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# $q$-ADDITION THEOREMS FOR THE $q$-APPELL POLYNOMIALS AND THE ASSOCIATED CLASSES OF $q$-POLYNOMIALS EXPANSIONS

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Abstract. Several addition formulas for a general class of $q$-Appell sequences are proved. The $q$-addition formulas, which are derived, involved not only the generalized $q$-Bernoulli, the generalized $q$-Euler and the generalized $q$-Genocchi polynomials, but also the $q$-Stirling numbers of the second kind and several general families of hypergeometric polynomials. Some $q$-umbral calculus generalizations of the addition formulas are also investigated.

## 1. Introduction

Throughout this paper, we adopt the following notations:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}.$$

The classical $q$-Bernoulli polynomials $B_n(x; q)$, the classical $q$-Euler polynomials $E_n(x; q)$ and the classical $q$-Genocchi polynomials $G_n(x; q)$ together with their generalizations $B_n^{(\alpha)}(x; q)$, $E_n^{(\alpha)}(x; q)$ and $G_n^{(\alpha)}(x; q)$ of (real or complex) order $\alpha$, are usually defined by means of the following generating functions (see for details, [11], and the references therein):

$$\left(\frac{t}{e_q(t) - 1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} B_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < 2\pi;\ 1^{\alpha} := 1), \tag{1.1}$$

$$\left(\frac{2}{e_q(t) + 1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} E_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < \pi;\ 1^{\alpha} := 1), \tag{1.2}$$

and

$$\left(\frac{2t}{e_q(t) + 1}\right)^{\alpha} e_q(xt) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (|t| < \pi;\ 1^{\alpha} := 1), \tag{1.3}$$

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so that, obviously, the $q$-Bernoulli polynomials $B_{n,q}(x)$, the $q$-Euler polynomials $E_{n,q}(x)$ and the $q$-Genocchi polynomials $G_{n,q}(x)$ are given respectively, by

$$B_{n,q}(x) := B_{n,q}^{(1)}(x), \quad E_{n,q}(x) := E_{n,q}^{(1)}(x), \quad \text{and} \quad G_{n,q}(x) := G_{n,q}^{(1)}(x), \quad (n \in \mathbb{N}_0).$$

For the $q$-Bernoulli numbers $B_{n,q}$, the $q$-Euler numbers $E_{n,q}$ and the $q$-Genocchi numbers $G_{n,q}$ of order $n$, we have

$$B_{n,q} = B_{n,q}(0), \quad E_{n,q} = E_{n,q}(0), \quad \text{and} \quad G_{n,q} = G_{n,q}(0), \quad (n \in \mathbb{N}_0),$$

respectively.

The Roger Szégo polynomials $H_n(x;q)$ (see [3, Eq. (1)]) and the Al-Salam Carlitz polynomials $U_n^{(a)}(x;q)$ (see [9, p. 534]) are defined by the generating functions

$$e_q(t)e_q(xt) = \sum_{n=0}^{\infty} H_n(x;q)\frac{t^n}{[n]_q!}, \tag{1.4}$$

and

$$\frac{e_q(xt)}{e_q(t)e_q(at)} = \sum_{n=0}^{\infty} U_n^{(a)}(x;q)\frac{t^n}{[n]_q!}, \tag{1.5}$$

respectively.

In these definitions, $[n]_q$ is the so-called $q$-number defined by [7, 9]

$$[n]_q = \frac{1-q^n}{1-q},$$

$e_q$ is the $q$-exponential function defined by [7, 9]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \tag{1.6}$$

where $[n]_q!$ denotes the so-called $q$-factorial

$$[n]_q! = \prod_{k=1}^{n} [k]_q, \quad \text{and} \quad [0]_q! = 1.$$

There is another $q$-exponential function $E_q(x)$ defined by [7, 9]

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} x^n. \tag{1.7}$$

Both $e_q(x)$ and $E_q(x)$ satisfy the fundamental relation $e_q(x)E_q(-x) = 1$.

The $q$-analogue of the binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad 0 \le k \le n,$$

here, $(q;q)_n$ is the so-called $q$-Pochhammer symbol defined by

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^k) \text{ for } n \geq 1, \quad (a;q)_0 = 1.$$

Various interesting and potentially useful properties and relations involving the Bernoulli, Euler, Genocchi, Roger-Szégo and Al-Salam Carlitz polynomials have been investigated in the literature.

In [12], the authors gave several addition formulas for a general class of Appell polynomials. In this work, we extend these results to a general class of $q$-Appell sequences.

**Definition 1.1** (see [1])**.** A polynomial sequence $\{P_n(x)\}_{n\in\mathbb{N}_0}$ is said to be a $q$-Appell sequence if

$$D_qP_0(x) = 0 \quad \text{and} \quad D_qP_n(x) = [n]_qP_{n-1}(x), \ (n \in \mathbb{N}), \tag{1.8}$$

or equivalently, if

$$A(t)e_q(xt) = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{[n]_q!}, \tag{1.9}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{[n]_q!},$$

is a formal power series with $a_0 \neq 0$.

From this definition, it is clear that the $q$-Bernoulli polynomials $B_{n,q}(x)$, the $q$-Euler polynomials $E_{n,q}(x)$, the $q$-Genocchi polynomials $G_{n,q}(x)$, the Roger-Szégo polynomials $H_n(x;q)$ and the Al-Salam Carlitz polynomials $U_n^{(a)}(x;q)$ are $q$-Appell sequences. Other definitions and notations for $q$-Appell sequences can be found in the literature (see for example [20]).

**Definition 1.2.** Let $a$ and $b$ two real or complex numbers. Then, the Ward $q$-addition of $a$ and $b$ is given by

$$(a \oplus_q b)^n := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^k b^{n-k}, \quad n = 0, 1, 3, \ldots. \tag{1.10}$$

The following $q$-Stirling numbers will be also needed.

**Definition 1.3** (see [4, p. 173])**.** The $q$-Stirling numbers of the first kind $s_q(n,k)$ and the $q$-Stirling numbers of the second kind $S_q(n,k)$ are defined by

$$(x)_{n,q} := \sum_{k=0}^{n} s_q(n,k)x^k, \tag{1.11}$$

and

$$x^n = \sum_{k=0}^{n} S_q(n,k)(x)_{k,q}, \tag{1.12}$$

where the polynomial $(x)_{k,q}$ is defined by

$$(x)_{k,q} = \prod_{m=0}^{k-1} (x - [m]_q).$$

## 2. Some $q$-addition theorems

Let $\{f_n^{(\alpha)}(x)\}$ ($\alpha \in \mathbb{C}$) be the *one-parameter* $q$-Appell sequence generated by

$$(f(t))^\alpha e_q(xt) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{t^n}{[n]_q!}, \quad (f(0) \neq 0;\;\; 1^\alpha = 1). \tag{2.1}$$

It is not difficult to see that comparing to (1.9), we have $f_n^{(1)}(x) = f_n(x)$, $(n \in \mathbb{N}_0)$. Also, replacing $\alpha$ by 0 in (2.1) and use the series expansion (1.6), we obtain

$$f_n^{(0)}(x) = x^n, \quad n \in \mathbb{N}_0.$$

Now we state the following important lemma.

**Lemma 2.1.** *For the one-parameter* $\{f_n^{(\alpha)}(x)\}$ *generated by* (2.1)*, the following* $q$*-addition formula holds*:

$$f_n^{(\alpha+\beta)}(x \oplus_q y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k^{(\alpha)}(x) f_{n-k}^{(\beta)}(y). \tag{2.2}$$

*Proof.* Using the generating function (2.1), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} f_n^{(\alpha+\beta)}(x \oplus_q y) \frac{t^n}{[n]_q!} &= (f(t))^{\alpha+\beta} e_q((x \oplus y)t) \\
&= (f(t))^\alpha e_q(xt) \times (f(t))^\beta e_q(yt) \\
&= \left( \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} f_n^{(\beta)}(y) \frac{t^n}{[n]_q!} \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k^{(\alpha)}(x) f_{n-k}^{(\beta)}(y) \right) \frac{t^n}{[n]_q!}.
\end{aligned}$$

This proves the lemma. □

As a direct consequence of this lemma the following proposition holds.

**Proposition 2.2.** *For the one-parameter* $\{f_n^{(\alpha)}(x)\}$ *generated by* (2.1), *the following q-addition equation applies*

$$f_n^{(\alpha)}(x\oplus_q y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q f_{n-k}^{(\alpha)}(y)x^k=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q f_k^{(\alpha)}(y)x^{n-k}. \tag{2.3}$$

Next, we need the following inversion formulas for the Roger-Szégo and the Al-Salam Carlitz polynomials.

**Proposition 2.3** (see [1, 19])**.** *The following inversion formulas hold for the Roger-Szégo polynomials* $H_k(x;q)$ *and the Al-Salam Carlitz polynomials* $U_n^{(a)}$ $(x;q)$:

$$x^n=\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}H_k(x;q), \tag{2.4}$$

$$(x\ominus 1)_q^n=\sum_{k=0}^{n}a^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q U_k^{(a)}(x;q), \tag{2.5}$$

$$x^n=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\sum_{i=0}^{n-k}\begin{bmatrix}n-k\\i\end{bmatrix}_q a^i\right)U_k^{(a)}(x;q). \tag{2.6}$$

*Proof.* From the generating function (1.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}x^n\frac{t^n}{[n]_q!}&=E_q(-t)e_q(t)e_q(xt)\\&=\left(\sum_{n=0}^{\infty}\frac{(-1)^nq^{\binom{n}{2}}}{[n]_q!}t^n\right)\left(\sum_{n=0}^{\infty}H_n(x;q)\frac{t^n}{[n]_q!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{n-k}{2}}H_k(x;q)\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

This prove the first equation. For the second one, we first remark that [18, (5.19)]

$$(x\ominus y)_q^n=\sum_{k=0}^{n}(-y)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q x^k.$$

Next, taking into account that $e_q(x)E_q(-x)=1$ and multiplying the generating function (1.5) by $e_q(at)$, the left-hand side gives

$$\begin{aligned}\frac{e_q(xt)}{e_q(t)}&=e_q(xt)E_q(-t)\\&=\left(\sum_{k=0}^{\infty}\frac{x^nt^n}{[n]_q!}\right)\left(\sum_{k=0}^{\infty}\frac{q^{\binom{n}{2}}}{[n]_q!}(-t)^n\right)\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \right) \frac{t^n}{[n]_q!}$$
$$= \sum_{n=0}^{\infty} (x \ominus 1)_q^n \frac{t^n}{[n]_q!},$$

the right-hand side gives

$$e_q(at) \sum_{n=0}^{\infty} U_n^{(a)}(x;q) \frac{t^n}{[n]_q!} = \left( \sum_{n=0}^{\infty} a^n \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} U_n^{(a)}(x;q) \frac{t^n}{[n]_q!} \right)$$
$$= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x;q) \right) \frac{t^n}{[n]_q!}.$$

Hence we have

$$\sum_{n=0}^{\infty} (x \ominus 1)_q^n \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q U_k^{(a)}(x;q) \right) \frac{t^n}{[n]_q!}.$$

So (2.5) is proved. Note that this result is proved in [19] using the Verma's $q$-extension [21] of Filds and Wimp inversion formula [5]. □

**Lemma 2.4** ($q$-Analogue of [10, p. 5707, Lemma 2])**.** *The following relation between the q-Genocchi polynomials and the q-Euler polynomials holds true*:

$$E_{n,q}^{(\ell)}(x) = \frac{[n]_q!}{[n+\ell]_q!} G_{n+\ell,q}^{(\ell)}(x), \quad n, \ell \in \mathbb{N}_0, \quad 0 \le \ell \le n. \tag{2.7}$$

*Proof.* Let $\ell$ such that $0 \le \ell \le n$. Then, from the generating functions (1.2) and (1.3), we have

$$\sum_{n=0}^{\infty} G_{n,q}^{(\ell)}(x) \frac{t^n}{[n]_q!} = t^\ell \left( \frac{2}{e_q(t)+1} \right)^\ell e_q(xt)$$
$$= \sum_{n=0}^{\infty} E_{n,q}^{(\ell)}(x) \frac{t^{n+\ell}}{[n]_q!}$$
$$= \sum_{n=\ell}^{\infty} E_{n-\ell,q}^{(\ell)}(x) \frac{t^n}{[n-\ell]_q!}$$
$$= \sum_{n=0}^{\infty} \frac{[n]_q!}{[n-\ell]_q!} E_{n-\ell,q}^{(\ell)}(x) \frac{t^n}{[n]_q!},$$

where we set $E_{k,q}^{\ell}(x) = 0$ for $k < 0$. Comparing the coefficients of $t^n$ provides the result. □

**Lemma 2.5.** *Each of the following expansion formulas holds true*:

$$x^n = \frac{1}{[n+1]_q} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q B_{k,q}(x), \tag{2.8}$$

$$x^n = \frac{1}{2}\left[E_{n,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}(x)\right], \tag{2.9}$$

*and*

$$x^n = \frac{1}{2[n+1]_q}\left[G_{n+1,q}(x) + \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q G_{k+1,q}(x)\right]. \tag{2.10}$$

*Proof.* Form the generating function (1.1) with $\alpha = 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} x^n \frac{t^n}{[n]_q!} &= \frac{e_q(t)-1}{t}\frac{t}{e_q(t)-1}e_q(xt) \\
&= \left(\sum_{n=0}^{\infty} \frac{t^n}{[n+1]_q!}\right)\left(\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{[n]_q! B_{k,q}(x)}{[k]_q![n+1-k]_q!}\right)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{[n+1]_q}\begin{bmatrix} n+1 \\ k \end{bmatrix}_q B_{k,q}(x)\right)\frac{t^n}{[n]_q!}.
\end{aligned}$$

This proves the relation for $q$-Bernoulli polynomials. The other results are obtained similarly.

A second proof of (2.10). It is easy to see from the generating function (1.3) that

$$G_{n,q}^{(\alpha+\beta)}(x \oplus_q y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{(\alpha)}(x) G_{n-k,q}^{(\beta)}(y),$$

which in the special case when $y = 1$ and $\beta = 0$, yields

$$G_{n,q}^{(\alpha)}(x \oplus_q 1) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k,q}^{\alpha}(x). \tag{2.11}$$

Moreover,

$$\begin{aligned}
\sum_{n=0}^{\infty}(G_{n,q}(x \oplus_q 1) + G_{n,q}(x))\frac{t^n}{[n]_q!} &= \frac{2t}{e_q(t)+1}\left(e_q((x \oplus_q 1)t) + e_q(xt)\right) \\
&= 2te_q(xt) = \sum_{n=0}^{\infty} 2[n+1]_q x^n \frac{t^{n+1}}{[n+1]_q!},
\end{aligned}$$

which implies that

$$G_{n+1,q}(x \oplus_q 1) + G_{n+1,q}(x) = 2[n+1]_q x^n. \tag{2.12}$$

Combining (2.11) with $\alpha = 1$ and (2.12) yields the result.

A third proof of (2.10). Taking $\ell = 1$ in (2.7), we obtain

$$E_{n,q}(x) = \frac{1}{[n+1]_q} G_{n+1,q}(x).$$

Equation (2.9) becomes

$$\begin{aligned} x^n &= \frac{1}{2}\left[\frac{1}{[n+1]_q} G_{n+1,q}(x) + \sum_{k=0}^{n} \frac{1}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q G_{k+1,q}(x)\right] \\ &= \frac{1}{2}\left[\frac{1}{[n+1]_q} G_{n+1,q}(x) + \frac{1}{[n+1]_q} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q G_{k+1,q}(x)\right], \end{aligned}$$

which is the required result. □

**Theorem 2.6.** *Let* $\{f_n^{(\alpha)}(x)\}_{n\in\mathbb{N}_0}$ *be a one-parameter sequence of q-Appell polynomials generated by* (2.1). *Then each of the following addition formulas holds true*:

$$f_n^{(\alpha)}(x \oplus_q y) = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} (-1)^{k-j} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} f_{n-k}^{(\alpha)}(y)\right] H_j(x;q),$$

$$f_n^{(\alpha)}(x \oplus_q y) = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \frac{1}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k+1 \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y)\right] B_{j,q}(x),$$

$$f_n^{(\alpha)}(x \oplus_q y) = \frac{1}{2}\sum_{j=0}^{n} \left[\begin{bmatrix} n \\ j \end{bmatrix}_q f_{n-j}^{(\alpha)}(y) + \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y)\right] E_{j,q}(x),$$

$$f_n^{(\alpha)}(x \oplus_q y) = \frac{1}{2}\sum_{j=0}^{n} \frac{1}{[j+1]_q} \left[\begin{bmatrix} n \\ j \end{bmatrix}_q f_{n-j}^{(\alpha)}(y) + \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k+1 \\ j+1 \end{bmatrix}_q\right] G_{j+1,q}(x),$$

$$f_n^{(\alpha)}(x \oplus_q y) = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q f_{n-k}^{(\alpha)}(y) \sum_{\ell=0}^{k-j} \begin{bmatrix} k-j \\ \ell \end{bmatrix}_q a^{\ell}\right] U_j^{(a)}(x;q),$$

*and*

$$f_n^{(\alpha)}(x \oplus_q y) = \sum_{j=0}^{n} \left[\sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q S_q(k,j) f_{n-k}^{(\alpha)}(y)\right] (x)_{j,q}. \tag{2.13}$$

*Proof.* The proof of this theorem uses (2.3), Proposition 2.3, Lemma 2.5 and the summation formula

$$\sum_{k=0}^{n} A_k \sum_{j=0}^{k} B_j = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} A_k\right) B_j. \tag{2.14}$$

□

## 3. $q$-addition formulas involving $q$-hypergeometric polynomials

The basic hypergeometric or $q$-hypergeometric function ${}_r\phi_s$ is defined by the series

$$ {}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix} \middle| q; z\right) := \sum_{k=0}^{\infty} \frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q;q)_k}, $$

where

$$ (a_1,\dots,a_r)_k := (a_1;q)_k\cdots(a_r;q)_k, $$

with

$$ (a_i;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-a_iq^j) & \text{if } k=1,2,3,\dots \\ 1 & \text{if } k=0. \end{cases} $$

For two general families of $q$-hypergeometric polynomials, Verma [21] derived the following expansion formulas:

$$ {}_{r+t}\phi_{s+u}\left(\begin{matrix} (a_r),(c_t) \\ (b_s),(d_u) \end{matrix} \middle| q; y\omega\right) = \sum_{j=0}^{\infty} \frac{((c_t),(e_k);q)_j}{(q,(d_u),\gamma q^j;q)_j} y^j \left[(-1)^j q^{\binom{j}{2}}\right]^{u+3-t-k} $$
$$ \cdot\, {}_{t+k}\phi_{u+1}\left(\begin{matrix} (c_tq^j),(e_kq^j) \\ \gamma q^{2j+1},(d_uq^j) \end{matrix} \middle| q; yq^{j(u+2-t-k)}\right) $$
$$ \cdot\, {}_{r+2}\phi_{s+k}\left(\begin{matrix} q^{-j},\gamma q^j,(a_r) \\ (b_s),(e_k) \end{matrix} \middle| q; \omega q\right) \tag{3.1} $$

in powers of $y\omega$ as given in [6, (3.7.9)] to find the solution of the inversion problem for polynomials of the Askey scheme and its $q$-analogue. Here, the notation $(a_r)$ means $r$ parameters of the type $a_1$, $a_2$,..., $a_r$ and the notation $(a_rq^j)$ means $r$ parameters of the form $a_1q^j$, $a_2q^j$,..., $a_rq^j$. The method is the following.

We choose $u=t=0$, and $k=1$ in (3.1). Then for $\omega=x$ and $\gamma=0$, we obtain

$$ {}_r\phi_s\left(\begin{matrix} (a_r) \\ (b_s) \end{matrix} \middle| q; yx\right) = \sum_{j=0}^{\infty} \frac{[(-1)^j q^{\binom{j}{2}}]^2}{(q;q)_j} y^j\, {}_1\phi_1\left(\begin{matrix} 0 \\ 0 \end{matrix} \middle| q; q^jy\right) $$
$$ \cdot\, {}_{r+1}\phi_s\left(\begin{matrix} q^{-j},(a_r) \\ (b_s) \end{matrix} \middle| q; qx\right). $$

Expanding the left-hand side, the coefficient of $y^n$ is

$$ \frac{((a_r);q)_n}{(q;q)_n((b_s);q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r+1} x^n. \tag{3.2} $$

Moreover, the right-hand side can be rewritten as

$$\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\left(\frac{q^{jh}}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^2 \frac{\left[(-1)^h q^{\binom{h}{2}}\right]}{(q;q)_h} y^{h+j}\right) {}_{r+1}\phi_s\left(\begin{matrix} q^{-j},(a_r) \\ (b_s)\end{matrix}\;\middle|\; q;qx\right),$$

so that the coefficient of $y^n$ in this expression is now

$$\sum_{\ell=0}^{n}\frac{(-1)^{n-\ell}q^{2\binom{\ell}{2}}q^{\binom{n-\ell}{2}}q^{(n-\ell)\ell}}{(q;q)_\ell (q;q)_{n-\ell}} {}_{r+1}\phi_s\left(\begin{matrix} q^{-\ell},(a_r) \\ (b_s)\end{matrix}\;\middle|\; q;qx\right). \tag{3.3}$$

From (3.2) and (3.3) we get

$$\begin{aligned}&\frac{((-1)^n q^{\binom{n}{2}})^{s-r}(a_2,\dots,a_{r+1};q)_n}{(b_1,b_2,\dots,b_s;q)_n}x^n \\ &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k\end{bmatrix}_q q^{\binom{k}{2}} {}_{r+1}\phi_s\left(\begin{matrix} q^{-k},a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_s\end{matrix}\;\middle|\; q;qx\right),\end{aligned} \tag{3.4}$$

already given in [2, (3.3)].

It should be mentioned that until now, the coefficients $a_R$ and $b_S$ appearing in (3.3) are independent of the summation index $k$. However, in some families belonging to the Askey scheme and its $q$-analogue, one of the numerator parameters depends on $k$ in the form $a_2+k$ (Askey scheme) or $a_2q^k$ ($q$-analogue). In these situations and in case of polynomials belonging to the $q$-analogue of the Askey scheme, the following formula (see [2, (3.5)]) should be used:

$$\begin{aligned}&\frac{((-1)^n q^{\binom{n}{2}})^{s-r}(a_3,\dots,a_{r+1})_n}{(b_1,b_2,\dots,b_s;q)_n}x^n \\ &= \sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q \frac{(-1)^k q^{\binom{k}{2}}}{(a_2q^k,a_2q^{2k+1};q)_k} {}_{r+1}\phi_s\left(\begin{matrix} q^{-k},a_2q^k,a_3,\dots,a_{r+1} \\ b_1,b_2,\dots,b_s\end{matrix}\;\middle|\; q;qx\right).\end{aligned} \tag{3.5}$$

By applying the expansion formula (2.3) and using relation (2.14), it is not difficult to prove the following theorem.

**Theorem 3.1.** *Let $\{f_n^{(\mu)}(x)\}_{n\in\mathbb{N}_0}$ be a one-parameter sequence of $q$-Appell polynomials generated by* (2.1) *with the parameter $\alpha$ replaced by $\mu$. Then each of the following $q$-addition formulas holds true.*

$$\begin{aligned}f_n^{(\mu)}(x\oplus_q y) = &\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q \begin{bmatrix} k \\ j\end{bmatrix}_q \frac{((-1)^k q^{\binom{k}{2}})^{r-s}\prod_{\ell=1}^{s}(b_\ell;q)_k}{\prod_{\ell=2}^{r+1}(a_\ell;q)_k} f_{n-k}^{(\mu)}(y)\right) \\ &\times (-1)^j q^{\binom{j}{2}} {}_{r+1}\phi_s\left(\begin{matrix} q^{-j},a_2,\dots,a_{r+1} \\ b_1,b_2,\dots,b_s\end{matrix}\;\middle|\; q;qx\right)\end{aligned} \tag{3.6}$$

*and*

$$f_n^{(\mu)}(x \oplus_q y) = \sum_{j=0}^{n} \left( \sum_{k=j}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{((-1)^k q^{\binom{k}{2}})^{r-s} \prod_{\ell=1}^{s} (b_\ell; q)_k}{\prod_{\ell=3}^{r+1} (a_\ell; q)_k} f_{n-k}^{(\mu)}(y) \right)$$

$$\times \frac{(-1)^j q^{\binom{j}{2}}}{(a_2 q^j, a_2 q^{2j+1}; q)_j} {}_{r+1}\phi_s \left( \begin{matrix} q^{-j}, a_2 q^j, a_3, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_s \end{matrix} \middle| q; qx \right). \tag{3.7}$$

*Remark* 3.2. Corollary 3.3 below, which involves such classical $q$-orthogonal polynomials as the Little $q$-Laguerre, the Little $q$-Legendre, the Little $q$-Laguerre, the $q$-Laguerre, the $q$-Bessel and the Stieltjes-Wigert polynomials, can be deduced by suitably specializing Theorem 3.1 or (alternatively) by directly applying (2.1) in conjunction with the following known polynomial expansions (see [19]):

- *the Little q-Jacobi polynomials*

$$x^k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (aq; q)_k}{(abq^{j+1}; q)_j (abq^{2j+2}; q)_{k-j}} p_j(x; a, b|q). \tag{3.8}$$

- *the Little q-Legendre polynomials*

$$x^k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}} (q; q)_k}{(q^{j+1}; q)_j (q^{2j+2}; q)_{k-j}} P_j(x|q). \tag{3.9}$$

- *the Little q-Laguerre polynomials*

$$x^k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} (aq; q)_k p_j(x; a|q). \tag{3.10}$$

- *the q-Laguerre polynomials*

$$x^k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(2\alpha+3j+k+1)}{2}} \frac{(q; q)_j}{q^{j(j+\alpha)}} (q^{j+\alpha+1}; q)_{k-j} L_j^{(\alpha)}(x; q). \tag{3.11}$$

- *the q-Bessel polynomials*

$$x^k = \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j}{2}}}{(-aq^j; q)_j (-aq^{2j+1}; q)_{k-j}} y_j(x; a|q). \tag{3.12}$$

- *the Stieltjes-Wigert polynomials*

$$x^k = \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{(j-k)(3j+k+1)}{2} - j^2} (q; q)_j S_j(x; q). \tag{3.13}$$

**Corollary 3.3.** *Let* $\{f_n^{(\mu)}(x)\}_{n \in \mathbb{N}_0}$ *be a one-parameter sequence of q-Appell polynomials generated by* (2.1) *with the parameter* $\alpha$ *replaced by* $\mu$*. Then each* *of the following q-addition formulas holds true.*

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\frac{(-1)^jq^{\binom{j}{2}}(aq;q)_kf^{(\mu)}(y)}{(abq^{j+1};q)_j(abq^{2j+2};q)_{k-j}}\right)p_j(x;a,b|q),$$

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\frac{(-1)^jq^{\binom{j}{2}}(q;q)_kf^{(\mu)}(y)}{(q^{j+1};q)_j(q^{2j+2};q)_{k-j}}\right)P_j(x|q),$$

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q(-1)^jq^{\binom{j}{2}}(aq;q)_kf^{(\mu)}(y)\right)p_j(x;a|q),$$

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q q^{\frac{(j-k)(2\alpha+3j+k+1)}{2}}\frac{(q;q)_j}{q^{j(j+\alpha)}}(q^{j+\alpha+1};q)_{k-j}\right)\cdot L_j^{(\alpha)}(x;q),$$

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q\frac{(-1)^jq^{\binom{j}{2}}}{(-aq^j;q)_j(-aq^{2j+1};q)_{k-j}}\right)y_j(x;a|q),$$

*and*

$$f_n^{(\mu)}(x\oplus_q y)=\sum_{j=0}^{n}\left(\sum_{k=j}^{n}(-1)^j\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\j\end{bmatrix}_q q^{\frac{(j-k)(3j+k+1)}{2}-j^2}(q;q)_j\right)S_j(x;q).$$

## 4. A $q$-umbral-calculus generalization of the addition theorems

In 1978, Roman and Rota vewed the classical umbral calculus from a new perspective and proposed an interesting approach based on a simple but innovative indication for effect of linear functional on polynomials, which Roman later called it the modern umbral calculus [17]. Roman, also, proposed a similar approach under the area of nonclassical umbral calculus which is called $q$-umbral calculus, [13–16]. In what follows, we adopt the notations of [8].

Let $\mathbb{C}$ be the field of complex numbers and $\mathcal{F}$ the set of all formal power $q$-series in the variable $t$ over $\mathbb{C}$. In other words, $f(t)$ in an element of $\mathcal{F}$ if

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{[k]_q!}t^k, \tag{4.1}$$

where $a_k\in\mathbb{C}$. Let $\mathcal{P}$ be the algebra of all polynomials in the variable $x$ over $\mathbb{C}$ and $\mathcal{P}^\star$ be the vector space of all linear functionals on $\mathcal{P}$. The formal power series (4.1) defines a linear functional on $\mathcal{P}^\star$ by setting

$$\langle f(t)|x^n\rangle=a_n\quad(n\in\mathbb{N}_0). \tag{4.2}$$

**Lemma 4.1** (see [8]). *Let* $\{h_n(x)\}_{n\in\mathbb{N}_0}$ *be a q-Appell sequence for the function* $h(t)$. *Then, for any polynomial* $\mathfrak{p}(x)$,

$$\mathfrak{p}(x) = \sum_{j\geq 0} \frac{\langle h(t)|\mathfrak{p}^{(j)}(x)\rangle}{[j]_q!} h_j(x), \tag{4.3}$$

*where* $\mathfrak{p}^{(j)}(x)$ *denotes the q-derivative of* $\mathfrak{p}(x)$ *of order* $j$.

**Theorem 4.2.** *Let* $\{f_n(x)\}_{n\in\mathbb{N}_0}$ *and* $\{g_n(x)\}_{n\in\mathbb{N}}$ *be the q-Appell sequences corresponding to the functions* $f(t)$ *and* $g(t)$, *respectively. Then*

$$f_n(x \oplus_q y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q f_k(y) \sum_{j=0}^{n-k} \frac{\langle g(t)|D_q^j[x^{n-j}]\rangle}{[j]_q!} g_j(x). \tag{4.4}$$

*Proof.* The proof follows directly from Lemma 4.1 and Proposition 2.2. □

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