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EULER SUMS OF GENERALIZED HYPERHARMONIC NUMBERS

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Abstract. The generalized hyperharmonic numbers $h_n^{(m)}(k)$ are defined by means of the multiple harmonic numbers. We show that the hyperharmonic numbers $h_n^{(m)}(k)$ satisfy certain recurrence relation which allow us to write them in terms of classical harmonic numbers. Moreover, we prove that the Euler-type sums with hyperharmonic numbers:

$$S\left(k,m;p\right) := \sum_{n=1}^{\infty} \frac{h_n^{(m)}\left(k\right)}{n^p} \quad \left(p \geq m+1,\ k=1,2,3\right)$$

can be expressed as a rational linear combination of products of Riemann zeta values and harmonic numbers. This is an extension of the results of Dil [10] and Mező [19]. Some interesting new consequences and illustrative examples are considered.

1. Introduction

Let $\mathbb{N} := \{1,2,3,\ldots\}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{N} \setminus \{1\} := \{2,3,4,\ldots\}$. Hyperharmonic numbers and their generalizations are classically defined by

$$h_n^{(m)}\left(k\right) := \sum_{\substack{1 \leq n_{m+k-1} < \cdots < n_m \\ \leq n_{m-1} \leq \cdots \leq n_1 \leq n}} \frac{1}{n_m n_{m+1} \cdots n_{m+k-1}}, \tag{1.1}$$

$$h_n^{(m)}\left(1\right) \equiv h_n^{(m)} := \sum_{1 \leq n_m \leq \cdots \leq n_1 \leq n} \frac{1}{n_m}, \tag{1.2}$$

where $k,m,n \in \mathbb{N}$ and for any $n < k$, we set $h_n^{(m)}\left(k\right) := 0$. When $k=1$ in (1.1), the number $h_n^{(m)}\left(1\right) \equiv h_n^{(m)}$ is called the classical hyperharmonic number (see [2, 9–12, 19]). In particular, the hyperharmonic number $h_n^{(1)}$ is simply called the classical harmonic number, which is the sum of the reciprocals of the first

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n natural numbers:

$$h_n^{(1)} \equiv H_n := \sum_{k=1}^{n} \frac{1}{k}.$$

Moreover, in [19], Mező and Dil showed that $h_n^{(m)}$ can be expressed by binomial coefficients and classical harmonic numbers:

$$h_n^{(m)} = \binom{n+m-1}{m-1} \left(H_{n+m-1} - H_{m-1}\right).$$

The n-th generalized harmonic numbers of order k, denoted by $H_n^{(k)}$, is defined by

$$H_n^{(k)} := \sum_{j=1}^{n} \frac{1}{j^k},\ n, k \in \mathbb{N}, \tag{1.3}$$

where the empty sum $H_0^{(k)}$ is conventionally understood to be zero, and $H_n^{(1)} \equiv H_n$. The limit as n tends to infinity exists if $k > 1$. In the limit of $n \to \infty$, the generalized harmonic number converges to the Riemann zeta value:

$$\lim_{n\to\infty} H_n^{(k)} = \zeta(k),\ \Re\,(k) > 1,\ k \in \mathbb{N},$$

where the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},\ \Re(s) > 1. \tag{1.4}$$

In general, for $r \in \mathbb{N}$, $\mathbf{s} := (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r$, and a non-negative integer n, the multiple harmonic number is defined by ([15])

$$H_n^{(s_1,s_2,\ldots,s_r)} := \sum_{1\le n_r<n_{r-1}<\cdots<n_1\le n} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_r^{s_r}}. \tag{1.5}$$

By convention, we put $H_n^{(\mathbf{s})} = 0$, if $n < r$, and $H_n^{(\emptyset)} = 1$. The limit cases of multiple harmonic numbers give rise to multiple zeta values ([25, 26]):

$$\zeta\left(s_1, s_2, \ldots, s_r\right) = \lim_{n\to\infty} H_n^{(s_1,s_2,\ldots,s_r)}$$

defined for $s_2, s_3, \ldots, s_r \ge 1$ and $s_1 \ge 2$ to ensure convergence of the series. Here, $w := s_1 + \cdots + s_r$ and r are called the weight and the multiplicity, respectively. To simplify the reading of such formulas, when a string of arguments is repeated an exponent is used. In other words, we treat string multiplication as concatenation. For example,

$$H_n^{\left(\underbrace{1,\ldots,1}_{r}\right)} = H_n^{(\{1\}^r)},\ H_n^{\left(\underbrace{2,\ldots,2}_{p},\underbrace{3,\ldots,3}_{r}\right)} = H_n^{(\{2\}^p,\{3\}^r)}.$$

With these notations, then the definition of hyperharmonic number $h_n^{(m)}(k)$ of formula (1.1) can be rewritten as

$$h_n^{(m)}(k) := \sum_{1\le n_m\le n_{m-1}\le\cdots\le n_1\le n} \frac{H_{n_m-1}^{(\{1\}^{k-1})}}{n_m}, \tag{1.6}$$

where $H_{n_m-1}^{(\{1\}^k)}$ is the multiple harmonic number $H_n^{(\{1\}^k)}$ with $n = n_m - 1$.

The subject of this paper is Euler-type sums $S(k,m;p)$, which is the infinite sum whose general term is a product of hyperharmonic numbers and a power of n^{-1}. Here, $p > m$ is both necessary and sufficient for the sum $S(k,m;p)$ to converge. The classical linear Euler sum is defined by ([13, 14])

$$S_{p,q} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}, \quad p \in \mathbb{N},\ q \in \mathbb{N}\setminus\{1\}. \tag{1.7}$$

The number $w = p + q$ is defined as the weight of $S_{p,q}$. The evaluation of $S_{p,q}$ in terms of values of Riemann zeta function at positive integers is known when $p = 1,\ p = q,\ (p,q) = (2,4), (4,2)$ or $p+q$ is odd (see [1,3,7,13]). For example, Euler discovered the following formula

$$S_{1,k} = \sum_{n=1}^{\infty} \frac{H_n}{n^k} = \frac{1}{2}\left\{(k+2)\zeta(k+1) - \sum_{i=1}^{k-2}\zeta(k-i)\zeta(i+1)\right\}. \tag{1.8}$$

Related series were studied by Borwein et al. [4,6], Markett [17], Mező [18], Sofo [20] and Xu et al. [21–23], for instance. Similarly, it has been discovered in the course of the years that many Euler type sums $S(k,m;p)$ admit expressions involving finitely the zeta values, that is to say values of the Riemann zeta function at the positive integer arguments, for more details, see for instance [10, 19]. For example, Dil and Boyadzhiev [10] gave explicit reductions to zeta values and (unsigned) Stirling numbers of the first kind for all sums $S(k,m;p)$ with $k = 1$. Here, the (unsigned) Stirling number of the first kind $\left[{n \atop k} \right]$ is defined by [8, 9]

$$n!x(1+x)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^{k+1} \tag{1.9}$$

with $\left[{n \atop k} \right] = 0$, if $n < k$ and $\left[{n \atop 0} \right] = \left[{0 \atop k} \right] = 0,\ \left[{0 \atop 0} \right] = 1$, or equivalently, by the generating function:

$$\log^k(1-x) = (-1)^k k! \sum_{n=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!},\ x \in [-1,1). \tag{1.10}$$

Moreover, the (unsigned) Stirling numbers $\left[{n \atop k} \right]$ of the first kind satisfy a recurrence relation in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}. \tag{1.11}$$

By the definition of $\left[{n \atop k}\right]$, we see that we may rewrite (1.9) as

$$\begin{aligned}\sum_{k=0}^{n}\begin{bmatrix}n+1\\k+1\end{bmatrix}x^k &= n!\exp\left\{\sum_{j=1}^{n}\ln\left(1+\frac{x}{j}\right)\right\}\\ &= n!\exp\left\{\sum_{j=1}^{n}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{kj^k}\right\}\\ &= n!\exp\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_n^{(k)}x^k}{k}\right\}.\end{aligned}$$

Therefore, we know that $\left[{n \atop k}\right]$ is a rational linear combination of products of harmonic numbers. Moreover, we deduce the following identities

$$\begin{aligned}&\begin{bmatrix}n\\1\end{bmatrix}=(n-1)!,\ \begin{bmatrix}n\\2\end{bmatrix}=(n-1)!H_{n-1},\ \begin{bmatrix}n\\3\end{bmatrix}=\frac{(n-1)!}{2}\left[H_{n-1}^2-H_{n-1}^{(2)}\right],\\ &\begin{bmatrix}n\\4\end{bmatrix}=\frac{(n-1)!}{6}\left[H_{n-1}^3-3H_{n-1}H_{n-1}^{(2)}+2H_{n-1}^{(3)}\right],\\ &\begin{bmatrix}n\\5\end{bmatrix}=\frac{(n-1)!}{24}\left[H_{n-1}^4-6H_{n-1}^{(4)}-6H_{n-1}^2H_{n-1}^{(2)}+3(H_{n-1}^{(2)})^2+8H_{n-1}H_{n-1}^{(3)}\right].\end{aligned}$$

In this paper we are interested in Euler-type sums with hyperharmonic numbers $S(k,m;p)$. Such series could be of interest in analytic number theory. We will prove that the generalized hyperharmonic number $h_n^{(m)}(k)$ can be expressed as a rational linear combination of products of harmonic numbers. Furthermore, we also provide an explicit evaluation of $S(k,m;p)$ with $k=2,3$ in a closed form in terms of zeta values and Stirling numbers of the first kind. The results which we present here can be seen as an extension of Mező and Dil's work.

2. Main theorems and their proof

In this section, we will show that the hyperharmonic number $h_n^{(m)}(k)$ is expressible in terms of harmonic numbers and give recurrence formula. We need the following lemma.

Lemma 2.1. *For positive integers n and k, then the following identity holds*:

$$\begin{bmatrix}n\\k\end{bmatrix}=(n-1)!H_{n-1}^{\left(\{1\}^{k-1}\right)}. \tag{2.1}$$

Proof. By considering the generating function (1.10), we know that we need to prove the following identity:

$$\log^k(1-x)=(-1)^k k!\sum_{n=1}^{\infty}H_{n-1}^{\left(\{1\}^{k-1}\right)}\frac{x^n}{n}. \tag{2.2}$$

To prove the identity we proceed by induction on k. Obviously, it is valid for $k=1$. For $k>1$ we use the integral identity

$$\log^{k+1}(1-x) = -(k+1)\int_0^x \frac{\log^k(1-t)}{1-t}dt$$

and apply the induction hypothesis, by using Cauchy product of power series, we arrive at

$$\begin{aligned}\log^{k+1}(1-x) &= -(k+1)\int_0^x \frac{\log^k(1-t)}{1-t}dt\\ &= (-1)^{k+1}(k+1)!\sum_{n=1}^{\infty}\frac{1}{n+1}\sum_{i=1}^{n}\frac{H_{i-1}^{\left(\{1\}^{k-1}\right)}}{i}x^{n+1}\\ &= (-1)^{k+1}(k+1)!\sum_{n=1}^{\infty}\frac{H_n^{\left(\{1\}^{k}\right)}}{n+1}x^{n+1}.\end{aligned}$$

Nothing that $H_n^{\left(\{1\}^{k}\right)}=0$ when $n<k$. Hence, we can deduce (2.2) holds. Thus, comparing the coefficients of x^n in (1.10) with (2.2), we obtain formula (2.1). The proof of Lemma 2.1 is completed. □

By using (1.6) and (2.2), we find that the generating function of hyperharmonic number $h_n^{(m)}(k)$ is given as

$$\sum_{n=1}^{\infty} h_n^{(m)}(k)\, z^n = \frac{(-1)^k}{k!}\frac{\log^k(1-z)}{(1-z)^m},\ z\in[-1,1)\,. \tag{2.3}$$

On the other hand, we note that the function on the right hand side of (2.3) is equal to

$$\frac{(-1)^k}{k!}\frac{\log^k(1-z)}{(1-z)^m} = \frac{1}{k!}\lim_{x\to m}\frac{\partial^k}{\partial x^k}\left(\frac{1}{(1-z)^x}\right)\ (k,m\in\mathbb{N}_0). \tag{2.4}$$

Therefore, the relations (2.3) and (2.4) yield the following result:

$$\sum_{n=1}^{\infty} h_n^{(m)}(k)\, z^n = \frac{1}{k!}\lim_{x\to m}\frac{\partial^k}{\partial x^k}\left(\frac{1}{(1-z)^x}\right). \tag{2.5}$$

Moreover, we know that the generating function of $(1-z)^{-x}$ is given as

$$\frac{1}{(1-z)^x} = \sum_{n=0}^{\infty}\frac{(x)_n}{n!}z^n,\ z\in(-1,1)\,, \tag{2.6}$$

where $(x)_n$ represents the Pochhammer symbol (or the shifted factorial) given by

$$(x)_n := x(x+1)\cdots(x+n-1) \tag{2.7}$$

with $(x)_0 := 1$. Hence, upon differentiating both members of (2.6) k times with respect to x then setting $x = m$, and combining (2.5), we readily arrive at the following relationship:

$$h_n^{(m)}(k) = \frac{1}{k!n!} \lim_{x\to m} \frac{\partial^k (x)_n}{\partial x^k}, \ k \in \mathbb{N}. \tag{2.8}$$

By convention, from (2.8), we define that

$$h_n^{(m)}(0) := \frac{1}{n!}(m)_n = \binom{m+n-1}{m-1}. \tag{2.9}$$

By simple calculation, the $\frac{\partial^k (x)_n}{\partial x^k}$ satisfy a recurrence relation in the form

$$\frac{\partial^k (x)_n}{\partial x^k} = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{\partial^i (x)_n}{\partial x^i} \left[\psi^{(m-i-1)}(x+n) - \psi^{(m-i-1)}(x)\right], \ k \in \mathbb{N}. \tag{2.10}$$

Here, $\psi^{(m)}(x)$ stands for the polygamma function of order m defined as the $(m+1)$ th derivative of the logarithm of the gamma function:

$$\psi^{(m)}(x) := \frac{d^m}{dx^m}\psi(x) = \frac{d^{m+1}}{dx^{m+1}} \log \Gamma(x).$$

Thus

$$\psi^{(0)}(x) = \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

holds where $\psi(x)$ is the digamma function and $\Gamma(x)$ is the gamma function. $\psi^{(m)}(x)$ satisfy the following relations

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z}\right), \ z \notin \mathbb{N}_0^- := \{0, -1, -2, \ldots\},$$

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} 1/(z+k)^{n+1}, \ n \in \mathbb{N},$$

$$\psi(x+n) = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n-1} + \psi(x), \ n \in \mathbb{N}.$$

Here, γ denotes the Euler-Mascheroni constant, defined by

$$\gamma := \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n\right) = -\psi(1) \approx 0.577215664901532860606512....$$

Hence, combining (2.8), (2.9) and (2.10), we obtain the recurrence relation

$$h_n^{(m)}(k) = \frac{(-1)^{k-1}}{k} \sum_{i=0}^{k-1} (-1)^i h_n^{(m)}(i) \left\{H_{m+n-1}^{(k-i)} - H_{m-1}^{(k-i)}\right\}. \tag{2.11}$$

By (2.11), we give the following description of hyperharmonic number $h_n^{(m)}(k)$.

Theorem 2.2. *For positive integers n and k, the hyperharmonic number $h_n^{(m)}(k)$ can be expressed in terms of ordinary harmonic numbers.*

For example, setting $m = 1, 2, 3, 4$ in the above equation (2.11) we obtain

$$h_n^{(m)}(1) = \binom{n+m-1}{m-1} (H_{n+m-1} - H_{m-1}), \tag{2.12}$$

$$h_n^{(m)}(2) = \frac{1}{2}\binom{n+m-1}{m-1} \left\{ (H_{n+m-1} - H_{m-1})^2 - \left(H_{n+m-1}^{(2)} - H_{m-1}^{(2)} \right) \right\}, \tag{2.13}$$

$$h_n^{(m)}(3) = \frac{1}{3!}\binom{n+m-1}{m-1} \left\{ \begin{array}{l} (H_{n+m-1} - H_{m-1})^3 + 2\left(H_{n+m-1}^{(3)} - H_{m-1}^{(3)} \right) \\ -3(H_{n+m-1} - H_{m-1}) \left(H_{n+m-1}^{(2)} - H_{m-1}^{(2)} \right) \end{array} \right\}, \tag{2.14}$$

$$h_n^{(m)}(4) = \frac{1}{4!}\binom{n+m-1}{m-1} \left\{ \begin{array}{l} (H_{n+m-1} - H_{m-1})^4 \\ +3\left(H_{n+m-1}^{(2)} - H_{m-1}^{(2)} \right)^2 - 6\left(H_{n+m-1}^{(4)} - H_{m-1}^{(4)} \right) \\ -6(H_{n+m-1} - H_{m-1})^2 \left(H_{n+m-1}^{(2)} - H_{m-1}^{(2)} \right) \\ +8(H_{n+m-1} - H_{m-1}) \left(H_{n+m-1}^{(3)} - H_{m-1}^{(3)} \right) \end{array} \right\}. \tag{2.15}$$

By replacing x by n and n by r in (1.9), we deduce that

$$\binom{n+r}{r} = \frac{1}{r!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1}. \tag{2.16}$$

Therefore, the relations (2.13), (2.14) and (2.16) yield the following results:

$$\begin{aligned} h_n^{(r+1)}(2) &= \frac{1}{2}\binom{n+r}{r} \left\{ (H_{n+r} - H_r)^2 - \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \right\} \\ &= \frac{1}{2!r!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1} \left\{ (H_{n+r} - H_r)^2 - \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \right\}, \end{aligned} \tag{2.17}$$

$$\begin{aligned} h_n^{(r+1)}(3) &= \frac{1}{3!}\binom{n+r}{r} \left\{ \begin{array}{l} (H_{n+r} - H_r)^3 + 2\left(H_{n+r}^{(3)} - H_r^{(3)} \right) \\ -3(H_{n+r} - H_r) \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \end{array} \right\} \\ &= \frac{1}{3!r!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1} \left\{ \begin{array}{l} (H_{n+r} - H_r)^3 + 2\left(H_{n+r}^{(3)} - H_r^{(3)} \right) \\ -3(H_{n+r} - H_r) \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \end{array} \right\}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} h_n^{(r+1)}(4) &= \frac{1}{4!}\binom{n+r}{r} \left\{ \begin{array}{l} (H_{n+r} - H_r)^4 + 3\left(H_{n+r}^{(2)} - H_r^{(2)} \right)^2 \\ -6\left(H_{n+r}^{(4)} - H_r^{(4)} \right) \\ -6(H_{n+r} - H_r)^2 \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \\ +8(H_{n+r} - H_r) \left(H_{n+r}^{(3)} - H_r^{(3)} \right) \end{array} \right\} \\ &= \frac{1}{4!r!} \sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1} \left\{ \begin{array}{l} (H_{n+r} - H_r)^4 + 3\left(H_{n+r}^{(2)} - H_r^{(2)} \right)^2 \\ -6\left(H_{n+r}^{(4)} - H_r^{(4)} \right) \\ -6(H_{n+r} - H_r)^2 \left(H_{n+r}^{(2)} - H_r^{(2)} \right) \\ +8(H_{n+r} - H_r) \left(H_{n+r}^{(3)} - H_r^{(3)} \right) \end{array} \right\}. \end{aligned} \tag{2.19}$$

Furthermore, using (2.17) and (2.18), by a direct calculation, we can give the following corollary.

Corollary 2.3. *For integers $r \in \mathbb{N}_0$ and $n \in \mathbb{N}$, we have*

(2.20)
$$h_n^{(r+1)}(2) = \frac{1}{2!r!}\sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1}\Big\{\Big(H_{n+r}^2 - H_{n+r}^{(2)}\Big) - 2H_r H_{n+r} + H_r^2 + H_r^{(2)}\Big\},$$

(2.21)
$$h_n^{(r+1)}(3) = \frac{1}{3!r!}\sum_{k=1}^{r+1} \begin{bmatrix} r+1 \\ k \end{bmatrix} n^{k-1}\left\{ \begin{array}{l} \Big(H_{n+r}^3 - 3H_{n+r}H_{n+r}^{(2)} + 2H_{n+r}^{(3)}\Big) - 3H_r\Big(H_{n+r}^2 - H_{n+r}^{(2)}\Big) \\ +3\Big(H_r^2 + H_r^{(2)}\Big)H_{n+r} - \Big(H_r^3 + 3H_r H_r^{(2)} + 2H_r^{(3)}\Big) \end{array} \right\}.$$

Moreover, from the definition of harmonic numbers $H_n^{(k)}$, we get

$$H_{n+r}^{(k)} = H_n^{(k)} + \sum_{j=1}^{r} \frac{1}{(n+j)^k}, \quad k, n \in \mathbb{N}. \tag{2.22}$$

By simple calculation, the following identities are easily derived

$$\begin{aligned} H_{n+r}^2 - H_{n+r}^{(2)} &= \left(H_n + \sum_{j=1}^{r}\frac{1}{n+j}\right)^2 - \left(H_n^{(2)} + \sum_{j=1}^{r}\frac{1}{(n+j)^2}\right) \\ &= H_n^2 - H_n^{(2)} + 2H_n\left(\sum_{j=1}^{r}\frac{1}{n+j}\right) + 2\sum_{1\le i<j\le r}\frac{1}{(n+i)(n+j)}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} &H_{n+r}^3 - 3H_{n+r}H_{n+r}^{(2)} + 2H_{n+r}^{(3)} \\ &= \left(H_n + \sum_{j=1}^{r}\frac{1}{n+j}\right)^3 + 2\left(H_n^{(3)} + \sum_{j=1}^{r}\frac{1}{(n+j)^3}\right) \\ &\quad - 3\left(H_n + \sum_{j=1}^{r}\frac{1}{n+j}\right)\left(H_n^{(2)} + \sum_{j=1}^{r}\frac{1}{(n+j)^2}\right) \\ &= H_n^3 - 3H_nH_n^{(2)} + 2H_n^{(3)} + 3\Big(H_n^2 - H_n^{(2)}\Big)\left(\sum_{j=1}^{r}\frac{1}{n+j}\right) \\ &\quad + 6H_n\left(\sum_{1\le i<j\le r}\frac{1}{(n+i)(n+j)}\right) \\ &\quad + 6\sum_{1\le i<j<k\le r}\frac{1}{(n+i)(n+j)(n+k)}. \end{aligned} \tag{2.24}$$

Now, we use the notation $W_{k,r}(m)$ to stands for the sum

$$W_{k,r}(m) := (k-1)! \sum_{n=1}^{\infty} \frac{\begin{bmatrix} n+r+1 \\ k \end{bmatrix}}{(n+r)!n^m}, \tag{2.25}$$

where $k \in \mathbb{N},\ r \in \mathbb{N}_0$ and $m \in \mathbb{N} \setminus \{1\}$. By using the above notation, we obtain

$$\begin{aligned} W_{1,r}(m) &= \zeta(m), \\ W_{2,r}(m) &= \sum_{n=1}^{\infty} \frac{H_{n+r}}{n^m}, \\ W_{3,r}(m) &= \sum_{n=1}^{\infty} \frac{H_{n+r}^2 - H_{n+r}^{(2)}}{n^m}, \\ W_{4,r}(m) &= \sum_{n=1}^{\infty} \frac{H_{n+r}^3 - 3H_{n+r}H_{n+r}^{(2)} + 2H_{n+r}^{(3)}}{n^m}. \end{aligned}$$

The following lemma will be useful in the development of the main theorem. Noting that when $r = 0$ and $k > 1$ in (2.25), then using (1.11), which can be rewritten as

$$\begin{aligned} W_{k,0}(m) :&= (k-1)! \sum_{n=1}^{\infty} \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}}{n!n^m} \\ &= (k-1)! \left(\sum_{n=1}^{\infty} \frac{\begin{bmatrix} n \\ k-1 \end{bmatrix}}{n!n^m} + \sum_{n=1}^{\infty} \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!n^{m-1}} \right) \\ &= (k-1)! \left(\zeta\left(m+1, \{1\}^{k-2}\right) + \zeta\left(m, \{1\}^{k-1}\right) \right). \end{aligned} \tag{2.26}$$

On the other hand, the Aomoto-Drinfel'd-Zagier formula reads

$$\sum_{n,m=1}^{\infty} \zeta\left(m+1, \{1\}^{n-1}\right) x^m y^n = 1 - \exp\left(\sum_{n=2}^{\infty} \zeta(n) \tfrac{x^n + y^n - (x+y)^n}{n} \right), \tag{2.27}$$

which implies that for any $m,\ n \in \mathbb{N}$, the multiple zeta value $\zeta\left(m+1, \{1\}^{n-1}\right)$ can be represented as a polynomial of zeta values with rational coefficients (see [5, 16]), and we have the duality formula

$$\zeta\left(n+1, \{1\}^{m-1}\right) = \zeta\left(m+1, \{1\}^{n-1}\right).$$

In particular, one can find explicit formulas for small weights $w := n + m$.

$$\zeta(2,\{1\}^m) = \zeta(m+2),$$
$$\zeta(3,\{1\}^m) = \frac{m+2}{2}\zeta(m+3) - \frac{1}{2}\sum_{k=1}^{m}\zeta(k+1)\zeta(m+2-k).$$

Hence, we know that for $m,\ k \in \mathbb{N}$, the sums $W_{k,0}(m)$ can be expressed as a rational linear combination of zeta values. For example, from [13, 24]

$$W_{2,0}(m) = \frac{1}{2}\left\{(m+2)\zeta(m+1) - \sum_{i=1}^{m-2}\zeta(m-i)\zeta(i+1)\right\},$$
$$W_{3,0}(m) = mW_{2,0}(m+1) - \frac{m(m+1)}{6}\zeta(m+2) + \zeta(2)\zeta(m).$$

Lemma 2.4 ([23]). *For integers $k \in \mathbb{N}$ and $p \in \mathbb{N}\setminus\{1\}$, the following identity holds*:

$$(p-1)!\sum_{n=1}^{\infty}\frac{\begin{bmatrix} n+1 \\ p \end{bmatrix}}{n!n(n+k)} = \frac{1}{k}\left\{(p-1)!\zeta(p) + \frac{Y_p(k)}{p} - \frac{Y_{p-1}(k)}{k}\right\}, \tag{2.28}$$

where $Y_k(n) = Y_k\left(H_n, 1!H_n^{(2)}, 2!H_n^{(3)}, \ldots, (r-1)!H_n^{(r)}, \ldots\right)$, $Y_k(x_1, x_2, \ldots)$ *stands for the complete exponential Bell polynomial defined by* (*see* [8])

$$\exp\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{k\geq 1} Y_k(x_1, x_2, \ldots)\frac{t^k}{k!}. \tag{2.29}$$

From the definition of the complete exponential Bell polynomial, we have

$$Y_1(n) = H_n,\ Y_2(n) = H_n^2 + H_n^{(2)},\ Y_3(n) = H_n^3 + 3H_nH_n^{(2)} + 2H_n^{(3)},$$
$$Y_4(n) = H_n^4 + 8H_nH_n^{(3)} + 6H_n^2H_n^{(2)} + 3(H_n^{(2)})^2 + 6H_n^{(4)},$$
$$Y_5(n) = H_n^5 + 10H_n^3H_n^{(2)} + 20H_n^2H_n^{(3)} + 15H_n(H_n^{(2)})^2 + 30H_nH_n^{(4)} + 20H_n^{(2)}H_n^{(3)} + 24H_n^{(5)}.$$

In fact, $Y_k(n)$ is a rational linear combination of products of harmonic numbers. Putting $p = 2, 3, 4$ in (2.28), we obtain the corollary.

Corollary 2.5. *For integer $k > 0$, we have*

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+k)} = \frac{1}{k}\left(\frac{1}{2}H_k^2 + \frac{1}{2}H_k^{(2)} + \zeta(2) - \frac{H_k}{k}\right), \tag{2.30}$$

$$\sum_{n=1}^{\infty}\frac{H_n^2 - H_n^{(2)}}{n(n+k)} = \frac{1}{k}\left\{2\zeta(3) + \frac{H_k^3 + 3H_kH_k^{(2)} + 2H_k^{(3)}}{3} - \frac{H_k^2 + H_k^{(2)}}{k}\right\}, \tag{2.31}$$

(2.32)

$$\sum_{n=1}^{\infty}\frac{H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}}{n\,(n+k)}=\frac{1}{k}\left\{\begin{array}{c}\frac{H_k^4+8H_kH_k^{(3)}+6H_k^2H_k^{(2)}+3(H_k^{(2)})^2+6H_k^{(4)}}{4}\\ -\frac{H_k^3+3H_kH_k^{(2)}+2H_k^{(3)}}{k}+6\zeta\left(4\right)\end{array}\right\}.$$

Hence, combining (2.22)-(2.25), (2.30) and (2.31), we deduce the following identities

$$\begin{aligned}W_{2,r}\left(m\right)&=\sum_{n=1}^{\infty}\frac{H_n}{n^m}+\sum_{j=1}^{r}\sum_{n=1}^{\infty}\frac{1}{n^m\left(n+j\right)}\\ &=W_{2,0}\left(m\right)+\sum_{l=1}^{m-1}\left(-1\right)^{l-1}\zeta\left(m+1-l\right)H_r^{(l)}+\left(-1\right)^{m-1}\sum_{j=1}^{r}\frac{H_j}{j^m},\end{aligned}\tag{2.33}$$

$$\begin{aligned}W_{3,r}\left(m\right)&=\sum_{n=1}^{\infty}\frac{H_n^2-H_n^{(2)}}{n^m}+2\sum_{j=1}^{r}\sum_{n=1}^{\infty}\frac{H_n}{n^m\left(n+j\right)}\\ &\quad+2\sum_{1\le i<j\le r}\sum_{n=1}^{\infty}\frac{1}{n^m\left(n+i\right)\left(n+j\right)}\\ &=W_{3,0}\left(m\right)+2\sum_{i=1}^{m-1}\left(-1\right)^{i-1}H_r^{(i)}W_{2,0}\left(m+1-i\right)\\ &\quad+2\sum_{l=1}^{m-1}\left(-1\right)^{l-1}\zeta\left(m+1-l\right)\sum_{1\le i<j\le r}\frac{1}{i^l\left(j-i\right)}\\ &\quad-2\sum_{l=1}^{m-1}\left(-1\right)^{l-1}\zeta\left(m+1-l\right)\sum_{1\le i<j\le r}\frac{1}{j^l\left(j-i\right)}\\ &\quad+\left(-1\right)^{m-1}\left\{\begin{array}{l}\sum_{j=1}^{r}\frac{H_j^2+H_j^{(2)}}{j^m}+2\zeta\left(2\right)H_r^{(m)}-2\sum_{j=1}^{r}\frac{H_j}{j^{m+1}}\\ +2\sum_{1\le i<j\le r}\frac{H_i}{i^m\left(j-i\right)}-2\sum_{1\le i<j\le r}\frac{H_j}{j^m\left(j-i\right)}\end{array}\right\},\end{aligned}\tag{2.34}$$

$$\begin{aligned}W_{4,r}\left(m\right)&=W_{4,0}\left(m\right)+3\sum_{j=1}^{r}\sum_{n=1}^{\infty}\frac{H_n^2-H_n^{(2)}}{n^m\left(n+j\right)}\\ &\quad+6\sum_{1\le i<j\le r}\sum_{n=1}^{\infty}\frac{H_n}{n^m\left(n+i\right)\left(n+j\right)}\\ &\quad+6\sum_{1\le i<j<k\le r}\sum_{n=1}^{\infty}\frac{1}{n^m\left(n+i\right)\left(n+j\right)\left(n+k\right)}\end{aligned}$$

$$
\begin{aligned}
&= W_{4,0}(m) + 3\sum_{l=1}^{m-1}(-1)^{l-1}H_r^{(l)}W_{3,0}(m+1-l)\\
&\quad + 6\sum_{l=1}^{m-1}(-1)^{l-1}W_{2,0}(m+1-l)\sum_{1\le i<j\le r}\frac{1}{j-i}\left(\frac{1}{i^l}-\frac{1}{j^l}\right)\\
&\quad + (-1)^{m-1}\left\{\begin{array}{l} 6\zeta(3)H_r^{(m)} - 3\sum\limits_{j=1}^{r}\dfrac{H_j^2+H_j^{(2)}}{j^{m+1}}\\ +\sum\limits_{j=1}^{r}\dfrac{H_j^3+3H_jH_j^{(2)}+2H_j^{(3)}}{j^m}\end{array}\right\}\\
&\quad + 6(-1)^{m-1}\sum_{1\le i<j\le r}\frac{1}{j-i}\left\{\begin{array}{l}\dfrac{1}{i^m}\left(\dfrac{H_i^2+H_i^{(2)}}{2}+\zeta(2)-\dfrac{H_i}{i}\right)\\ -\dfrac{1}{j^m}\left(\dfrac{H_j^2+H_j^{(2)}}{2}+\zeta(2)-\dfrac{H_j}{j}\right)\end{array}\right\}\\
&\quad + 6\sum_{l=1}^{m-1}(-1)^{l-1}\zeta(m+1-l)\sum_{1\le i<j<k\le r}\left\{\begin{array}{l}\frac{1}{i^l(j-i)(k-i)}+\frac{1}{j^l(i-j)(k-j)}\\ +\frac{1}{k^l(j-k)(i-k)}\end{array}\right\}\\
&\quad + 6(-1)^{m-1}\sum_{1\le i<j<k\le r}\left\{\begin{array}{l}\dfrac{H_i}{i^m(j-i)(k-i)}+\dfrac{H_j}{j^m(i-j)(k-j)}\\ +\dfrac{H_k}{k^m(j-k)(i-k)}\end{array}\right\}.
\end{aligned}
\tag{2.35}
$$

Therefore, the sums of harmonic numbers $W_{k,r}(m)$, for $k = 1,2,3,4$, have been successfully represented in terms of zeta values and harmonic numbers. In fact, the other case of $W_{k,r}(m)$ can be evaluated in a similar fashion. Next, we shall present a closed form evaluation of the following sum:

$$S(k,m;p) := \sum_{n=1}^{\infty}\frac{h_n^{(m)}(k)}{n^p},\ p\ge m+1,\ k=2,3.$$

By using the definitions of $S(k,m;p)$ and $W_{k,r}(m)$, then combining (2.20) and (2.21), we obtain the following theorem.

Theorem 2.6. *For positive integers r and $p \ge r+1$, the following identities hold*:

(2.36)

$$S(2,r;p) = \frac{1}{2!(r-1)!}\sum_{k=1}^{r}\begin{bmatrix} r\\ k\end{bmatrix}\left\{\begin{array}{l} W_{3,r-1}(p+1-k) - 2H_{r-1}W_{2,r-1}(p+1-k)\\ +\left(H_{r-1}^2+H_{r-1}^{(2)}\right)\zeta(p+1-k)\end{array}\right\},$$

(2.37)

$$S(3,r;p) = \frac{1}{3!(r-1)!}\sum_{k=1}^{r}\begin{bmatrix} r\\ k\end{bmatrix}\left\{\begin{array}{l} W_{4,r-1}(p+1-k) - 3H_{r-1}W_{3,r-1}(p+1-k)\\ +3\left(H_{r-1}^2+H_{r-1}^{(2)}\right)W_{2,r-1}(p+1-k)\\ -\left(H_{r-1}^3+3H_{r-1}H_{r-1}^{(2)}+2H_{r-1}^{(3)}\right)\zeta(p+1-k)\end{array}\right\}.$$

From (2.33)-(2.37), we know that the sums $S(2,r;p)$ and $S(3,r;p)$ can be evaluated in terms of harmonic numbers and zeta values whenever $p \geq r+1$. A simple example is as follows:

$$S(2,2;3) = 4\zeta(5) - 2\zeta(2)\zeta(3) + \frac{5}{4}\zeta(4) - 2\zeta(3) + \zeta(2).$$

It may also be possible to represent the sums $S(k,m;p)$ for $4 \leq k \in \mathbb{N}$ in closed form, this work is currently under investigation. It does appear however, that there is a difficulty with the representation of $W_{k,r}(p)$ for $5 \leq k \in \mathbb{N}$ in closed form.

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