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ON Φ-FLAT MODULES AND Φ-PRÜFER RINGS

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Abstract. Let R be a commutative ring with non-zero identity and let $NN(R) = \{I \mid I$ is a nonnil ideal of $R\}$. Let M be an R-module and let ϕ-tor$(M) = \{x \in M \mid Ix = 0$ for some $I \in NN(R)\}$. If ϕ-tor$(M) = M$, then M is called a ϕ-torsion module. An R-module M is said to be ϕ-flat, if $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is an exact R-sequence, for any exact sequence of R-modules $0 \to A \to B \to C \to 0$, where C is ϕ-torsion.

In this paper, the concepts of NRD-submodules and NP-submodules are introduced, and the ϕ-flat modules over a ϕ-Prüfer ring are investigated.

1. Introduction

Throughout this paper, it is assumed that all rings are commutative and associative with non-zero identity and all modules are unitary. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Nil(R)$ denotes the set of its nilpotent elements, and $Z(R)$ denotes the set of zero-divisors of R. An ideal I of a ring R is said to be a *nonnil ideal* if $I \nsubseteq Nil(R)$. Recall from [15] and [4] that a prime ideal P of R is called *divided* if $P \subset (x)$ for each $x \in R \backslash P$. Set $\mathcal{H} = \{R \mid R$ is a commutative ring and $Nil(R)$ is a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. If $R \in \mathcal{H}$ and $Nil(R) = Z(R)$, then R is called a strongly ϕ-ring, and denoted by $R \in S\phi R$. Recall from [5] that for a ring $R \in \mathcal{H}$ with total quotient ring $T(R)$, the map $\phi : T(R) \to R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for $a \in R$ and $b \notin Z(R)$ is a ring homomorphism from $T(R)$ into $R_{Nil(R)}$, and ϕ restricted to R is also a ring homomorphism from R into $R_{Nil(R)}$ given by $\phi(x) = x/1$ for each $x \in R$.

Recently, the authors in [1,2,14], and [20] generalized the concept of Prüfer domains, Bezout domains, Dedekind domains, Krull domains, Mori domains, and strongly Mori domains to the context of rings that are in the class $\mathcal{H}$.

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Also, the authors in [4–8], and [10] investigated the following classes of rings: ϕ-CR, ϕ-PVR, and ϕ-$ZPUI$. Furthermore, in [12], the authors investigated going-down ϕ-rings. The authors in [9, 13] and [18], introduced the notion of nonnil-Noetherian rings (later called ϕ-Noetherian rings). This notion was also extended to noncommutative rings in [21]. The authors in [11], stated many of the main results on ϕ-rings.

In order to investigate modules and ϕ-rings, the authors in [24], introduced ϕ-torsion modules and ϕ-torsion free modules, and investigated ϕ-flat modules and ϕ-von Neumann regular rings. The authors in [3] gave the concepts of nonnil-coherent rings and ϕ-coherent rings.

We recall that a valuation domain is a commutative integral domain such that for any two elements r and s, either r divides s or s divides r. This clearly implies that any finitely generated ideal is principal (and hence flat) and that for any two ideals I and J, either $I \subseteq J$ or $J \subseteq I$. In particular, a valuation domain is a local ring. A ring R is said to be a chained ring if for every $a, b \in R$, either $a|b$ or $b|a$ in R. Recall from [7] that a ring $R \in \mathcal{H}$ is called a ϕ-chained ring (ϕ-CR) if $x^{-1} \in \phi(R)$ for every $x \in R_{Nil(R)} \setminus \phi(R)$; equivalently, if for every $a, b \in R \setminus Nil(R)$, either $a|b$ or $b|a$ in R. The author in [23] showed that a finitely presented module over a valuation domain is a direct sum of cyclically presented modules. In this paper, the following result is shown.

Theorem. *A finitely presented ϕ-torsion module over a ϕ-chain ring is a direct sum of cyclically presented ϕ-torsion modules.*

In this paper, a submodule N of an R-module M is said to be *nonnil relatively divisible* in M, if $rN = N \cap rM$ holds for any $r \in R \setminus Nil(R)$. We denote briefly that N is an NRD-*submodule* of M. A submodule N of an R-module M is said to be *nonnil pure* in M, if $IN = N \cap IM$ holds for any $I \in NN(R)$. We denote briefly that N is an NP-*submodule* of M.

A Prüfer domain is an integral domain such that every finitely generated ideal is invertible (and hence projective). It is well known that a local domain is a Prüfer domain if and only if it is a valuation domain, and therefore, R is a Prüfer domain if and only if for each maximal ideal m, R_m is a valuation domain. A ring R is called a Prüfer ring, in the sense of [17], if every finitely generated regular ideal of R is invertible. Recall from [1] that R is called a ϕ-Prüfer ring if every finitely generated nonnil ideal of R is ϕ-invertible. This generalized the definition of Prüfer domain in $\mathcal{H}$. Here a nonnil ideal I of R is ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. The author in [23] showed that over Prüfer rings, relative divisibility and purity are equivalent. In this paper, the following result is shown, which generalizes the result in [16].

Theorem. *Over ϕ-Prüfer rings, nonnil relative divisibility and nonnil purity are equivalent.*

Anderson and Badawi showed in [1] that the following statements are equivalent for a ring R.

(1) R is a ϕ-Prüfer ring.
(2) $\phi(R)$ is a Prüfer ring.
(3) $\phi(R)/Nil(\phi(R))$ is a Prüfer domain.
(4) R_P is a ϕ-CR for each prime ideal P of R.
(5) $R_P/Nil(R_P)$ is a valuation domain for each prime ideal P of R.
(6) $R_M/Nil(R_M)$ is a valuation domain for each maximal ideal M of R.
(7) R_M is a ϕ-CR for each maximal ideal M of R.

In this paper, the ϕ-flat modules and ϕ-Prüfer rings are investigated, and the following result is shown.

Theorem. *Let $R \in \mathcal{H}$ and $Nil(R) = Z(R)$. The following statements are equivalent.*

(1) *R is a ϕ-Prüfer ring.*
(2) *All ϕ-torsion free R-modules are ϕ-flat.*
(3) *Each submodule of a ϕ-flat R-module is ϕ-flat.*
(4) *Each nonnil ideal of R is a ϕ-flat R-module.*
(5) *Each finitely generated nonnil ideal of R is a ϕ-flat R-module.*
(6) *If M is a ϕ-torsion R-module and N is a ϕ-torsion free R-module, then*
$$\mathrm{Tor}_1^R(M, N) = 0.$$
(7) *If M is a ϕ-torsion R-module and I is a nonnil ideal of R, then*
$$\mathrm{Tor}_1^R(M, I) = 0.$$
(8) *If M is a ϕ-torsion R-module and I is a finitely generated nonnil ideal of R, then*
$$\mathrm{Tor}_1^R(M, I) = 0.$$

2. On ϕ-torsion modules and ϕ-flat modules

Let R be a ϕ-ring. Set $Ker(\phi) = \{x \in R \,|\, xy = 0$ for some $y \in Z(R)$ and $y \notin Nil(R)\}$, then $\phi(R) = R/Ker(\phi)$. Observe that if $R \in \mathcal{H}$, then $\phi(R) \in \mathcal{H}$, $Ker(\phi) \subseteq Nil(R)$, $Nil(T(R)) = Nil(R)$, $Nil(R_{Nil(R)}) = \phi(Nil(R)) = Nil(\phi(R)) = Z(\phi(R))$, $T(\phi(R)) = R_{Nil(R)}$ is quasilocal with the maximal ideal $Nil(\phi(R))$, and $R_{Nil(R)}/Nil(\phi(R)) = T(\phi(R))/Nil(\phi(R))$ is the quotient field of $\phi(R)/Nil(\phi(R)) \cong R/Nil(R)$.

Proposition 2.1. *Let $R \in \mathcal{H}$ and $\phi : R \to R_{Nil(R)}$ such that $\phi(a) = a/1$ for $a \in R$. Then ϕ is a monomorphism if and only if $Ker(\phi) = 0$, if and only if $Nil(R) = Z(R)$.*

Proof. Since $Nil(R)$ is a prime ideal of R, we have that $Ker(\phi) = 0$ if and only if $Nil(R) = Z(R)$. □

Set $NN(R) = \{I \,|\, I$ is a nonnil ideal of ring $R\}$. Let M be an R-module. We define
$$\phi - \mathrm{tor}(M) = \{x \in M \,|\, Ix = 0 \text{ for some } I \in NN(R)\}.$$

If ϕ-tor$(M) = M$, then M is called a *ϕ-torsion module*, and if ϕ-tor$(M) = 0$, then M is called a *ϕ-torsion free module*. Clearly, submodules and quotient modules of ϕ-torsion modules are still ϕ-torsion; submodules of ϕ-torsion free modules are still ϕ-torsion free.

Proposition 2.2. *Let R be a commutative ring with prime nil radical. Then R is a ϕ-torsion free R-module if and only if $Nil(R) = Z(R)$.*

Proof. Observe that $I \in NN(R)$ if and only if there is an element $r \in I \backslash Nil(R)$. Thus R is a ϕ-torsion free R-module if and only if $Ker(\phi) = 0$, if and only if $Nil(R) = Z(R)$. □

Example 2.3. If S is the multiplicative set of all non-zero-divisors in the ring R, then $S^{-1}R/R$ is a ϕ-torsion R-module. If the nil radical of R is prime, then $R_{Nil(R)}/R$ is ϕ-torsion R-module.

If $Nil(R)$ is a prime ideal, then ϕ-tor(M) is a submodule of M which is called the *total ϕ-torsion* submodule of M. Set $T = \phi$-tor(M). Then T is always ϕ-torsion and M/T is always ϕ-torsion free. If R is a commutative ring with prime nil radical, then

(1) A module T is ϕ-torsion if and only if $\mathrm{Hom}_R(T, F) = 0$ for any ϕ-torsion free module F.

(2) A module F is ϕ-torsion free if and only if $\mathrm{Hom}_R(T, F) = 0$ for any ϕ-torsion module T.

Proposition 2.4. *Let R be a commutative ring with prime nil radical and $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then B is ϕ-torsion if and only if A and C are both ϕ-torsion. Moreover, $\bigoplus_{i\in\Gamma} M_i$ is a ϕ-torsion module if and only if each M_i is a ϕ-torsion module.*

Proof. We only need to consider the long exact sequence

$$0 \to \mathrm{Hom}_R(C, F) \to \mathrm{Hom}_R(B, F) \to \mathrm{Hom}_R(A, F) \to \mathrm{Ext}^1_R(C, F) \to \cdots .$$ □

Recall from [24] that an R-module M said to be *ϕ-flat*, if $0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$ is an exact R-sequence, for any exact sequence of R-modules $0 \to A \to B \to C \to 0$, where C is ϕ-torsion. The following conditions are shown to be equivalent for an R-module M.

(a) M is ϕ-flat.

(b) $\mathrm{Tor}^R_1(P, M) = 0$ for any ϕ-torsion R-module P.

(c) $\mathrm{Tor}^R_1(R/I, M) = 0$ for any nonnil ideal I of R.

(d) $0 \to I \otimes_R M \to R \otimes_R M$ is an exact R-sequence for any nonnil ideal I of R.

(e) $I \otimes_R M \cong IM$ for any nonnil ideal I of R.

(f) $0 \to N \otimes_R M \to F \otimes_R M \to C \otimes_R M \to 0$ is an exact R-sequence, for any exact sequence of R-modules $0 \to N \to F \to C \to 0$, where N, F, C are finitely generated, C is ϕ-torsion, and F is free.

(g) $0 \to N \otimes_R M \to F \otimes_R M \to C \otimes_R M \to 0$ is an exact R-sequence, for any exact sequence of R-modules $0 \to N \to F \to C \to 0$, where C is ϕ-torsion, and F is free.

(h) $\mathrm{Tor}_1^R(R/I, M) = 0$ for any finitely generated nonnil ideal I of R.

(i) $0 \to I \otimes_R M \to R \otimes_R M$ is an exact R-sequence for any finitely generated nonnil ideal I of R.

(j) $I \otimes_R M \cong IM$ for any finitely generated nonnil ideal I of R.

(k) $\mathrm{Ext}_R^1(I, M^+) = 0$ for any nonnil ideal I of R, where M^+ denotes the character R-module $\mathrm{Hom}_Z(M, Q/Z)$.

(l) Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \cap FI = IK$ for any nonnil ideal I of R.

(m) Let $0 \to K \to F \xrightarrow{g} M \to 0$ be an exact sequence of R-modules, where F is free. Then $K \cap FI = IK$ for any finite generated nonnil ideal I of R.

Proposition 2.5. (a) *Let R be a commutative ring with prime nil radical and $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. If A and C is ϕ-flat, then B is ϕ-flat.*

(b) *Let R be a strongly ϕ-ring. Then each ϕ-flat R-module is ϕ-torsion free.*

Proof. (a) We only need to consider the long exact sequence

$$\cdots \to \mathrm{Tor}_1^R(C, F) \to A \otimes_R F \to B \otimes_R F \to C \otimes_R F \to 0.$$

(b) If R is a strongly ϕ-ring, then R is a ϕ-torsion free R-module. $R_{Nil(R)}/R$ being a ϕ-torsion R-module implies that

$$0 \to M = R \otimes_R M \to R_{Nil(R)} \otimes_R M = M_{Nil(R)}$$

is exact sequence for an R-module M. If $J \in NN(R)$ and $x \in M$ such that $Jx = 0$, then there is an element $s \in R$, $s \notin Nil(R)$ such that $x = \frac{x}{1} = \frac{sx}{s} = 0$. Hence M is ϕ-torsion free. □

3. On NRD-submodules and NP-submodules

Recalled from [23] that a submodule N of an R-module M is said to be relatively divisible in M, if $rN = N \cap rM$ holds for any $r \in R$. Analogously, we have

Definition 3.1. A submodule N of an R-module M is said to be nonnil relatively divisible in M, if $rN = N \cap rM$ holds for any $r \in R \setminus Nil(R)$. We denote briefly that N is an NRD-*submodule* of M.

As the inclusion $rN \subseteq N \cap rM$ holds for all submodules N of M, nonnil relatively divisibility holding amounts to the reverse inclusion, i.e., if for any $r \in R \setminus Nil(R)$, the equation $rx = a \in N$ has a solution for x in M, then it is solvable in N as well. It is clear that a relatively divisible submodule N of R-module M is also nonnil relatively divisible in M, but the converse may be not true. For example, $Ker(\phi)$ is nonnil relatively divisible in R but not relatively divisible in R. The following properties are clear.

(a) Nonnil relatively divisibility is also transitive: if L is an NRD-submodule of N and N is an NRD-submodule of M, then L is an NRD-submodule of M.

(b) If $L \subseteq N \subseteq M$ and N is an NRD-submodule of M, then N/L is an NRD-submodule of M/L.

(c) If $L \subseteq N \subseteq M$ and L is an NRD-submodule of M, then N/L being an NRD-submodule of M/L implies N is an NRD-submodule of M.

Theorem 3.2. *Let $0 \to N \to M \overset{\beta}{\to} C \to 0$ be a short exact sequence of R-modules.*

(a) *If C is ϕ-torsion free, then N is an* NRD*-submodule of M.*

(b) *If M is ϕ-torsion free and N is an* NRD*-submodule of M, then C is ϕ-torsion free.*

Proof. (a) For any $r \in R\backslash Nil(R)$ and $rx = a \in N, x \in M$, we have $r\beta(x) = 0$ in C. Set $I = Rr \in NN(R)$, C being a ϕ-torsion free R-module implies $\beta(x) = 0$, and hence $x \in N$. So N is an NRD-submodule of M.

(b) If $I \in NN(R)$ and $Ix = 0$ in C, there is an element $y \in M$ such that $x = \beta(y)$. We have $Iy \subseteq N$, and there exists $r \in R \setminus Nil(R)$ such that $ry = a \in N$. N being an NRD-submodule of M implies that there is an element $z \in N$ such that $rz = a$. Hence $r(y - z) = 0$, so $y = z \in N$, and $x = \beta(y) = 0$. Therefore C is ϕ-torsion free. □

Theorem 3.3. *Let $0 \to N \to M \overset{\beta}{\to} C \to 0$ be a short exact sequence of R-modules. If the natural homomorphism* $\mathrm{Hom}_R(R/Rr, M) \to \mathrm{Hom}_R(R/Rr, C)$ *is surjective for any $r \in R \setminus Nil(R)$, then N is an* NRD*-submodule of M. Moreover, if M is ϕ-torsion free, the converse holds.*

Proof. For any $r \in R \setminus Nil(R)$ and $rx = a \in N, x \in M$, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (r) & \xrightarrow{i} & R & \xrightarrow{\pi} & R/(r) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{\beta} & C & \longrightarrow & 0, \end{array}$$

where π is the natural homomorphism, $f(r) = a, g(1) = x$, and h is the homomorphism induced by the left square. If the natural homomorphism $\mathrm{Hom}_R(R/Rr, M) \to \mathrm{Hom}_R(R/Rr, C)$ is surjective for any $r \in R \setminus Nil(R)$, then there exists a homomorphism $\rho : R/(r) \to M$ such that $h = \beta\rho$. By lemma 8.4 in [16], there is a homomorphism $\sigma : R \to N$ such that $f = \sigma i$. Set $\sigma(1) = c \in N$, we have $rc = a$. Hence N is an NRD-submodule of M.

Now assume that M is ϕ-torsion free. If $r \in R\backslash Nil(R)$ and $h \in \mathrm{Hom}_R(R/Rr, C)$, the projective property of R implies that there is a homomorphism $g : R \to M$ such that $\beta g = h\pi$. Hence the right square induces a homomorphism f. Set $f(r) = a, g(1) = x$, so $rx = a \in N, x \in M$. □

Theorem 3.4. *Let* $0 \to N \to M \to C \to 0$ *be a short exact sequence of* R*-modules. Then* N *is an* NRD*-submodule of* M *if and only if the natural homomorphism* $R/rR \otimes_R N \to R/rR \otimes_R M$ *is injective for any* $r \in R \setminus Nil(R)$.

Proof. Because of the natural isomorphism $R/Rr \otimes_R M \cong M/rM$, we only consider the homomorphism $N/rN \xrightarrow{f} M/rM$ with $f : x + rN \to x + rM$. If $x + rM = 0$, i.e., $x = ry$ for some $y \in M$, N being an NRD-submodule of M implies $x = ry'$ for some $y' \in N$, and hence $x + rN = 0$, so f is injective.

For the converse, $x = ry, y \in M, x \in N$ implies $x + rM = 0$ in M/rM. If the homomorphism f is injective, then $x + rN = 0$ in N/rN. Therefore, $x = ry'$ for some $y' \in N$, and hence N is an NRD-submodule of M. □

Theorem 3.5. *An* R*-module* N *is an* NRD*-submodule of* R*-module* M *if and only if* N_m *is an* NRD*-submodule of* M_m *as* R_m*-module for any* $m \in \mathrm{Max}(R)$.

Proof. We have that N is an NRD-submodule of M if and only if the natural homomorphism $R/rR \otimes_R N \to R/rR \otimes_R M$ is injective, if and only if $R/rR \otimes_R N \otimes R_m \to R/rR \otimes_R M \otimes R_m$ is injective for any maximal ideal m of R, if and only if N_m is an NRD-submodule of M_m for any m. □

Definition 3.6. A submodule N of an R-module M is said to be nonnil pure in M, if $IN = N \cap IM$ holds for any $I \in NN(R)$. We denote briefly that N is an NP-*submodule* of M.

As the inclusion $IN \subseteq N \cap IM$ holds for all modules N of M, nonnil relatively divisibility holding amounts to the reverse inclusion, i.e., if for any $I \in NN(R)$, the equation $\sum_{i=1}^{n} r_i x_i = a \in N$ has a solution for x_i in M, then it is solvable in N as well. It is clear that N being an NP-submodule of M implies N being an NRD-submodule of M.

Theorem 3.7. *Let* $0 \to N \to M \xrightarrow{\beta} C \to 0$ *be a short exact sequence of* R*-modules.*

(a) *If* C *is* ϕ*-flat, then* N *is an* NP*-submodule of* M.

(b) *If* M *is* ϕ*-flat and* N *is an* NP*-submodule of* M, *then* C *is* ϕ*-flat.*

Proof. (a) Consider the following homomorphism

$$\beta_0 : IM \to IC,\ \beta_0(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i g(x_i),$$

where $a_i \in I, x_i \in M$. It is clear that $\ker(\beta_0) = N \cap IM$, and there is a short exact sequence

$$0 \to N \cap IM \to IM \to IC \to 0.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} N & \longrightarrow & I \otimes N & \longrightarrow & I \otimes M & \longrightarrow & I \otimes C & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & N \cap IM & \longrightarrow & IM & \xrightarrow{\beta_0} & IC & \longrightarrow & 0, \end{array}$$

where f, g, h are the natural homomorphisms. The R-module C being ϕ-flat implies by Theorem 3.2 in [24] that h is an isomorphism for any nonnil ideal I of R. The Snake lemma implies that f is an epimorphism. So $N \cap IM = IN$, and hence N is an NP-submodule of M.

(b) If N is an NP-submodule of M, then $N \cap IM = IN$ for any nonnil ideal I of R. There is a short exact sequence

$$0 \to IN \to IM \to IC \to 0.$$

Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & I \otimes N & \longrightarrow & I \otimes M & \longrightarrow & I \otimes C & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & IN & \longrightarrow & IM & \longrightarrow & IC & \longrightarrow & 0. \end{array}$$

The R-module M being ϕ-flat implies that g is an isomorphism for any nonnil ideal I of R. Therefore, h is an isomorphism, and hence C is ϕ-flat. □

Theorem 3.8. *Let $0 \to N \to M \to C \to 0$ be a short exact sequence of R-modules. Then N is an* NP*-submodule of M if and only if the natural homomorphism $T \otimes_R N \to T \otimes_R M$ is injective for any finitely presented ϕ-torsion R-module T.*

Proof. We suppose N is an NP-submodule of M, so C is a ϕ-flat R-module, hence $\mathrm{Tor}_1^R(T, C) = 0$ implies that the natural homomorphism $T \otimes_R N \to T \otimes_R M$ is injective for any finitely presented ϕ-torsion R-module T.

For the converse, if T is a finitely presented ϕ-torsion R-module, then there is a short exact sequence of R-modules $0 \to K \to F \to T \to 0$, where F, K are finitely generated and F is free. If the natural homomorphism $T \otimes_R N \to T \otimes_R M$ is injective for any finitely presented ϕ-torsion R-module T, i.e., $\mathrm{Tor}_1^R(T, C) = 0$, then C is ϕ-flat by theorem 3.2 in [24], hence N is a NP-submodule of M. □

Theorem 3.9. *An R-module N is an* NP*-submodule of an R-module M if and only if N_m is an* NP*-submodule of M_m as an R_m-module for any $m \in \mathrm{Max}(R)$.*

Proof. We have that N is an NP-submodule of M if and only if the natural homomorphism $R/I \otimes_R N \to R/I \otimes_R M$ is injective, if and only if $R/I \otimes_R N \otimes R_m \to R/I \otimes_R M \otimes R_m$ is injective for any maximal ideal m of ring R. Noted that for every nonnil ideal J of R_m, there is a nonnil ideal I of R such that $J = I_m$. This implies that N_m is an NP-submodule of M_m for any m. □

4. On ϕ-Prüfer rings

A valuation domain is a commutative integral domain such that for any two elements r and s, either r divides s or s divides r. A ring R is said to be a chained ring if for every $a, b \in R$, either $a|b$ or $b|a$ in R. Recall from [7] that a ring $R \in \mathcal{H}$ is called a ϕ-chained ring (ϕ-CR) if $x^{-1} \in \phi(R)$ for every $x \in R_{Nil(R)}$ $\phi(R)$. The author in [23] showed that a finitely presented module over a valuation domain is a direct sum of cyclically presented modules. Similarly, we have the following result.

Theorem 4.1. *A finitely presented ϕ-torsion module over a ϕ-chain ring is a direct sum of cyclically presented ϕ-torsion modules.*

Proof. The proof is completed by the following several steps.

(1) If R is a ϕ-chain ring, then $R/Nil(R)$ is a valuation domain. Hence the nilradical $Nil(R)$ is the only minimal prime ideal and the Jacobson radical $J = J(R)$ is the only maximal ideal of R. If M is a finitely presented ϕ-torsion R-module, then M/JM is a finitely generated R/J-module. Set

$$M/JM = \sum_{i=1}^{n} R/J \cdot y_i,$$

where $y_i = x_i + JM$, and $x_i \in M$ are representative elements of y_i for $1 \leqslant i \leqslant n$. By Nakayama lemma, we have $M = \sum_{i=1}^{n} R \cdot x_i$.

(2) We show that a finitely generated module M over $R \in \mathcal{H}$ is ϕ-torsion if and only if the annihilator $Ann(M) \supset Nil(R)$. If $Ann(M) \supset Nil(R)$, then there is an element $r \notin Nil(R)$ such that $rM = 0$, and hence M is ϕ-torsion. For the converse, if $M = \sum_{i=1}^{n} R \cdot x_i$ is ϕ-torsion, then there are elements $r_i \notin Nil(R)$ such that $r_i x_i = 0$, and hence $r = \prod_{i=1}^{n} r_i \notin Nil(R)$ (note $Nil(R)$ is a prime ideal of R) such that $rM = 0$, so $Ann(M) \supset Nil(R)$.

(3) We show that there exists a coset y_i, say y_1, such that for any representative element a of y_1 ($y_i = a + JM$), $Ann(M) = Ann(a)$. Otherwise, for any y_i, there exists $a_i \in M$ such that $Ann(a_i) \supset Ann(M) \supset Nil(R)$ for all $1 \leqslant i \leqslant n$. R being a ϕ-chain ring implies a contradiction to $Ann(M) = \cap_{i=1}^{n} Ann(a_i)$.

(4) We show that $M_1 = Ra$ is an NRD-submodule of M. Suppose that $r \notin Nil(R), rx = sa \in Ra, sa \neq 0$, then $s \notin Nil(R)$ by $Ann(M) \supset Nil(R)$. If $s = rt$ for some $t \in R$, then $x = ta \in Ra$ is a solution, and hence M_1 is an NRD-submodule of M. If $r = sp$ for some $p \in J(R)$, then $s(a - px) = 0$, so $s \in Ann(a - px) = Ann(M) = Ann(a)$, this is a contradiction to $sa \neq 0$.

(5) We continue with an induction on the number of generators. Applying the induction hypothesis to M/M_1, we note that the preimages of NRD-submodules of M/M_1 are NRD-submodules in M. Therefore, there exists a finite chain

$$0 = M_0 < M_1 < \cdots < M_n = M$$

of submodules such that each M_i is an NRD-submodule of M, and the factor M_{i+1}/M_i is a cyclic ϕ-torsion R-module for each $0 \leqslant i \leqslant n-1$.

(6) Let T be a finitely presented cyclic ϕ-torsion R-module. We show that $T \cong R/(a)$ for some $a \notin Nil(R)$. Because, there is a short exact sequence

$$0 \to K \to R \to T \to 0,$$

where $K = Ann(a)$ is a finitely generated nonnil ideal of R. R being a ϕ-chain ring implies that K is a principal ideal, say $K = Ra, a \notin Nil(R)$, hence $T \cong R/(a)$.

(7) Consider the short exact sequence

$$0 \to M_{n-1} \to M \to M/M_{n-1} \to 0.$$

The projective property of M/M_{n-1} relative to this exact sequence implies that M/M_{n-1} is a summand of M, i.e., $M \cong M_{n-1} \bigoplus M/M_{n-1}$. Here M_{n-1} is likewise finitely generated and has a smaller number of generators, so induction infers that

$$M \cong \bigoplus_{i=1}^{n} R/Ra_i, a_i \notin Nil(R).$$

□

A Prüer domain is an integral domain such that every finitely generated ideal is invertible. A domain R is a Prüer domain if and only if for each maximal ideal m, R_m is a valuation domain. A ring R is called a Prüfer ring, in the sense of [17], if every finitely generated regular ideal of R is invertible. Recall from [1] that R is called a ϕ-Prüfer ring if every finitely generated nonnil ideal of R is ϕ-invertible. This generalized the definition of Prüfer domain in $\mathcal{H}$. The author in [23] showed that over Prüfer rings, relative divisibility and purity are equivalent. Similarly, by Theorem 4.1 we have the following result.

Theorem 4.2. *Over ϕ-Prüfer rings, nonnil relative divisibility and nonnil purity are equivalent.*

Proof. By passing to the local case, we may as well assume that R is a ϕ-chain ring. We show that an NRD-submodule A is also an NP-submodule of B in the exact sequence $0 \to A \to B \to C \to 0$. For any finitely presented ϕ-torsion R-module T, we have that

$$T \cong \bigoplus_{i=1}^{n} R/Ra_i$$

for some $a_i \notin Nil(R)$. Therefore,

$$\mathrm{Tor}_1^R(T, C) \cong \mathrm{Tor}_1^R(\bigoplus_{i=1}^{n} R/Ra_i, C) \cong \bigoplus_{i=1}^{n} \mathrm{Tor}_1^R(R/Ra_i, C) = 0.$$

So C is ϕ-flat, and hence A is an NP-submodule of B. □

We know from [22] that the following statements are equivalent for a domain.
(1) R is a Prüfer domain;
(2) R_M is a valuation domain for each maximal ideal M of R;
(3) All torsion free R-modules are flat;
(4) Each submodule of a flat R-module is flat;
(5) Each ideal of R is flat;
(6) Each finitely generated ideal of R is flat.

Anderson and Badawi showed in [1] that the following statements are equivalent for a ϕ-ring.
(1) R is a ϕ-Prüfer ring;
(2) $\phi(R)$ is a Prüfer ring;

(3) $\phi(R)/Nil(\phi(R))$ is a Prüfer domain;
(4) R_P is a ϕ-CR for each prime ideal P of R;
(5) $R_P/Nil(R_P)$ is a valuation domain for each prime ideal P of R;
(6) $R_M/Nil(R_M)$ is a valuation domain for each maximal ideal M of R;
(7) R_M is a ϕ-CR for each maximal ideal M of R.

Theorem 4.3. *Let $R \in \mathcal{H}$ and $Nil(R) = Z(R)$. Then the following statements are equivalent.*

(1) *R is a ϕ-Prüfer ring.*
(2) *All ϕ-torsion free R-modules are ϕ-flat.*
(3) *Each submodule of a ϕ-flat R-module is ϕ-flat.*
(4) *Each nonnil ideal of R is a ϕ-flat R-module.*
(5) *Each finitely generated nonnil ideal of R is a ϕ-flat R-module.*
(6) *If M is a ϕ-torsion R-module and N is a ϕ-torsion free R-module, then $\mathrm{Tor}_1^R(M,N) = 0$.*
(7) *If M is a ϕ-torsion R-module and I is a nonnil ideal of R, then $\mathrm{Tor}_1^R(M,I) = 0$.*
(8) *If M is a ϕ-torsion R-module and I is a finitely generated nonnil ideal of R, then $\mathrm{Tor}_1^R(M,I) = 0$.*

Proof. (1) ⇒ (2) Let $R \in \mathcal{H}$ with $Nil(R) = Z(R)$. Then R is ϕ-torsion free as an R-module, and all ϕ-flat R-modules are ϕ-torsion free. Consider the exact sequence $0 \to K \to F \to M \to 0$, where F is a ϕ-torsion free R-module, we infer that M is ϕ-flat if and only if K is an NP-submodule of F, if and only if K is an NRD-submodule of F, if and only if M is ϕ-torsion free.

(2) ⇒ (3) Let K be a submodule of a ϕ-flat R-module F. Then F is a ϕ-torsion free R-module. So K is also ϕ-torsion free, and hence K is ϕ-flat.

(3) ⇒ (4) ⇒ (5) Notice that R is a ϕ-torsion free R-module.

(5) ⇒ (1) For each finitely generated nonzero ideal J of $R/Nil(R)$, there exists a finitely generated nonnil ideal I of R such that $J = I + Nil(R)$. Owing to I being ϕ-flat, we have J is a flat $R/Nil(R)$-module. Therefore $R/Nil(R)$ is a Prüfer domain, and hence R is a ϕ-Prüfer ring.

(2) ⇒ (6) ⇒ (7) ⇒ (8) It is clear.

(8) ⇒ (2) Observe Lemma 2.5. □

Notice that if R is not a strongly ϕ-ring, then the above results may not be true, because R is not a ϕ-torsion free R-module.

Also, we have the following result.

Theorem 4.4. *Let $R \in \mathcal{H}$. If each finitely generated nonnil ideal of R is flat, then R is a ϕ-Prüfer ring.*

Proof. It is true that I being flat implies that $J = I + Nil(R)$ is a flat $R/Nil(R)$-module. □

Recall from [19] that a ring R is said to be von Neumann regular if every R-module is flat and R is said to be π-regular if for each $r \in R$ there is a positive integer n and an element $x \in R$ such that $r^{2n}x = r^n$. The authors in [24] defined a ϕ-ring R to be a ϕ-von Neumann regular ring if every R-module is ϕ-flat. They showed that a ϕ-ring R is ϕ-von Neumann regular if and only if R is π-regular if and only if $R/Nil(R)$ is von Neumann regular. By above theorem, all ϕ-von Neumann regular rings are regarded as rings of dimension zero, and all ϕ-Prüfer rings are regarded as rings of dimension one.

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