

STABILITY OF WEAK MEASURE EXPANSIVE DIFFEOMORPHISMS

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ABSTRACT. A notion of measure expansivity for homeomorphisms was introduced by Morales recently as a generalization of expansivity, and he obtained many interesting dynamic results of measure expansive homeomorphisms in [8]. In this paper, we introduce a concept of weak measure expansivity for homeomorphisms which is really weaker than that of measure expansivity, and show that a diffeomorphism f on a compact smooth manifold is C^1 -stably weak measure expansive if and only if it is Ω -stable. Moreover we show that C^1 -generically, if f is weak measure expansive, then f satisfies both Axiom A and the no cycle condition.

1. Introduction

The notion of expansivity for a homeomorphism on a compact metric space introduced by Utz [12] plays an important role in the qualitative study of dynamical systems. The phenomenon of expansivity occurs when the orbits of nearby points are separated by the dynamical system. Recently Morales [8] introduced a notion of *measure expansivity*, generalizing the usual concept of expansivity. Several interesting properties of measure expansivity have been obtained elsewhere [2, 8–10]. In particular, Artigue and Carrasco-Olivera [2] characterized the homeomorphisms for which all probability measures are expansive as those which are *countably-expansive*.

In this paper we introduce a notion of weak measure expansivity for homeomorphisms which is really weaker than that of measure expansivity, and study the stability of weak measure expansive diffeomorphisms on a compact C^∞ -manifold.

Let X be a compact metric space with a metric d , and let f be a homeomorphism from X to X . A homeomorphism $f : X \rightarrow X$ is called *expansive* if there is $\delta > 0$ such that for any distinct points $x, y \in X$ there exists $i \in \mathbb{Z}$ such that $d(f^i(x), f^i(y)) > \delta$. Given $x \in X$ and $\delta > 0$, we define the *dynamic δ -ball of f at x* ,

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$$\Gamma_\delta^f(x) = \{y \in X : d(f^i(x), f^i(y)) \leq \delta \text{ for all } i \in \mathbb{Z}\}.$$

(Denote $\Gamma_\delta(x)$ by $\Gamma_\delta^f(x)$ for simplicity if there is no confusion.) Then we can see that f is expansive if there is $\delta > 0$ such that $\Gamma_\delta(x) = \{x\}$ for all $x \in X$.

Let β be the Borel σ -algebra on X . Denote by $\mathcal{M}(X)$ the set of Borel probability measures on X endowed with weak* topology. We say that $\mu \in \mathcal{M}(X)$ is *atomic* if there exists a point $x \in X$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(X)$ be the set of nonatomic measures $\mu \in \mathcal{M}(X)$ and let $\mathcal{M}_f^*(X)$ be the set of f -invariant measures $\mu \in \mathcal{M}^*(X)$. For any $\mu \in \mathcal{M}^*(X)$, a homeomorphism $f : X \rightarrow X$ is said to be μ -*expansive* (or μ is expansive with respect to f) if there is $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$. Here δ is called an *expansive constant* of μ with respect to f . Note that

$$\Gamma_\delta(x) = \bigcap_{i \in \mathbb{Z}} f^{-i}(B_\delta[f^i(x)]),$$

where $B_\delta[x]$ denotes the closed δ -ball centered at x . A homeomorphism $f : X \rightarrow X$ is said to be *measure expansive* if f is μ -expansive for all $\mu \in \mathcal{M}^*(X)$.

Above all, we prepare and check some preliminaries of weak measure expansivity on a compact metric space X in Section 2. Because it is the first paper which introduce the weak measure expansivity for homeomorphisms, it is meaningful that including fundamentals of weak measure expansivity. Subsequently, we characterize the hyperbolicity of the nonwandering sets of diffeomorphisms using the notion of weak measure expansivity in Section 3.

2. Preliminaries of weak measure expansivity

In this section, we introduce a concept of weak measure expansivity generalizing the notion of measure expansivity which is based on the concept of measure-sensitive partition in [7]. To do this, we say that a finite collection $P = \{A_1, A_2, \dots, A_n\}$ of subsets of X is a *finite δ -partition* ($\delta > 0$) of X if

- (i) A_i 's are disjoint, and $\cup_{i=1}^n A_i = X$;
- (ii) each A_i is measurable, $\text{int}(A_i) \neq \emptyset$ and $\text{diam} A_i \leq \delta$ for all $i = 1, 2, \dots, n$.

It can be easily checked that for any $\delta > 0$ there is a finite δ -partition $P = \{A_1, A_2, \dots, A_n\}$ of X . In fact, let $\mathcal{O} = \{B_\delta(x) : x \in X\}$ be an open cover of X , where $B_\delta(x)$ is the open δ -ball centered at x . Then there are $x_1, x_2, \dots, x_n \in X$ such that $X = \cup_{i=1}^n B_\delta(x_i)$. Put $U_i = B_\delta(x_i)$. Then we can make a finite δ -partition $\{A_i\}_{i=1}^n$ of X by letting $A_i = U_i \setminus (\cup_{j=1}^{i-1} U_j)$ for each $i = 1, 2, \dots, n$. For convenience, let's skip " δ " and we say that P is a finite partition of X . Because if a finite partition P exists such that $\mu(\Gamma_P(x)) = 0$ for all $x \in X$, then we can take $\delta = \text{diam}(X)$ to get $\text{diam}(A_i) \leq \delta$ for every $A_i \in P$. Hence P is a finite δ -partition of X . However, if exact constants are needed, we will use " $\delta > 0$ " and so on.

Now we introduce the notion of weak measure expansivity by using a finite partition as follow.

Definition. For some $\mu \in \mathcal{M}(X)$, a homeomorphism $f : X \rightarrow X$ is said to be *weak μ -expansive* if there is a finite partition $P = \{A_1, A_2, \dots, A_n\}$ of X such that $\mu(\Gamma_P^f(x)) = 0$ for all $x \in X$, where

$$\Gamma_P^f(x) = \{y \in X : f^i(y) \in P(f^i(x)) \text{ for all } i \in \mathbb{Z}\}.$$

The set $\Gamma_P^f(x)$ is called the *dynamic P-ball of f centered at $x \in X$* , and $P(x)$ denotes the element of P containing x . Denote $\Gamma_P(x)$ by $\Gamma_P^f(x)$ for simplicity if there is no confusion. Note that

$$\Gamma_P(x) = \bigcap_{i \in \mathbb{Z}} f^{-i}(P(f^i(x))).$$

A homeomorphism $f : X \rightarrow X$ is called *weak measure expansive* if it is weak μ -expansive for all $\mu \in \mathcal{M}^*(X)$.

Note that if a homeomorphism $f : X \rightarrow X$ is weak μ -expansive for $\mu \in \mathcal{M}(X)$, then μ is clearly nonatomic. In fact, if a homeomorphism f is weak μ -expansive, then there exists a finite partition P of X such that $\mu(\Gamma_P(x)) = 0$ for all $x \in X$. Since $\{x\} \subset \Gamma_P(x)$ and $\mu(\{x\}) \leq \mu(\Gamma_P(x))$, we have $\mu(\{x\}) = 0$. This means that μ is nonatomic.

From now, we give basic properties of weak measure expansivity for homeomorphisms on a compact metric space X . As we can see in the following theorem (Theorem 2.2) and example (Example 2.3), the concept of the weak measure expansivity is really weaker than that of the measure expansivity.

Theorem 2.1. *For any $\mu \in \mathcal{M}^*(X)$, a homeomorphism f is weak μ -expansive if and only if there exists a finite partition $P = \{A_1, A_2, \dots, A_n\}$ of X such that $\mu(\Gamma_P(x)) = 0$ for a.e. $x \in X$.*

Proof. To prove this theorem, it is enough to show the “if part”. By the assumption, there is a finite partition $P = \{A_1, A_2, \dots, A_n\}$ of X such that $\mu(\Gamma_P(x)) = 0$ for a.e. $x \in X$. Let $Y = \{x \in X : \mu(\Gamma_P(x)) = 0\}$. Then $\mu(Y) = 1$. Suppose a homeomorphism f is not weak μ -expansive. Then there exists $x_0 \in X \setminus Y$ such that $\mu(\Gamma_P(x_0)) > 0$. Since $\mu(Y) = 1$ and $\mu(\Gamma_P(x_0)) > 0$, $Y \cap \Gamma_P(x_0) \neq \emptyset$. We can choose $y_0 \in Y \cap \Gamma_P(x_0)$. Since $y_0 \in \Gamma_P(x_0)$, $f^i(y_0) \in P(f^i(x_0))$ for all $i \in \mathbb{Z}$. Let $z \in \Gamma_P(y_0)$. Then $f^i(z) \in P(f^i(y_0))$ for all $i \in \mathbb{Z}$. Therefore, $f^i(z) \in P(f^i(x_0))$. This means $\Gamma_P(x_0) = \Gamma_P(y_0)$ and so $\mu(\Gamma_P(x_0)) = 0$. The contradiction completes the proof. \square

Theorem 2.2. *If a homeomorphism f is measure expansive, then it is weak measure expansive.*

Proof. Since a homeomorphism f is μ -expansive, there exists $\delta > 0$ such that $\mu(\Gamma_\delta(x)) = 0$ for all $x \in X$. Let P be a finite partition of X . Let $y \in \Gamma_P(x)$, then $f^i(y) \in P(f^i(x))$ for all $i \in \mathbb{Z}$. Since $\text{diam}P(f^i(x)) \leq \delta$, we have $d(f^i(x), f^i(y)) \leq \delta$ for all $i \in \mathbb{Z}$. Therefore $y \in \Gamma_\delta(x)$. That is, for any $x \in X$, we get $\Gamma_P(x) \subset \Gamma_\delta(x)$. Since $\mu(\Gamma_P(x)) \leq \mu(\Gamma_\delta(x))$, we have $\mu(\Gamma_P(x)) = 0$. Hence f is weak μ -expansive. \square

The following example shows that the converse of Theorem 2.2 does not hold. More precisely, we give a homeomorphism f on the unit circle to claim that f is weak measure expansive but not measure expansive.

Example 2.3. Let $f : S^1 \rightarrow S^1$ be an irrational rotation map. Then f is weak measure expansive. But f is not m -expansive, where $m \in \mathcal{M}^*(S^1)$ is the Lebesgue measure on S^1 .

Proof. First, we show that f is weak measure expansive. Regard S^1 as $[0, 2\pi)$ for convenience. For any small $0 < \delta < \frac{1}{2}$, let $P = \{A_i \subset [0, 2\pi) : i = 1, \dots, n\}$ be a finite partition of $[0, 2\pi)$ such that each A_i is a half-open interval with $\text{diam}A_i < \delta$. For any $x \in S^1$, let $x \in A_j$ for some $j \in \{1, 2, \dots, n\}$. We claim that $\Gamma_P(x) = \{x\}$. For this, take a point $y \in A_j$ with $x \neq y$, say $x < y$. Put $d(x, y) = \varepsilon > 0$. Since every rotation map is an isometry, $d(x, y) = d(f^i(x), f^i(y))$ for all $i \in \mathbb{Z}$. Take an end point z of A_k for some $k \in \{1, 2, \dots, n\}$, and an open ball $B_{\varepsilon/2}(z)$ containing z . Since every orbit of f is dense, there is $l \in \mathbb{Z}$ such that $f^l(x) \in A_{k-1} \cap B_{\varepsilon/2}(z)$. Since f is an orientation preserving map and an isometry, $f^l(y)$ must be an element of A_k . This means $\Gamma_P(x) = \{x\}$, that is $\mu(\Gamma_P(x)) = 0$ for all $\mu \in \mathcal{M}^*(S^1)$. So f is weak measure expansive.

On the other hand, let $m \in \mathcal{M}^*(S^1)$ be the Lebesgue measure on S^1 . Since f is an isometry, we can see that for every $x \in X$

$$\Gamma_\delta(x) = \{y \in S^1 : d(f^i(x), f^i(y)) \leq \delta \text{ for } i \in \mathbb{Z}\} = B_\delta[x].$$

Hence we get $m(\Gamma_\delta(x)) = m(B_\delta[x]) > 0$. This means that f is not m -expansive. □

Lemma 2.4. *The identity map on a compact metric space X is not weak measure expansive.*

Proof. Let Id be the identity map. Let $P = \{A_1, A_2, \dots, A_n\}$ be a finite partition of X . Then $\Gamma_P^{\text{Id}}(x) = P(x) = A_i$ for some $i = 1, \dots, n$. Choose $A_i \in P$ such that $\mu(A_i) > 0$. That means $\mu(\Gamma_P^{\text{Id}}(x)) > 0$ for all $x \in A_i$. Therefore Id is not weak measure expansive. □

Lemma 2.5. *A homeomorphism f is weak measure expansive if and only if f^n is weak measure expansive for $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. First, we prove the necessary part. Let f^n be weak measure expansive for $n \in \mathbb{Z} \setminus \{0\}$. This means that there exists a finite partition P of X such that $\mu(\Gamma_P^{f^n}(x)) = 0$ for all $x \in X$. And it is easy to check $\Gamma_P^f(x) \subset \Gamma_P^{f^n}(x)$. Indeed, if we take $y \in \Gamma_P^f(x)$, that is, $f^i(y) \in P(f^i(x))$ for all $i \in \mathbb{Z}$, then $(f^n)^i(y) \in P((f^n)^i(x))$ for all $i \in \mathbb{Z}$. Thus, we know that $y \in \Gamma_P^{f^n}(x)$. Therefore, $\mu(\Gamma_P^f(x)) \leq \mu(\Gamma_P^{f^n}(x)) = 0$.

Conversely, suppose that a homeomorphism f is weak measure expansive with a finite partition P of X , that is, $\mu(\Gamma_P^f(x)) = 0$ for all $x \in X$ and

$\mu \in \mathcal{M}^*(X)$. We consider $Q = \bigvee_{i=0}^n f^{-i}(P)$, then Q is a finite partition of X satisfying $Q(x) = \bigcap_{i=0}^n f^{-i}(P(f^i(x)))$. Here, $\bigvee_{i=0}^n f^{-i}(P)$ means the set $\{\bigcap_{i=0}^n \zeta_i : \zeta_i \in f^{-i}(P) \text{ for all } 0 \leq i \leq n\}$ and it is called the *join* of the partition P . Now, take $y \in \Gamma_Q^{f^n}(x)$, then clearly $y \in Q(x)$. From this, we can know that

$$f^i(y) \in P(f^i(x)) \text{ for every } 0 \leq i \leq n.$$

Take $k > n$, so $k = pn + i$ for some $p \in \mathbb{N}$ and $0 \leq i < n$. Since $y \in \Gamma_Q^{f^n}(x)$, we have $f^{pn}(y) \in Q(f^{pn}(x))$ and then

$$f^k(y) = f^{pn+i}(y) = f^i(f^{pn}(y)) \in P(f^i(f^{pn}(x))) = P(f^k(x))$$

for all $k \in \mathbb{N}$, i.e., $y \in \Gamma_P^f(x)$. Therefore we get $\Gamma_Q^{f^n}(x) \subset \Gamma_P^f(x)$ and so $\mu(\Gamma_Q^{f^n}(x)) = 0$ for all $x \in X$ and $\mu \in \mathcal{M}^*(X)$. It follows that f^n is weak measure expansive with a finite partition Q of X . □

Theorem 2.6. *A homeomorphism $f : X \rightarrow X$ is weak μ -expansive for some $\mu \in \mathcal{M}^*(X)$ if and only if there exists an f -invariant Borel set Y of f such that $f|_Y$ is weak ν -expansive for some $\nu \in \mathcal{M}^*(Y)$.*

Proof. Since the sufficiency is clear, it is enough to prove that necessary part. Suppose $f|_Y$ is weak ν -expansive. Then there exists a finite partition $P = \{A_1, A_2, \dots, A_k\}$ of Y such that $\nu(\Gamma_P^{f|_Y}(x)) = 0$ for all $x \in Y$. Let $Q = \{A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_l\}$ be a finite partition of X . Since Y is invariant, we have

$$\text{either } \Gamma_Q^f(x) \cap Y = \emptyset \text{ if } x \in X \setminus Y \text{ or } \Gamma_Q^f(x) = \Gamma_P^{f|_Y}(x) \text{ if } x \in Y.$$

Define $\mu(B) = \nu(B \cap Y)$ for any Borel set B of X . Then $\mu \in \mathcal{M}^*(X)$ and $\mu(\Gamma_Q^f(x)) = 0$ for all $x \in X$. Therefore, f is weak μ -expansive with a finite partition Q of X . □

Topological conjugacy is important in the study of iterated functions and more general dynamical systems. If the dynamics of one iterated function can be solved, then those for any topological conjugate function follow trivially. The next theorem implies that the property of having weak measure expansivity is a conjugacy invariant. Given a measure $\mu \in \mathcal{M}^*(X)$ and a homeomorphism $\phi : X \rightarrow Y$, we denote by $\phi_*(\mu)$ the *pullback measure of μ* defined by $\phi_*(\mu)(A) = \mu(\phi^{-1}(A))$ for all borelian A .

Theorem 2.7. *Let X and Y be compact metric spaces and a homeomorphism $f : X \rightarrow X$ be weak μ -expansive for all $\mu \in \mathcal{M}^*(X)$. If $\phi : X \rightarrow Y$ is a homeomorphism from X to Y , then $g : Y \rightarrow Y$ is weak $\phi_*(\mu)$ -expansive, where $g = \phi \circ f \circ \phi^{-1}$.*

Proof. Since a homeomorphism f is weak μ -expansive, there exist $\delta > 0$ and a finite δ -partition $P = \{A_1, A_2, \dots, A_n\}$ of X such that $\mu(\Gamma_P^f(x)) = 0$ for all $x \in X$ and all $\mu \in \mathcal{M}^*(X)$. Let

$$\delta' = \max\{\text{diam}\phi(A_i) : i = 1, \dots, n\} \text{ and } P' = \{\phi(A_1), \phi(A_2), \dots, \phi(A_n)\}.$$

Then P' is clearly a finite δ' -partition of Y . We claim that

$$\Gamma_{P'}^g(y) \subset \phi(\Gamma_P^f(\phi^{-1}(y))) \text{ for all } y \in Y.$$

Indeed, if $z \in \Gamma_{P'}^g(y)$, then $g^j(z) \in P'(g^j(y))$ for all j by the definition. This means that $g^j(z)$ and $g^j(y)$ are contained in the same element, say $\phi(A_k)$, of P' for some k . Since $g^j = \phi \circ f^j \circ \phi^{-1}$, we can rewrite the above statement as follows: $\phi \circ f^j \circ \phi^{-1}(z)$ and $\phi \circ f^j \circ \phi^{-1}(y)$ are elements of $\phi(A_k)$ for some k . This fact implies that

$$f^j \circ \phi^{-1}(z) \in A_k \text{ and } f^j \circ \phi^{-1}(y) \in A_k$$

for some $1 \leq k \leq n$. By the definition of the dynamic P -ball of f , we get $f^j(\phi^{-1}(z)) \in P(f^j(\phi^{-1}(y)))$. From this, we know $\phi^{-1}(z) \in \Gamma_P^f(\phi^{-1}(y))$. Therefore we get $z \in \phi(\Gamma_P^f(\phi^{-1}(y)))$. Consequently we have

$$\phi_*\mu(\Gamma_{P'}^g(y)) = \mu(\phi^{-1}(\Gamma_{P'}^g(y))) \leq \mu(\Gamma_P^f(\phi^{-1}(y))) = 0.$$

Hence $\phi_*(\mu)(\Gamma_{P'}^g(y)) = 0$ and g is weak $\phi_*(\mu)$ -expansive. □

3. Hyperbolicity of weak measure expansive diffeomorphisms

Let M be a compact C^∞ -manifold and $\text{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 -topology. Denote by d the distance on M induced from the Riemannian metric $\|\cdot\|$ on the tangent bundle TM .

Given a diffeomorphism $f : M \rightarrow M$ define the non-wandering set $\Omega(f)$ as the set of those $x \in M$ satisfying: for all $\varepsilon > 0$ there is $n \geq 1$ such that $B_\varepsilon(x) \cap f^n(B_\varepsilon(x)) \neq \emptyset$ where $B_\varepsilon(x)$ denotes the open ε -ball centered at x . Recall that f satisfies *Smale's Axiom A* if $\overline{\text{Per}(f)} = \Omega(f)$ and $\Omega(f)$ is hyperbolic. A diffeomorphism $f : M \rightarrow M$ is Ω -stable if there is a C^1 -neighborhood U of f such that for all $g \in U$ there is a homeomorphism $h : \Omega(f) \rightarrow \Omega(g)$ such that $h \circ f = g \circ h$. If f satisfies *Axiom A*, then $\Omega(f)$ decomposes into a finite disjoint union basic sets $\Omega(f) = \Lambda_1 \cup \dots \cup \Lambda_l$. A collection $\Lambda_{i_1}, \dots, \Lambda_{i_k}$ is called a cycle if there exist points $a_j \notin \Omega(f)$ for $j = 1, \dots, k$ such that $\omega(a_j) \subset \Lambda_{i_{j+1}}$ and $\alpha(a_j) \subset \Lambda_{i_j}$ (with $k+1 \equiv 1$). Here $\omega(x)$ is the set of cluster points of the forward orbit $\{f^n(x) : n \in \mathbb{N}\}$ of the iterated function f and is called *the ω -limit set of x* . Also $\alpha(x)$ is called *the α -limit set of x* and is defined in a similar fashion, but for the backward orbit. We say that f satisfies *the no cycle condition* (or has not cycles) if there are not cycles among the basic sets of $\Omega(f)$.

We say that a diffeomorphism f is *quasi-Anosov* if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded. R. M\u00e2n\u00e9 [6] proved that a diffeomorphism which is an element of the C^1 -interior of the set of all expansive

diffeomorphisms satisfying both Axiom *A* and the no cycle condition. K. Sakai *et al.* [10] proved that a diffeomorphism which is an element of the C^1 -interior of the set of all invariant measure expansive diffeomorphisms is quasi-Anosov. From the result, we know that the C^1 -interior of the set of all expansive diffeomorphisms is equal to the C^1 -interior of the set of all invariant measure expansive diffeomorphisms.

Now we state our first main result as follows.

Theorem 3.1. *Let \mathcal{WIE} be the set of all f -invariant weak measure expansive diffeomorphisms of M . Denote by $\text{int}(\mathcal{WIE})$ is a C^1 -interior of \mathcal{WIE} . A diffeomorphism $f \in \text{int}(\mathcal{WIE})$ if and only if f is Ω -stable.*

To prove Theorem 3.1, we need some definitions and lemmas. Let $f \in \text{Diff}(M)$, and let Λ be a closed f -invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df -invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E_x^s}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E_x^u}\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n > 0$. We say that $f \in \text{Diff}(M)$ is contained in $\mathcal{F}^1(M)$ if there exists a C^1 -neighborhood $\mathcal{U}(f)$ such that for all $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic.

Lemma 3.2 ([4]). *Let $f \in \text{Diff}(M)$. The following properties are mutually equivalent:*

- (i) f is Ω -stable,
- (ii) f satisfies both Axiom *A* and the no cycle condition, and
- (iii) $f \in \mathcal{F}^1(M)$.

Lemma 3.3 ([3, Franks' lemma]). *Let $\mathcal{U}(f)$ be any given C^1 -neighborhood of f . Then there exists $\delta > 0$ such that for a finite set $\{x_1, x_2, \dots, x_n\}$, a neighborhood U of $\{x_1, x_2, \dots, x_n\}$ and linear maps $L_i : T_{x_i}M \rightarrow T_{f(x_i)}M$ satisfying $\|L_i - D_{x_i}f\| < \delta$ for $1 \leq i \leq n$, there are $\varepsilon_0 > 0$ and $g \in \mathcal{U}(f)$ such that*

- (i) $g(x) = f(x)$ if $x \in M \setminus U$ and
- (ii) $g(x) = \exp_{f(x_i)} \circ L_i \circ \exp_{x_i}^{-1}(x)$ if $x \in B_{\varepsilon_0}(x_i)$ for all $1 \leq i \leq n$.

Observe that the assertion (ii) implies that

$$g(x) = f(x) \text{ if } x \in \{x_1, x_2, \dots, x_n\}$$

and that $D_{x_i}g = L_i$ for all $1 \leq i \leq n$.

Proof of Theorem 3.1. First of all, we will show if a diffeomorphism f is Ω -stable, then $f \in \text{int}(\mathcal{WIE})$. By [10], we know that if f satisfies both Axiom *A* and the no cycle condition, then f is contained in the C^1 -interior of the set of all f -invariant measure expansive diffeomorphisms. So this direction of proof is clear, because if a diffeomorphism f is measure expansive, then f is weak measure expansive.

But in this paper, we give a proof of this directly. Let f satisfy both Axiom A and the no cycle condition. Since the set of Ω -stable diffeomorphisms is open in $\text{Diff}(M)$, there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ satisfies both Axiom A and the no cycle condition. By [11], $g|_{\Omega(g)}$ is expansive, and thus, $g|_{\Omega(g)}$ is weak measure expansive, that is, there is a finite partition \mathcal{P} of M such that $\mu(\Gamma_{\mathcal{P}}^g(x)) = 0$ for all $x \in \Omega(g)$. Since $\mu(\Omega(g)) = 1$ because μ is g -invariant probability measure, $\mu(\Gamma_{\mathcal{P}}^g(x)) = 0$ for a.e. $x \in M$. Thus g is weak μ expansive for all $\mu \in \mathcal{M}_g^*(M)$ on M , that is, $g \in \mathcal{WIE}$. Hence $f \in \text{int}(\mathcal{WIE})$.

Second, we prove only if part. Note that f is weak measure expansive implies that f^n is weak measure expansive for all $n \in \mathbb{Z} \setminus \{0\}$ by Lemma 2.5. To prove this direction, we claim that if $f \in \text{int}(\mathcal{WIE})$, then $f \in \mathcal{F}^1(M)$. Let $f \in \text{int}(\mathcal{WIE})$ and assume $f \notin \mathcal{F}^1(M)$. Since $f \in \text{int}(\mathcal{WIE})$, there is a C^1 -neighborhood $\mathcal{U}(f)$ of f such that for all $g \in \mathcal{U}(f)$ and $\mu \in \mathcal{M}_g^*(M)$, g is weak μ -expansive. On the other hand, since $f \notin \mathcal{F}^1(M)$, there are $g \in \mathcal{U}(f)$ and non-hyperbolic periodic point p of g . By Lemma 3.3, we can assume that $D_p g^{\pi(p)}$ has either only one real eigenvalue λ with $|\lambda| = 1$, or only one pair of complex conjugated eigenvalues. Denote by E_p^c the eigenspace corresponding to λ .

At first, we consider the case $\dim E_p^c = 1$. In this case, suppose that $\lambda = 1$ for simplicity. Then by Lemma 3.3, there are $\varepsilon_0 > 0$ and $h \in \mathcal{U}(f)$ such that

- (i) $h^{\pi(p)}(p) = g^{\pi(p)} = p$, and
- (ii) $h(x) = \exp_{g^{i+1}(p)} \circ D_{g^i(p)} g \circ \exp_{g^i(p)}^{-1}(x)$,

if $x \in B_{\varepsilon_0}(g^i(p))$ for all $0 \leq i \leq \pi(p) - 1$. Since $\lambda = 1$, there is a small arc $\mathcal{I}_p \subset B_{\varepsilon_0}(p) \cap \exp_p(E_p^c(\varepsilon_0))$ with its center at p such that

- $h^i(\mathcal{I}_p) \cap h^j(\mathcal{I}_p) = \emptyset$ if $i \neq j$ for $0 \leq i, j \leq \pi(p) - 1$,
- $h^{\pi(p)}(\mathcal{I}_p) = \mathcal{I}_p$, and
- $h^{\pi(p)}|_{\mathcal{I}_p}$ is the identity map.

Let $\mathfrak{M}_{\mathcal{I}_p}$ be the normalized Lebesgue measure on \mathcal{I}_p . Define $\tilde{\mu} \in \mathcal{M}_h^*(M)$ by

$$\tilde{\mu}(B) = \frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} \mathfrak{M}_{\mathcal{I}_p} [h^{-j}(B \cap h^j(\mathcal{I}_p))]$$

for all Borel set B of M (this is well-defined). Since $h^{\pi(p)}|_{\mathcal{I}_p}$ is the identity map, $h^{\pi(p)}|_{\mathcal{I}_p}$ is not weak $\tilde{\mu}$ -expansive. Therefore h is not weak $\tilde{\mu}$ -expansive by Theorem 2.6. This is contradicts the fact that $h \in \mathcal{U}(f)$.

Similarly, we can prove the other case and complete the proof. □

We say that a subset $\mathcal{G} \subset \text{Diff}(M)$ is *residual* if \mathcal{G} contains the intersection of a countable family of open and dense subsets of $\text{Diff}(M)$. In this case \mathcal{G} is dense in $\text{Diff}(M)$. A property “ \mathcal{P} ” is said to be C^1 -*generic* if “ \mathcal{P} ” holds for all diffeomorphisms which belong to some residual subset of $\text{Diff}(M)$. We use the terminology for C^1 -*generic* f to express “there is residual subset of $\mathcal{G} \subset \text{Diff}(M)$ ”

such that for any $f \in \mathcal{G} \dots$. For expansivity, Yang and Gan [13] proved that C^1 -generically, if the homoclinic class which contains a hyperbolic periodic point is expansive, then it is hyperbolic. Arbieto [1] showed that C^1 -generically, if a diffeomorphism is expansive, then the diffeomorphism satisfies both Axiom A and the no-cycle condition. Moreover, Pacifico and Vieitez [9] claimed that diffeomorphisms in a residual subset far from homoclinic tangencies are measure expansive. Recently Lee [5] proved that C^1 -generically, if a diffeomorphism f does not present a homoclinic tangency, then it is weak Lebesgue measure expansive. In this direction we prove the following theorem which is the second main result of this paper.

Theorem 3.4. *For C^1 -generic $f \in \text{Diff}(M)$, if f is weak measure-expansive, then f satisfies both Axiom A and the no cycle condition.*

To prove the theorem, we need some definitions and lemmas. Denote by $P(f)$ the set of periodic points of f , and by $P_h(f)$ the set of hyperbolic periodic points of f . We say a hyperbolic periodic point p of f with period $\pi(p)$ said to have a δ -weak eigenvalue if there is an eigenvalue λ of $Df^{\pi(p)}(p)$ such that

$$(1 - \delta)^{\pi(p)} < |\lambda| < (1 + \delta)^{\pi(p)}.$$

We say that the periodic point p has *simple real spectrum* if all of its eigenvalues are real and have multiplicity one.

Definition. For $\eta > 0$ and $f \in \text{Diff}(M)$, a C^1 -curve \mathcal{L} is called η -periodic curve of f containing $p \in P_h(f)$ with period $\pi(p)$ if \mathcal{L} is diffeomorphic to $[0, 1]$, \mathcal{L} is periodic with period $\pi(p)$ and $l(f^i(\mathcal{L})) < \eta$ for any $i = 0, \dots, \pi(p) - 1$, where $l(\mathcal{L})$ denotes the length of \mathcal{L} .

Lemma 3.5. *There is a residual set $\mathcal{G} \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}$,*

- (a) *for any $p \in P_h(f)$ and $\delta > 0$, if for any sufficiently small C^1 -neighborhood $\mathcal{U}(f)$ of f there exists $g \in \mathcal{U}(f)$ such that g has a δ -periodic curve γ_g containing p_g with period $\pi(p)$, then f has a δ -periodic curve γ containing p .*
- (b) *if $g_n \rightarrow f$ and $p_{g_n} \in P_h(g_n)$ has a δ -weak eigenvalue, then there exists $p \in P_h(f)$ with 2δ -weak eigenvalue such that $p_{g_n} \rightarrow p$.*

Proof. (a) Let $K(M)$ be the space of nonempty closed subsets of M with the Hausdorff metric d_H . Take a countable basis $\beta = \{\mathcal{V}_1, \mathcal{V}_2, \dots\}$ of $K(M)$. For each $n \in \mathbb{N}$, we let $H_n(\delta) = \{f \in \text{Diff}(M) : f \text{ has a } \delta\text{-periodic curve } \gamma \text{ such that } O_f(\gamma) \in \mathcal{V}_n\}$. Then we know that $H_n(\delta)$ is open in $\text{Diff}(M)$ since every orbit $O_f(\gamma)$ of periodic curve γ has a continuation; i.e., there are a C^1 -neighborhood $\mathcal{U}(f)$ of f and a neighborhood U of $O_f(\gamma)$ such that for any $g \in \mathcal{U}(f)$, $\cap_{n \in \mathbb{Z}} g^n(U) = O_g(\gamma_g)$ is an g -orbit of periodic curve γ_g containing p_g . Let $N_n(\delta) = \text{Diff}(M) - \overline{H_n(\delta)}$. Then $N_n(\delta) \cup H_n(\delta)$ is an open and dense subset of $\text{Diff}(M)$. Let

$$\mathcal{G}(\delta) = \bigcap_{n \in \mathbb{N}} (N_n(\delta) \cup H_n(\delta)) \text{ and } \mathcal{G} = \bigcap_{\delta \in \mathbb{Q}^+} \mathcal{G}(\delta).$$

Then we can see that \mathcal{G} is a desired residual subset of $\text{Diff}(M)$.

In fact, for any $f \in \mathcal{G}$ and $p \in P_h(f)$, take $\mathcal{V}_n \in \beta$ containing $O_f(p)$ in $K(M)$ such that $O_f(p)$ is unique compact f -invariant set belongs to \mathcal{V}_n . Choose $\delta \in \mathbb{Q}^+$ satisfying

$$B_{d_H}(O_f(p), 2\delta) \subset \mathcal{V}_n,$$

where $B_{d_H}(O_f(p), 2\delta)$ is an open ball of $O_f(p)$ with radius $2/\delta$. Let $\mathcal{U}(f)$ be a C^1 -neighborhood of f such that for any $h \in \mathcal{U}(f)$,

$$d_H(O_f(p), O_h(p_h)) < \delta \text{ and } O_h(p_h) \in \mathcal{V}_n,$$

where p_h is the continuation of p under h . For the $\mathcal{U}(f)$, by assumption there exists $g \in \mathcal{U}(f)$ such that g has a δ -periodic curve γ_g containing p_g . Since $d_H(O_g(p_g), O_g(\gamma_g)) < \delta$, we have

$$\begin{aligned} d_H(O_f(p), O_g(\gamma_g)) &\leq d_H(O_f(p), O_g(p_g)) + d_H(O_g(p_g), O_g(\gamma_g)) \\ &< \delta + \delta = 2\delta. \end{aligned}$$

Hence we have $O_g(\gamma_g) \in \mathcal{V}_n$ and $g \in H_n(\delta)$. This means that $f \in \overline{H_n(\delta)}$ and so $f \notin N_n(\delta)$. Thus $f \in H_n(\delta)$, i.e., f has a δ -periodic curve γ such that $O_f(\gamma) \in \mathcal{V}_n$.

(b) It is straightforward by [1]. □

Lemma 3.6. *There is a residual set $\mathcal{G} \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}$, if f is weak measure expansive, then there exists $\delta > 0$ such that every $p \in P_h(f)$ has no δ -weak eigenvalue.*

Proof. Let \mathcal{G} be a residual subset of $\text{Diff}(M)$ which is obtain in Lemma 3.5. Suppose there exists a weak measure expansive diffeomorphism $f \in \mathcal{G}$ such that for any $\delta > 0$, there exists $p \in P_h(f)$ such that $D_p f^{\pi(p)}$ has a δ -weak eigenvalue λ . Since f is weak measure expansive, for any $\mu \in \mathcal{M}^*(M)$ there exist $e > 0$ and finite e -partition $P = \{A_1, A_2, \dots, A_n\}$ of M such that $\mu(\Gamma_P^f(x)) = 0$ for all $x \in M$. By Lemma 3.3, there exist $g \in \mathcal{U}(f)$ and $p_g \in P_h(g)$ such that $D_{p_g} g^{\pi(p_g)}$ has a simple real spectrum and an eigenvalue λ_g with $|\lambda_g| = 1$. By Lemma 3.3, there exist $\varepsilon_0 > 0$ and $h \in \mathcal{U}(g) \subset \mathcal{U}(f)$ such that

- (i) $h^{\pi(p_g)}(p_g) = g^{\pi(p_g)}(p_g) = p_g$, and
- (ii) $h(x) = \exp_{g^{i+1}(p_g)} \circ D_{g^i(p_g)} g \circ \exp_{g^i(p_g)}^{-1}(x)$,

if $x \in B_{\varepsilon_0}(g^i(p_g))$, $0 \leq i \leq \pi(p_g) - 1$. Put $\mathcal{K} = \{c \cdot v_g \mid -1 \leq c \leq 1\}$, where v_g is an eigenvector corresponding to λ_g . Then there exists \mathcal{I} , subarc of $\exp_{p_g}(\mathcal{K}) \cap B_{\varepsilon_0}(p_g)$, such that

- $h^i(\mathcal{I}) \cap h^j(\mathcal{I}) = \emptyset$, if $0 \leq i \neq j \leq \pi(p_g) - 1$, and
- $h^{\pi(p)}(\mathcal{I}) = \mathcal{I}$,

i.e., \mathcal{I} is a c -periodic curve of h . Then by Lemma 3.5(a), there exists c -periodic curve \mathcal{I}_f for f such that $f^{\pi(p)}|_{\mathcal{I}_f}$ is invariant. Put $J_f = \mathcal{I}_f \cap A_i (\neq \emptyset)$ for some

$A_i \in P$. Then for all $x \in J_f$, we get $J_f \subset \Gamma_P^{f^{\pi(p)}}(x)$. Define a new measure $\widehat{\mu}$ on M by

$$\widehat{\mu}(B) = \mathfrak{M}(B \cap J_f),$$

where B is a Borel set and \mathfrak{M} is a normalized Lebesgue measure on J_f .

Since $\widehat{\mu}(\Gamma_P^{f^{\pi(p)}}(x)) \geq \widehat{\mu}(J_f) > 0$, we arrive at the contradiction to the fact that $f^{\pi(p)}$ is weak measure expansive. The contradiction completes the proof. \square

End of proof of Theorem 3.4. Let $f \in \mathcal{G}$ be weak measure expansive. To derive a contradiction, we assume that $f \notin \mathcal{F}^1(M)$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n(f) = \{g \in \text{Diff}(M) : d_1(f, g) < \frac{1}{n}\}$ be the C^1 -neighborhood of f . Then there exist $g_n \in \mathcal{U}_n(f)$ and a periodic point p_{g_n} of g_n such that p_{g_n} has an eigenvalue λ_{g_n} with $|\lambda_{g_n}| = 1$. By Lemma 3.3, for any C^1 -neighborhood $\mathcal{U}(g_n)$ of g_n , we can take $\tilde{g}_n \in \mathcal{U}(g_n)$ and $p_{\tilde{g}_n} \in P_h(\tilde{g}_n)$ such that $p_{\tilde{g}_n}$ has a $\frac{\delta}{2}$ -weak eigenvalue. By Lemma 3.5(b), since $\tilde{g}_n \rightarrow f$, there exists $p \in P_h(f)$ such that p has a δ -weak eigenvalue. This is a contradiction to Lemma 3.6. And so $f \in \mathcal{F}^1(M)$. This means that f satisfies both Axiom A and the no cycle condition. \square

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