \frac{2\tau A(s)}{\beta(s)}\cos(B(s) + t\beta(s)) + \frac{ta(s)\left(\kappa - 4\tau^2\right)b'(s)}{\beta(s)^2} - \frac{2\tau a(s)\theta'(s)}{\beta(s)^2}$$

for some functions $A(s)$ and $B(s)$. We will insert these into MC in (20) and inspect its shape. Before we actually do it, we observe that

$$MC = \sum_{\ell,m,n} F_{\ell,m,n}(s) \times t^\ell \times \sin^m(B(s) + t\beta(s)) \times \cos^n(B(s) + t\beta(s))$$

for some functions $F_{\ell,m,m}$ of s only, where

$$0 \le \ell \le 2, \qquad 0 \le m \le 2, \qquad 0 \le n \le 3.$$

Considering the Fourier expansions of the products of sines and cosines, we conclude that MC is a linear combination of the linearly independent functions

$$t^{\ell_1}, \qquad t^{\ell_2}\sin m(B(s) + t\beta(s)), \qquad t^{\ell_3}\cos n(B(s) + t\beta(s))$$

for

$$0 \le \ell_1, \ell_2, \ell_3 \le 3, \qquad 1 \le m \le 2, \qquad 1 \le n \le 5$$

whose coefficients are functions of s. By inspection, we see that the coefficient function of $\cos 3(B(s) + t\beta(s))$ is

$$\frac{4\tau a(s)A(s)^3 b(s)\left(4\tau^2 - \beta(s)^2\right)}{\beta(s)^3}.$$

Since this must be 0, we conclude that $A(s) \equiv 0$. Plugging this into MC, we see that the coefficient of t^3 is

$$\frac{a(s)^2 b(s)\left(\kappa - 4\tau^2\right)^3 b'(s)^3\left(8\tau a(s)^2 + \beta(s)^2\right)}{\tau\beta(s)^6}.$$

Since this must be 0, we see that $b'(s) \equiv 0$. But then the MC is equal to

$$-\frac{8\tau^2 a(s)^2 b(s)\theta'(s)^3\left(8\tau a(s)^2 + \beta(s)^2\right)}{\beta(s)^6}.$$

Since this must be 0, we conclude that $\theta'(s) \equiv 0$. But this implies that $f \equiv 0$, $g \equiv 0$, hence $X_s(s,t) \equiv 0$, which is a contradiction.

Case (ii): By continuity, we may assume that $\kappa a^2(s) + 4\tau^2 b^2(s) < 0$ around $s = 0$. Set

$$\gamma(s) := \sqrt{-(\kappa a^2(s) + 4\tau^2 b^2(s))}.$$

The general solutions f and g of the integrability equation (19) are given by

$$f(s,t) = A(s)e^{t\gamma(s)} + B(s)e^{-t\gamma(s)} - \frac{2\tau b'(s)}{a(s)\gamma(s)^2},$$
$$g(s,t) = -\frac{2\tau A(s)e^{t\gamma(s)}}{\gamma(s)} + \frac{2\tau B(s)e^{-t\gamma(s)}}{\gamma(s)} - \frac{ta(s)\left(\kappa - 4\tau^2\right)b'(s)}{\gamma(s)^2} + \frac{2\tau a(s)\theta'(s)}{\gamma(s)^2}$$

for some functions $A(s)$ and $B(s)$. We will plug these into MC and inspect its shape. Before we proceed further, we observe that MC is a linear combination of the linearly independent functions

$$t^n e^{mt\gamma(s)}, \qquad 0 \le n \le 3, \quad -3 \le m \le 3$$

whose coefficients are functions of s. By plugging the above expressions into MC, we see that the coefficients of t^3, $e^{3t\gamma(s)}$, $e^{-3t\gamma(s)}$, in particular, are

$$-\frac{a(s)^3b(s)\left(\kappa-4\tau^2\right)^3b'(s)^3\left(8\tau a(s)^2-\gamma(s)^2\right)}{\tau\gamma(s)^4}, \qquad -\frac{16\tau a(s)^2A(s)^3b(s)\left(\gamma(s)^2+4\tau^2\right)}{\gamma(s)},$$

and

$$\frac{16\tau a(s)^2b(s)B(s)^3\left(\gamma(s)^2+4\tau^2\right)}{\gamma(s)}$$

respectively. Since they must be all 0, we must have $A(s) \equiv B(s) \equiv b'(s) \equiv 0$. Plugging these into $MC = 0$, we obtain

$$a(s)^2b(s)\theta'(s)^3\left(a(s)^2\left(\kappa - 4\tau^2 + 8\tau\right) + 4\tau^2\right) = 0,$$

which implies that $\theta'(s) \equiv 0$. Then $f(s,t) \equiv g(s,t) \equiv 0$, and in turn $X_s(s,t) \equiv 0$, which is a contradiction.

Case (iii): Because of cases (i) and (ii), we may assume without loss of generality that $\kappa a^2(s) + 4\tau^2 b^2(s) \equiv 0$ around $s = 0$. Combined with the facts that $a^2(s) + b^2(s) = 1$ and that $a(s) > 0$, this yields $a(s) = \frac{2\tau}{\sqrt{-\kappa+4\tau^2}}$, $b(s) = \pm\frac{\mu}{\sqrt{-\kappa+4\tau^2}}$. Then the general solutions f and g of the integrability equation (19) are

$$f(s,t) = ta(s)\theta'(s) + c_0(s), \qquad g(s,t) = -\tau t^2 a(s)\theta'(s) - 2\tau t c_0(s) + c_1(s)$$

for some functions $c_0(s)$ and $c_1(s)$ of s only. Plugging these into $MC = 0$, we obtain a 6-th order polynomial of t, the coefficient of t^6 of which is $-4\tau^3a(s)^4b(s)\theta'(s)^3$. Since this must be 0, we conclude that $\theta'(s) \equiv 0$. Plugging this into MC again we immediately see that $c_1(s) \equiv c_2(s) \equiv 0$, hence $f(s,t) \equiv g(s,t) \equiv 0$, hence $X_s(s,t) \equiv 0$, a contradiction.

Since we obtain contradictions in all cases (i), (ii), (iii), we conclude that either $a(0) = 0$ or $b(0) = 0$. Then by Lemma 4.1 we may conclude that the geodesic $t \mapsto X(0,t)$ is either horizontal or vertical.

Let us call a ruled surface *horizontally ruled* if all of its ruling geodesics are horizontal and *vertically ruled* if all of its ruling geodesics are vertical. We have:

Proposition 4.4. *Every ruled minimal surface in $SU(1,1)$ is either vertically ruled or horizontally ruled.*

4.3. Classification of horizontally ruled minimal surfaces

We now consider the horizontally ruled surfaces, that is, when $b \equiv 0$ in (16) in which case $a \equiv 1$. Then, from (17),

$$\phi(s,t) = \phi(s) = -\theta(s).$$

Recall that $\mu = \sqrt{-\kappa}$. Then, the solutions of the integrability equation (19) are

$$\begin{aligned} f(s,t) &= A(s)e^{-\mu t} + B(s)e^{\mu t}, \\ g(s,t) &= \frac{2\tau A(s)e^{-\mu t}}{\mu} - \frac{2\tau B(s)e^{\mu t}}{\mu} + \frac{2\tau\theta'(s)}{\mu^2}. \end{aligned}$$

Inserting this into the ruled minimal surface equation in (14), we obtain

$$\begin{aligned} &\frac{4\tau}{\mu}\left(A'(s)B(s) - A(s)B'(s)\right) + e^{-\mu t}\frac{2\tau}{\mu^2}\left(A(s)\theta''(s) - A'(s)\theta'(s)\right) \\ &+ e^{\mu t}\frac{2\tau}{\mu^2}\left(B(s)\theta''(s) - B'(s)\theta'(s)\right) = 0, \end{aligned}$$

which implies

$$A(s) = -\underline{a}\ \theta'(s) = \underline{a}\ \phi'(s), \quad B(s) = -\underline{b}\ \theta'(s) = \underline{b}\ \phi'(s)$$

for some constants $\underline{a}, \underline{b} \in \mathbb{R}$. Now recalling (16), (17) and (18), one has

$$\begin{aligned} X_t &= \cos\phi(s)\,\mathbf{e}_1 - \sin\phi(s)\,\mathbf{e}_2, \\ X_s &= fV + gW \\ &= f\left(\sin\phi(s)\ \mathbf{e}_1 + \cos\phi(s)\ \mathbf{e}_2\right) + g\mathbf{e}_3 \\ &= \phi'(s)\Big[\left(\underline{a}e^{-\mu t} + \underline{b}e^{\mu t}\right)\left(\sin\phi(s,t)\,\mathbf{e}_1 + \cos\phi(s,t)\,\mathbf{e}_2\right) \\ &\qquad + \left(\frac{2\tau\underline{a}e^{-\mu t}}{\mu} - \frac{2\tau\underline{b}e^{\mu t}}{\mu} + \frac{2\tau}{\mu^2}\right)\mathbf{e}_3\Big]. \end{aligned}$$

Now taking the reparametrization $\tilde{s} = \phi(s)$ and abusing the notation as $\tilde{s} = s$, one has

$$\begin{aligned} X_t &= \cos s\,\mathbf{e}_1 - \sin s\,\mathbf{e}_2, \\ X_s &= \left(\underline{a}e^{-\mu t} + \underline{b}e^{\mu t}\right)\left(\sin s\,\mathbf{e}_1 + \cos s\,\mathbf{e}_2\right) + \left(\frac{2\tau\underline{a}e^{-\mu t}}{\mu} - \frac{2\tau\underline{b}e^{\mu t}}{\mu} + \frac{2\tau}{\mu^2}\right)\mathbf{e}_3. \end{aligned}$$

We can rewrite these equations as

$$X^{-1}X_t = T, \tag{21}$$

$$X^{-1}X_s = S, \tag{22}$$

where

$$T := \frac{\mu}{2}\begin{pmatrix} 0 & e^{-is} \\ e^{is} & 0 \end{pmatrix},$$

$$S := \frac{\mu}{2}\begin{pmatrix} -i\left(\underline{a}e^{-\mu t} - \underline{b}e^{\mu t}\right) & ie^{-is}\left(\underline{a}e^{-\mu t} + \underline{b}e^{\mu t}\right) \\ -ie^{is}\left(\underline{a}e^{-\mu t} + \underline{b}e^{\mu t}\right) & i\left(\underline{a}e^{-\mu t} - \underline{b}e^{\mu t}\right) \end{pmatrix} + \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

Let

$$T_1 = T_1(t) := \begin{pmatrix} \cosh\frac{\mu}{2}t & \sinh\frac{\mu}{2}t \\ \sinh\frac{\mu}{2}t & \cosh\frac{\mu}{2}t \end{pmatrix}, \qquad T_2 = T_2(s) := \begin{pmatrix} e^{is/2} & 0 \\ 0 & e^{-is/2} \end{pmatrix}.$$

Then T_1T_2 is a particular solution of (21), so

$$X(s,t) = Z(s)T_1(t)T_2(s)$$

for some $Z(s) \in \mathrm{SU}(1,1)$. By direct calculations, which involves (22), we see

$$Z^{-1}Z_s = T_1T_2\left(S - T_2^{-1}(T_2)_s\right)(T_1T_2)^{-1} = \frac{\mu}{2}\begin{pmatrix} i(\underline{b}-\underline{a}) & i(\underline{b}+\underline{a}) \\ -i(\underline{b}+\underline{a}) & -i(\underline{b}-\underline{a}) \end{pmatrix}.$$

So it is enough to solve this equation. Solutions $Z(s)$ of this equation satisfies

$$Z(s) = Y\left(\frac{\mu}{2}s\right),$$

where $Y = Y(s)$ satisfies

$$Y^{-1}(s)Y(s)_s = \begin{pmatrix} i(\underline{b}-\underline{a}) & i(\underline{b}+\underline{a}) \\ -i(\underline{b}+\underline{a}) & -i(\underline{b}-\underline{a}) \end{pmatrix}. \tag{23}$$

We want to find a general solution of this in $\mathrm{SU}(1,1)$.

Case 1) $\underline{ab} < 0$: In this case, define

$$\alpha := -\underline{b}e^{\tanh^{-1}\left(\frac{\underline{a}+\underline{b}}{\underline{a}-\underline{b}}\right)}, \qquad \beta := \frac{1}{2}\tanh^{-1}\left(\frac{\underline{a}+\underline{b}}{\underline{a}-\underline{b}}\right).$$

Then $Y^{-1}Y_s = 2i\alpha M_1$ where $M_1 := \begin{pmatrix} -\cosh 2\beta & \sinh 2\beta \\ -\sinh 2\beta & \cosh 2\beta \end{pmatrix}$. Since

$$M_1 = P\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}P^{-1}, \qquad P := \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix},$$

it is easy to see that

$$Y_p(s) := \begin{pmatrix} e^{-2i\alpha s} & 0 \\ 0 & e^{2i\alpha s} \end{pmatrix}\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}$$

is a particular solution of (23).

Case 2) $\underline{a}\underline{b} > 0$: In this case, define

$$\alpha := \underline{b}e^{\tanh^{-1}\left(\frac{\underline{a}-\underline{b}}{\underline{a}+\underline{b}}\right)}, \qquad \beta := \frac{1}{2}\tanh^{-1}\left(\frac{\underline{a}-\underline{b}}{\underline{a}+\underline{b}}\right).$$

Then $Y^{-1}Y_s = 2i\alpha M_2$ where $M_2 := \begin{pmatrix} -\sinh 2\beta & \cosh 2\beta \\ -\cosh 2\beta & \sinh 2\beta \end{pmatrix}$. Since

$$M_2 = P\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}P^{-1}, \qquad P := \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

it is easy to see that the general solution of (23) is

$$A_1\begin{pmatrix} e^{-2\alpha s} & 0 \\ 0 & e^{2\alpha s} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}, \qquad A_1 \in \mathrm{GL}(2,\mathbb{C}).$$

Note that in general this is not in $\mathrm{SU}(1,1)$. Fortunately we notice that

$$\begin{aligned} &\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^{-1}\begin{pmatrix} e^{-2\alpha s} & 0 \\ 0 & e^{2\alpha s} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \cosh 2\alpha s & i\sinh 2\alpha s \\ -i\sinh 2\alpha s & \cosh 2\alpha s \end{pmatrix} \in \mathrm{SU}(1,1), \end{aligned}$$

hence

$$Y_p(s) := \begin{pmatrix} \cosh 2\alpha s & i\sinh 2\alpha s \\ -i\sinh 2\alpha s & \cosh 2\alpha s \end{pmatrix}\begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix}$$

is a particular solution of (23).

Case 3) $\underline{a} = 0$ or $\underline{b} = 0$: In this case, define Y_p as

$$Y_p(s) := \begin{pmatrix} 1+i\underline{b}s & i\underline{b}s \\ -i\underline{b}s & 1-i\underline{b}s \end{pmatrix} \text{ if } \underline{a} = 0, \text{ or } Y_p(s) := \begin{pmatrix} 1-i\underline{a}s & i\underline{a}s \\ -i\underline{a}s & 1+i\underline{a}s \end{pmatrix} \text{ if } \underline{b} = 0.$$

Then Y_p is a particular solution of (23).

Proposition 4.5. *Let $\Lambda(s)$ denote any of*

$$\Lambda_e(-2\alpha s),\ \Lambda_{h1}(2\alpha s),\ \Lambda_{p1}(\underline{b}s),\ \Lambda_p(-\underline{a}s),$$

where α, $\underline{a}$, $\underline{b}$ are arbitrary real numbers. Then, for an arbitrary $A \in \mathrm{SU}(1,1)$, the following map

$$X : \mathcal{U} \to SU(1,1), \qquad X(s,t) := A\Lambda(s)\Lambda_h\left(\frac{\mu}{2}t\right)\Lambda_e\left(\frac{s}{2}\right) \tag{24}$$

becomes an immersion for a horizontally ruled minimal surface in $\mathrm{SU}(1,1)$, where

$$\mathcal{U} := \begin{cases} \mathbb{R}\times\mathbb{R}^+ \text{ or } \mathbb{R}\times\mathbb{R}^- & \text{if } \Lambda(s) = \Lambda_e(-2\alpha s) \text{ and } \alpha = \frac{1}{4}, \\ \mathbb{R}\times\mathbb{R} & \text{otherwise.} \end{cases} \tag{25}$$

Conversely, any horizontally ruled minimal surface in $\mathrm{SU}(1,1)$ is of this form.

Proof. It follows from above considerations. We just remark that

$$\Lambda_h(-\beta)\Lambda_h\left(\frac{\mu}{2}t\right) = \Lambda_h\left(\frac{\mu}{2}(t - \frac{2}{\mu}\beta)\right),$$

hence by reparametrizing t if necessary, we may assume $\beta = 0$ in the above cases. We also remark that with $X(s,t) := \Lambda_e\left(-\frac{s}{2}\right)\Lambda_h\left(\frac{\mu}{2}t\right)\Lambda_e\left(\frac{s}{2}\right)$, $X(s,0)$ is a single point for all s. This is the analogue of the plane in $\mathbb{E}^3$ obtained by rotating a line ℓ around a line m while ℓ and m are perpendicular, and we exclude it from the class of ruled minimal surface. □

The image of $X(s,t) := \Lambda_e\left(-\frac{s}{2}\right)\Lambda_h\left(\frac{\mu}{2}t\right)\Lambda_e\left(\frac{s}{2}\right)$ for $\mathbb{R}\times\mathbb{R}^+$ is the set of all points with $y = 0$ minus a single point, the identity. One may hope that the set of all points with $y = 0$ may be obtained as a helicoid in another way. But that is not the case as we see in the following.

Lemma 4.6. *The set of all points with $y = 0$ in $\mathrm{SU}(1,1)$ is not a helicoid.*

Proof. Let us by S denote the set of all points with $y = 0$ in $\mathrm{SU}(1,1)$. Fix a geodesic in S through the identity matrix I_2, say $t \mapsto \Lambda_h(t)$, and a one parameter family of isometries Ψ_s. Suppose that for some s, the image of the geodesic $t \mapsto \Psi_s \circ \Lambda_h(t)$ is still in S. Since $t \mapsto \Lambda_h(t)$ is horizontal for any t, the geodesic $t \mapsto \Psi_s \circ \Lambda_h(t)$ must also be horizontal for any t. (Recall from subsection 2.5 that the image of any horizontal vector by a member of one parameter group of isometries is again horizontal.) But it turn out that any horizontal vector tangent to S must be radial. To show this, we will use the coordinate system (3). $y = 0$ is equivalent to $\phi = 0$. Now we see

$$\begin{aligned} M^{-1}(s,\theta,0)\partial_s M(s,\theta,0) &= \tfrac{2}{\mu}\cos\theta e_1 + \tfrac{2}{\mu}\sin\theta e_2, \\ M^{-1}(s,\theta,0)\partial_\theta M(s,\theta,0) &= -\tfrac{2}{\mu}\sin\theta\cosh s\sinh s e_1 + \tfrac{2}{\mu}\cos\theta\cosh s\sinh s e_2 \\ &\quad + \tfrac{4\tau}{\kappa}\sinh^2 s e_3. \end{aligned}$$

So among the vectors tangent to S at $M(s,\theta,0)$ only $M_s(s,\theta,0)$, which is the radial vector, is horizontal. Then we can conclude that for any $p \in S\backslash\{I_2\}$, the geodesic in S through p must go through I_2. So $X(s,t) := \Lambda_e\left(-\frac{s}{2}\right)\Lambda_h\left(\frac{\mu}{2}t\right)\Lambda_e\left(\frac{s}{2}\right)$ for $\mathbb{R}\times\mathbb{R}$ is the only way to obtain the entire S, which is not a helicoid because of the branch points at $(s,0)$. □

It is clear that for $\Lambda(s)$ as in the above lemma,

$$f : \mathrm{SU}(1,1) \to \mathrm{SU}(1,1), \qquad f(M) := \Lambda(s)M\Lambda_e(s)$$

is an analogue of screw motions in $\mathbb{E}^3$.

4.4. Classification of vertically ruled minimal surfaces

We consider the case that the ruling geodesics are all vertical around $X(0,0)$. Then one has $a \equiv 0$, $b \equiv 1$ on an interval, say I, containing 0. One then has

$$X_t(s,t) = \mathbf{e}_3, \quad X_s(s,t) = F(s,t)\mathbf{e}_1 + G(s,t)\mathbf{e}_2$$

for some functions $F(s,t), G(s,t)$. The compatibility equation (13) gives

$$F_t = \frac{\kappa}{2\tau} G, \qquad G_t = -\frac{\kappa}{2\tau} F$$

and the ruled minimal surface equation (14) gives

$$F_s G - F G_s = 0. \tag{26}$$

The general solutions of the compatibility equation are

$$F(s,t) = A(s) \sin\left(\frac{\kappa}{2\tau} t + B(s)\right), \quad G(s,t) = A(s) \cos\left(\frac{\kappa}{2\tau} t + B(s)\right).$$

Inserting these into the ruled minimal surface equation (26) yields

$$A(s)^2 B'(s) = 0$$

on I. If $A(s_0) = 0$ for some $s_0 \in I$, one has $X_s(s_0, t) = 0$, which is a contradiction. Hence A is nowhere 0 and B is constant on I. Now, by changing the parameters appropriately, we have

$$X_t = \mathbf{e}_3, \quad X_s = \sin\left(\frac{\kappa}{2\tau} t\right) \mathbf{e}_1 + \cos\left(\frac{\kappa}{2\tau} t\right) \mathbf{e}_2.$$

But then for each fixed t, the curve $s \mapsto X(s,t)$ is a horizontal geodesic, hence X is in fact horizontally ruled. So it is one of the horizontally ruled surfaces studied in the previous subsection. More explicitly we see that

$$X(s,t) = A \Lambda_{h1}(\tfrac{\mu}{2} s) \Lambda_e(\tfrac{\kappa}{4\tau} t), \qquad A \in \mathrm{SU}(1,1). \tag{27}$$

Note that this is obtained by moving the geodesic $\Lambda_{h1}(\frac{\mu}{2} s)$ by $M \mapsto M \Lambda_e(\frac{\kappa}{4\tau} t)$, which is an analogue of the translation along the fiber.

We would like to remark that there are no totally geodesic surfaces in $\mathrm{SU}(1,1)$ (cf. [13]).

4.5. Classification of ruled minimal surfaces

By combining the results of the previous subsections, we have

Theorem 4.7. *Any ruled minimal surface in* $SL_2\mathbb{R}$ *is given by* (24) *and* (25).

Proof. The proof follows from the results of the previous three subsections. □

In §3, we showed that every helicoid is a ruled minimal surface (cf. Theorem 3.5). Now we show the converse for ruled minimal surfaces in $\mathrm{SL}_2\mathbb{R}$. The key property to check is if K vanishes at some points or not.

Theorem 4.8. *In* $SL_2\mathbb{R}$*, every ruled minimal surface is a helicoid.*

Proof. One can easily check that $(X^{-1}X_s)(s,t)$ does not vanish and that all the conditions of Definition 3.4 for helicoids are satisfied. □

Now we can show that the helicoids in [1] are special cases of ours. In [1], helicoids are defined as the minimal surfaces given as the graph of functions of the form $x_3 = h\left(\frac{x_2}{x_1}\right)$. The intersection of these surfaces with $x_3 =$ constant has geodesics of the form $ax_1 + bx_2 = 0$, and the x_3-axis is the axis of the screw motion. This corresponds to (24) with $\Lambda(s) = \Lambda_e(-2\alpha s)$.

5. Helicoidal Killing fields and helicoids in Sol(3)

In §3 we showed that in $E(\kappa,\tau)$ every Killing field is helicoidal. One may ask if any Killing field is helicoidal in an arbitrary manifold. One of the results in this section is that this is not true in general.

Sol(3) is one of the 8 geometries dealt with Thurston, which is not represented by $E(\kappa,\tau)$. To emphasize this fact, we use not x_1, x_2, x_3 but x, y, z to denote the variables for a coordinate system of Sol(3), which is $\mathbb{R}^3$ equipped with the Lie group structure

$$(x_1, y_1, z_1)\cdot(x_2, y_2, z_2) = (x_1 + e^{-z_1}x_2, y_1 + e^{-z_1}y_2, z_1 + z_2)$$

and the left invariant metric

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

In this section, we will show that:

Theorem 5.1. *The followings hold*:

(i) *In Sol*(3)*, there are both helicoidal Killing fields and non-helicoidal Killing fields.*

(ii) *Any helicoid in Sol*(3) *is congruent to*

$$\underline{a}x + \underline{b}y + \underline{c} = 0, \tag{28}$$

where a, b, c *are arbitrary constants with* $\underline{a}^2 + \underline{b}^2 > 0$.

(iii) *In Sol*(3) *the following is a ruled minimal surface which is not a helicoid.*

$$y = xe^{2z}. \tag{29}$$

As we shall see in the proof, (29) is obtained from a certain Killing field *which is helicoidal in some direction.* From Proposition 3.2, we know that (29) are ruled minimal surfaces. So we know at least two kinds of ruled minimal surfaces in Sol(3). But at the moment, we do not know if these are all or not.

The rest of this section is devoted to a proof of the theorem. The connection is given by

$$\begin{aligned}
&\nabla_{\partial_x}\partial_x = -e^{2z}\partial_z, && \nabla_{\partial_y}\partial_x = 0, && \nabla_{\partial_z}\partial_x = \partial_x,\\
&\nabla_{\partial_x}\partial_y = 0, && \nabla_{\partial_y}\partial_y = e^{-2z}\partial_z, && \nabla_{\partial_z}\partial_y = -\partial_y,\\
&\nabla_{\partial_x}\partial_z = \partial_x, && \nabla_{\partial_y}\partial_z = -\partial_y, && \nabla_{\partial_z}\partial_z = 0.
\end{aligned}$$

The following

$$(x, y, z) \mapsto (-x, y, z), \quad (x, y, z) \mapsto (x, -y, z), \quad (x, y, z) \mapsto (y, x, -z)$$

are reflections, and the following

$$(x,y,z) \mapsto (x+\underline{c},y,z), \quad (x,y,z) \mapsto (x,y+\underline{c},z), \quad (x,y,z) \mapsto (e^{-\underline{c}}x, e^{\underline{c}}y, z+\underline{c})$$

are one-parameter groups of isometries of Sol(3), whose Killing fields are

$$K_1 := \partial_x, \qquad K_2 := \partial_y, \qquad K_3 := -x\partial_x + y\partial_y + \partial_z.$$

Any Killing field K can be written as $K = \underline{a}\partial_x + \underline{b}\partial_y + \underline{c}(-x\partial_x + y\partial_y + \partial_z)$ for some constants $\underline{a}, \underline{b}, \underline{c}$.

If $\underline{c} = 0$, then $K = \underline{a}\partial_x + \underline{b}\partial_y$, and $\nabla_K K = (-\underline{a}^2 e^{2z} + \underline{b}^2 e^{-2z})\partial_z$. For $\tilde{V} := \partial_z$, we see that $\nabla_{\tilde{V}}\tilde{V} \times \tilde{V} = 0$, hence K is helicoidal. From the process in §3 we get the helicoid $(\underline{a}s + \underline{a}_0, \underline{b}s + \underline{b}_0, t)$, which can be written as $-\underline{b}x + \underline{a}y = \underline{c}$.

If $\underline{c} \neq 0$, then $K = \underline{c}\Big(-(x - \frac{\underline{a}}{\underline{c}})\partial_x + (y + \frac{\underline{b}}{\underline{c}})\partial_y + \partial_z\Big)$ which is the same as $\underline{c}K_3$ up to isometries. Hence, the integral curves of $\nabla_K K$ is the same as the integral curves of $\nabla_{K_3} K_3$, up to isometries. Direct calculations show that $\nabla_{K_3} K_3 = -\tilde{V}_3$ where $\tilde{V}_3 := x\partial_x + y\partial_y + (x^2e^{2z} - y^2e^{-2z})\partial_z$, and

$$\begin{aligned}\nabla_{\tilde{V}_3}\tilde{V}_3 &= x(1 + 2x^2e^{2z} - 2y^2e^{-2z})\partial_x + y(1 - 2x^2e^{2z} + 2y^2e^{-2z})\partial_y \\ &\quad + (x^2e^{2z} - y^2e^{-2z})(1 + 2x^2e^{2z} + 2y^2e^{-2z})\partial_z.\end{aligned}$$

Then we see that $\nabla_{\tilde{V}_3}\tilde{V}_3 \times \tilde{V}_3 = 0$ is equivalent to

$$\begin{aligned}-4xy(x^2e^{2z} - y^2e^{-2z}) &= 0, \quad \text{and} \\ -4xy^2e^{-2z}(x^2e^{2z} - y^2e^{-2z}) &= 0, \quad \text{and} \\ 4yx^2e^{2z}(x^2e^{2z} - y^2e^{-2z}) &= 0,\end{aligned}$$

which is again equivalent to

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad x^2e^{2z} - y^2e^{-2z} = 0 \tag{30}$$

along the integral curve. One consequence of this observation is that K_3 is not helicoidal.

For integral curves $\gamma(t) := (x(t), y(t), z(t))$ of $\tilde{V}_3$, we have $\gamma'(t) = \tilde{V}_3$ or, in components

$$(x'(t), y'(t), z'(t)) = (x(t), y(t), x(t)^2e^{2z(t)} - y(t)^2e^{-2z(t)}).$$

If $x(t) = 0$ or $y(t) = 0$ or $x(t)^2e^{2z(t)} - y(t)^2e^{-2z(t)} = 0$, as in (30), then the above integrates to

$$\begin{aligned}&\gamma_1(t) := (0, \underline{b}e^t, \ln\sqrt{C - \underline{b}^2e^{2t}}) \quad \text{or} \\ &\gamma_2(t) := (\underline{a}e^t, 0, \ln\frac{1}{\sqrt{C - \underline{a}^2e^{2t}}}) \quad \text{or} \\ &\gamma_3(t) := (\underline{a}e^t, \underline{b}e^t, \underline{c}),\end{aligned}$$

respectively. These are the only geodesics among the integral curves of $\nabla_{K_3} K_3$.

Translations of each of the geodesics $\gamma_1(t)$, $\gamma_2(t)$, $\gamma_3(t)$ by the one parameter group of isometries $(x, y, z) \mapsto (e^{-s}x, e^s y, z + s)$ for X_3 gives

$$x = 0 \qquad \text{or} \quad y = 0 \qquad \text{or} \quad (s, t) \mapsto (\underline{a}e^{t-s}, \pm\underline{a}e^{t+2\underline{c}+s}, \underline{c} + s),$$

respectively. Note that the last one can be rewritten as $(s, t) \mapsto (\underline{a}e^{t-s}, \pm\underline{a}e^{t+s}, s)$, which is $xe^z = \pm ye^{-z}$ in nonparametric form. Therefore, up to isometry,

$$\underline{b}x - \underline{a}y + \underline{c} = 0 \qquad \text{or} \qquad y = \pm xe^{2z}$$

are all the helicoids in Sol(3). Among them, $x = x_0$ and $y = y_0$ are totally geodesic. In fact, they are the only totally geodesic surfaces in Sol(3) [9].

Remark 5.2. Some of the calculations in §4 and §5 are done with Mathematica®.

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