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VANISHING PROPERTIES OF p-HARMONIC ℓ-FORMS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we show several vanishing type theorems for p-harmonic ℓ-forms on Riemannian manifolds ($p \geq 2$). First of all, we consider complete non-compact immersed submanifolds M^n of N^{n+m} with flat normal bundle, we prove that any p-harmonic ℓ-form on M is trivial if N has pure curvature tensor and M satisfies some geometric conditions. Then, we obtain a vanishing theorem on Riemannian manifolds with a weighted Poincaré inequality. Final, we investigate complete simply connected, locally conformally flat Riemannian manifolds M and point out that there is no nontrivial p-harmonic ℓ-form on M provided that the Ricci curvature has suitable bound.

1. Introduction

Suppose that M is a complete noncompact oriented Riemannian manifold of dimension n. At a point $x \in M$, let $\{\omega_1, \ldots, \omega_n\}$ be a positively oriented orthonormal coframe on $T^*_x(M)$, the Hodge star operator acting on the space of smooth ℓ-forms $\Lambda^\ell(M)$ is given by

$$*(\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}) = \omega_{j_1} \wedge \cdots \wedge \omega_{j_{n-\ell}},$$

where $j_1, \ldots, j_{n-\ell}$ are selected such that $\{\omega_{i_1}, \ldots, \omega_{i_\ell}, \omega_{j_1}, \ldots, \omega_{j_{n-\ell}}\}$ gives a positive orientation. Let d be the exterior differential operator, so its dual operator δ is defined by

$$\delta = (-1)^{n(\ell+1)+1} * d * .$$

Then the Hogde-Laplace-Beltrami operator Δ acting on the space of smooth ℓ-forms $\Omega^\ell(M)$ is of form

$$\Delta = -(\delta d + d\delta).$$

When M is compact, it is well-known that the space of harmonic ℓ-forms is isomorphic to its ℓ-th de Rham cohomology group. This is not true when M is non-compact but the theory of L^2 harmonic forms still has some interesting

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applications. For further results, we refer the reader to [2,3]. In [16], Li studied Sobolev inequality on spaces of harmonic ℓ-forms on compact Riemannian manifolds. He gave estimates of the bottom of ℓ-spectrum and proved that the space of harmonic ℓ-forms is of finite dimension provided that the Ricci curvature is bounded from below. In [29], Tanno studied L^2 harmonic ℓ-forms on complete orientable stable minimal hypersurface M immersed in the Euclidean space $\mathbb{R}^{n+1}$. He showed that there are no non trivial L^2 harmonic ℓ-forms on M if $n \leq 4$. Later, Zhu generalized Tanno's results to manifolds with non-negative isotropic curvature (see [35]). He also proved in [36] that the Tanno's results are true if M^n, $n \leq 4$, is a complete noncompact strongly stable hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ or $\mathbb{S}^{n+1}$. Recently, in [20], Lin investigated spaces of L^2 harmonic ℓ-forms $H^\ell(L^2(M))$ on submanifolds in Euclidean space with flat normal bundle. Assumed that the submanifolds are of finite total curvature, Lin showed that the space $H^\ell(L^2(M))$ has finite dimension (see also [37] for the case $\ell = 2$). For further results in this direction, we refer to [21–23, 27, 35, 36] and the references therein. It is also worth to notice that the main tools to study the spaces of harmonic ℓ-forms are the Bochner type formulas and refined Kato type inequalities. In 2000, Calderbank et al. gave very general forms of Kato type inequalities in [1]. Then in [33], Wang used them to prove a vanishing property of the space of L^2 harmonic ℓ-forms on convex cocompact hyperbolic manifolds. Later, in [32], Wan and Xin studied L^2 harmonic ℓ-forms on conformally compact manifolds with a rather weak boundary regularity assumption. Recently, in [7], Cibotaru and Zhu introduced a proof of the mentioned results from [1] avoiding as much as possible representation theoretic technicalities. The refined Kato type inequalities they obtained also refined those used in [32, 33].

On the other hand, the p-Laplacian operator on a Riemannian manifold M is defined by

$$\Delta_p u := \mathrm{Div}(|\nabla u|^{p-2}\nabla u)$$

for any function $u \in W^{1,p}_{loc}(M)$ and $p > 1$, which arises as the Euler-Lagrange operator associated to the p-energy functional

$$E_p(u) := \int_M |\nabla u|^p.$$

Therefore, if u is a smooth p-harmonic function, then du is a p-harmonic 1-form. We refer the reader to [14, 24] for the connection between p-harmonic functions and the inverse mean curvature flow. Motivated by the above beautiful relationship between the space of harmonic ℓ-forms and the geometry of Riemannian manifolds, it is very natural for us to study the geometric structures of Riemannian manifolds by using the vanishing properties of p-harmonic ℓ-forms with finite L^q energy for some $p \geq 2$ and $q \geq 0$.

Recall that an ℓ-form ω on a Riemannian manifold M is said to be p-harmonic if ω satisfies $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$. When $p = 2$, a p-harmonic ℓ-form is exactly a harmonic ℓ-form. Some vanishing properties of the space of p-harmonic ℓ-forms were given by X. Zhang in [34]. In particular, Zhang showed that there are no nontrivial p-harmonic 1-forms in $L^q(M)$, $q > 0$ if the Ricci curvature on M is nonnegative. Motivated by Zhang's results, Chang et al., in [5] proved that any bounded set of p-harmonic 1-forms in $L^q(M)$, $0 < q < \infty$, is relatively compact with respect to the uniform convergence topology. Recently, it is showed that the set of p-harmonic 1-forms has closed relationship with the connectedness at infinity of the manifold, in particularly, with p-nonparabolic ends. In [10], the first author and Seo studied the connectedness at infinity of complete submanifolds by using the theory of p-harmonic functions. For lower-dimensional cases, they proved that if M is a complete orientable noncompact hypersurface in $\mathbb{R}^{n+1}$ and a δ-stable inequality holds on M, then M has at most one p-nonparabolic end. It was also proved that if M^n is a complete noncompact submanifold in $\mathbb{R}^{n+k}$ with sufficiently small L^n-norm of the traceless second fundamental form, then M has at most one p-nonparabolic end. For the reader's convenience, let us recall a definition of the p-nonparabolic ends. Let $E \subset M$ be an end of M, namely, E is a unbounded connected component of $M \setminus \Omega$ for a sufficiently large compact subset $\Omega \subset M$ with smooth boundary. As in usual harmonic function theory, we define the p-parabolicity and p-nonparabolicity of E as follows (see [4] and the references therein):

Definition 1.1. An end E of the Riemannian manifold M is called *p-parabolic* if for every compact subset $K \subset \overline{E}$

$$\mathrm{cap}_p(K, E) := \inf \int_E |\nabla f|^p = 0,$$

where the infimum is taken among all $f \in C_0^\infty(\overline{E})$ such that $f \geq 1$ on K. Otherwise, the end E is called *p-nonparabolic*.

The first main result in this paper is the below theorem.

Theorem 1.2. *Let M^n $(n \geq 3)$ be a complete non-compact immersed submanifold of N^{n+m}. Assume that M has flat normal bundle, N^{n+m} has pure curvature tensor and the $(1, n-1)$-curvature of N^{n+m} is not less than $-k$, $k \geq 0$. If one of the following conditions*

1. $$|A|^2 \leq \frac{n^2|H|^2 - 2k}{n-1}, \ \mathrm{vol}(M) = \infty;$$
2. $$\frac{n^2|H|^2 - 2k}{n-1} < |A|^2 \leq \frac{n^2|H|^2}{n-1} \text{ and } \lambda_1(M) > \frac{kp^2(n-1)}{4(p-1)(n+p-2)};$$
3. *the total curvature $\|A\|_n$ is bounded by*

$$\|A\|_n^2 < \min\left\{\frac{n^2}{(n-1)C_S},\ \frac{2}{(n-1)C_S}\left[\frac{4(p-1)(n+p-2)}{p^2(n-1)} - \frac{k}{\lambda_1(M)}\right]\right\};$$

4. $\sup_M |A|$ *is bounded and the fundamental tone satisfies*

$$\lambda_1(M) > \frac{p^2(n-1)(2k+(n-1)\sup_M |A|^2)}{8(p-1)(n+p-2)},$$

holds true, then every p-harmonic L^p 1-form on M is trivial. Therefore, M has at most one p-nonparabolic end. Here C_S is the Sobolev constant which depends only on n (see Lemma 2.1).

Here, we refer the reader to Section 2, for a definition of $(1, n-1)$-curvature. This results generalizes a work of the first author in [8]. On the other hand, we consider p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality. Recall that let (M^n, g) be a Riemannian manifold of dimension n and $\rho \in \mathcal{C}(M)$ be a positive function on M. We say that M has a weighted Poincaré inequality, if

$$\int_M \rho\varphi^2 \le \int_M |\nabla\varphi|^2 \tag{1.1}$$

holds true for any smooth function $\varphi \in \mathcal{C}_0^\infty(M)$ with compact support in M. The positive function ρ is called the weighted function. It is easy to see that if the bottom of the spectrum of Laplacian $\lambda_1(M)$ is positive, then M satisfies a weighted Poincaré inequality with $\rho \equiv \lambda_1$. This is because $\lambda_1(M)$ can be characterized by variational principle

$$\lambda_1(M) = \inf\left\{ \frac{\int_M |\nabla\varphi|^2}{\int_M \varphi^2} : \varphi \in \mathcal{C}_0^\infty(M) \right\}.$$

When M satisfies a weighted Poincaré inequality then M has many interesting properties concerning topology and geometry. For example, in [31], Vieira obtained vanishing theorems for L^2 harmonic 1-forms on complete Riemannian manifolds satisfying a weighted Poincaré inequality and having a certain lower bound of the bi-Ricci curvature. His theorems are an improvement of Li-Wang's and Lam's results (see [15, 18, 19]). Moreover, some applications to study geometric structures of minimal hypersurfaces are also given. We refer to [6, 11] and the references therein for further results on the vanishing property of the space of harmonic ℓ-forms. In the nonlinear setting, Chang et al. studied p-harmonic functions with finite L^q energy in [4], and proved some vanishing type theorems on Riemannian manifolds satisfying a weighted Poincaré inequality. Later, Sung and Wang, Dat and the first author used theory of p-harmonic functions to show some interesting rigidity properties of Riemannian manifolds with maximal p-spectrum (see [9, 28]). In this paper, we will investigate Riemannian manifolds with a weighted Poincaré inequality and prove some vanishing results for p-harmonic ℓ-forms on such these manifolds. Our results can be considered as a generalization of Vieira's and the first author's results (see [8, 31]). Finally, we are also interested in locally conformally flat Riemannian manifolds, our theorem is the following vanishing property.

Theorem 1.3. *Let (M^n, g), $n \geq 3$, be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold. If one of the following conditions*

1.
$$\|T\|_{n/2} + \frac{\|R\|_{n/2}}{\sqrt{n}} < \frac{4\left(p-1+\min\left\{1, \frac{(p-1)^2}{n-1}\right\}\right)}{Sp^2};$$

2. *the scalar curvature R is nonpositive and*
$$K_{p,n} := \frac{p-1+\min\left\{1, \frac{(p-1)^2}{n-1}\right\}}{p^2} - \frac{n-1}{\sqrt{n}(n-2)} > 0,$$
and
$$\|T\|_{n/2} < \frac{4K_{p,n}}{S} = 4K_{p,n}\mathcal{Y}(\mathbb{S}^n);$$

holds true, then every p-harmonic 1-form with finite $L^p (p \geq 2)$ norm on M is trivial, and M must has at most one p-nonparabolic end. Here T is the traceless Ricci tensor, S is the constant given in Lemma 5.1, and $\mathcal{Y}(\mathbb{S}^n)$ is the Yamabe constant of $\mathbb{S}^n$.

The rest of this paper is organized as follows. In Section 2, we recall some basic notations and useful backgrounds on theory of smooth ℓ-forms. Then, in Section 3, we study p-harmonic ℓ-forms in submanifolds of N^{n+m} with flat normal bundle and pure curvature tensor. We will give a proof of Theorem 1.2 in this section. In Section 4, we derive some vanishing properties for p-harmonic ℓ-forms on manifolds with a weighted Poincaré inequality. Finally, in Section 5, we consider locally conformally flat Riemannian manifolds and give a proof of Theorem 1.3.

2. Preliminary notations

In this paper for $n \geq 3,\ 1 \leq \ell \leq n-1$ and $p \geq 2,\ q \geq 0$, we denote
$$C_{n,\ell} := \max\{\ell, n-\ell\}$$
and
$$A_{p,n,\ell} = \begin{cases} 1 + \dfrac{1}{\max \ell, n-\ell}, & \text{if } p = 2, \\ 1 + \dfrac{1}{(p-1)^2} \min\left\{1, \dfrac{(p-1)^2}{n-1}\right\}, & \text{if } p > 2 \text{ and } \ell = 1, \\ 1, & \text{if } p > 2 \text{ and } 1 < \ell \leq n-1. \end{cases}$$
Hence,
$$(p-1)^2(A_{p,n,\ell} - 1) = \begin{cases} \dfrac{1}{\max\{\ell, n-\ell\}}, & \text{if } p = 2 \text{ and } 1 \leq \ell \leq n-1, \\ \min\left\{1, \dfrac{(p-1)^2}{n-1}\right\}, & \text{if } p > 2 \text{ and } \ell = 1, \\ 0, & \text{if } p > 2 \text{ and } 1 < \ell \leq n-1. \end{cases}$$

We will use the following Sobolev inequality.

Lemma 2.1 ([13]). *Let M^n $(n \geq 3)$ be an n-dimensional complete submanifold in a complete simply-connected manifold with nonpositive sectional curvature. Then for any $f \in W_0^{1,2}(M)$ we have*

$$\left(\int_M |f|^{\frac{2n}{n-2}} dv\right)^{\frac{n-2}{n}} \leq C_S \int_M \left(|\nabla f|^2 + |H|^2 f^2\right) dv,$$

where C_S is the Sobolev constant which depends only on n.

An important ingredient in our methods is the following refined Kato inequality. In order to state the inequality, let us recall some notations. Suppose that $\{e_1, \dots, e_n\}$ is an orthonormal basic of $\mathbb{R}^n$. Let $\theta_1 : \Lambda^{\ell+1}\mathbb{R}^n \to \mathbb{R} \otimes \Lambda^{\ell}\mathbb{R}^n$ given by

$$\theta_1(v_1 \wedge \cdots \wedge v_{\ell+1}) = \frac{1}{\sqrt{\ell+1}} \sum_{j=1}^{\ell+1} (-1)^j v_j \otimes v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_{\ell+1}$$

and $\theta_2 : \Lambda^{\ell-1}\mathbb{R}^n \to \mathbb{R} \otimes \Lambda^{\ell}\mathbb{R}^n$ by

$$\theta_2(\omega) = -\frac{1}{\sqrt{n+1-\ell}} \sum_{j=1}^{n} e_j \otimes (e_j \wedge \omega).$$

Lemma 2.2 ([1, 10, 17]). *For $p \geq 2, \ell \geq 1$, let ω be a p-harmonic ℓ-form on a complete Riemannian manifold M^n. The following inequality holds true*

$$\left|\nabla\left(|\omega|^{p-2}\omega\right)\right|^2 \geq A_{p,n,\ell} \left|\nabla|\omega|^{p-1}\right|^2. \tag{2.1}$$

Moreover, when $p = 2, \ell > 1$ then the equality holds if and only if there exists a 1-form α such that

$$\nabla\omega = \alpha \otimes \omega - \frac{1}{\sqrt{\ell+1}} \theta_1(\alpha \wedge \omega) + \frac{1}{\sqrt{n+1-\ell}} \theta_2(i_\alpha \omega).$$

Proof. The inequality (2.1) is well-known when $\ell = 1, p = 2$, for example, see [17]. When $\ell = 1, p > 2$, the inequality (2.1) was proved by Seo and the first author in [10]. Note that when $p = 2, \ell > 1$, the inequality (2.1) was proved by Calderbank et al. (see [1]) but we refer to [7] in a way convenient for our purpose without introducing abstract notation.

Finally, when $p > 2, \ell > 1$, the inequality (2.1) is standard (see [1]). □

Note that if M is not complete, then (2.1) is still true provided that ω is a harmonic field, see [7] for further discussion.

Suppose that M is a complete noncompact Riemannian manifold. Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on M with dual coframe $\{\omega_1, \dots, \omega_n\}$. Given an ℓ-form ω on M, the Weitzenböck curvature operator K_ℓ acting on ω is defined by

$$K_\ell = \sum_{j.k=1}^{n} \omega_k \wedge i_{e_j} R(e_k, e_j)\omega.$$

Using the Weitzenböck curvature operator, we have the following Bochner type formula for ℓ-forms.

Lemma 2.3 ([17]). *Let $\omega = \sum_I a_I \omega_I$ be a ℓ-form on M. Then*

$$\begin{aligned}\Delta|\omega|^2 &= 2\langle \Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2\langle E(\omega), \omega\rangle \\ &= 2\langle \Delta\omega, \omega\rangle + 2|\nabla\omega|^2 + 2K_\ell(\omega,\omega),\end{aligned}$$

where $E(\omega) = \sum\limits_{j,k=1}^{n} \omega_k \wedge i_{e_j} R(e_k, e_j)\omega$.

In order to estimate the K_ℓ, we need to define a new curvature which is appear naturally as a component of the Weitzenböck curvature operator (see [20, 22]).

Definition 2.4. Let M^n be a complete immersed submanifold in a Riemannian manifold N^{n+m} with flat normal bundle. Here the submanifold M is said to have flat normal bundle if the normal connection of M is flat, namely the components of the normal curvature tensor of M are zero. For any point $x \in N^{n+m}$, choose an orthonormal frame $\{e_i, \dots, e_n\}_{i=1}^{n+m}$ of the tangent space T_xN and define

$$\overline{R}^{(\ell,n-\ell)}([e_{i_1} \dots, e_{i_n}]) = \sum_{k=1}^{\ell} \sum_{h=\ell+1}^{n} \overline{R}_{i_k i_h i_k i_h}$$

for $1 \le \ell \le n-1$, where the indices $1 \le i_1, \dots i_n \le n+m$ are distinct with each other. We call $\overline{R}^{(\ell,n-\ell)}([e_{i_1} \dots, e_{i_n}])$ the $(\ell, n-\ell)$-curvature of N^{n+m}.

Assume that N has pure curvature tensor, namely for every $p \in N$ there is an orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pN such that $R_{ijrs} := \langle R(e_i, e_j)e_r, e_s\rangle = 0$ whenever the set $\{i, j, r, s\}$ contains more than two elements. Here R_{ijrs} denote the curvature tensors of N. It is worth to notice that all 3-manifolds and conformally flat manifolds have pure curvature tensor. It was proved in [20] that

$$K_\ell(\omega,\omega) \ge \frac{1}{2}(n^2|H|^2 - C_{n,\ell}|A|^2) + \inf_{i_1,\dots,i_n} \overline{R}^{(\ell,n-\ell)}([e_{i_1}, \dots, e_{i_n}])|\omega|^2.$$

Finally, to prove the vanishing property of p-harmonic ℓ-forms, we use the following useful estimate.

Lemma 2.5. *For any closed ℓ-form ω and $\varphi \in \mathcal{C}^\infty(M)$, we have*

$$|d(\varphi\omega)| = |d\varphi \wedge \omega| \le |d\varphi| \cdot |\omega|.$$

Proof. Let $\{X_i\}$ be a local orthonormal frame and $\{dx_i\}$ is the dual coframe. Since ω is closed, we have $d(\varphi\omega) = d\varphi \wedge \omega$. Suppose that

$$\omega = \sum_{|I|=\ell} \omega_I dx^I = \sum_{|K|=\ell-1} \sum_{j=1}^{n} \omega_{jK} dx_j \wedge dx^K,$$

where $\omega_{jK} = 0$ if $j \in K$. Therefore, denote $\varphi_i = \nabla_{X_i}\varphi$, we have

$$d\varphi \wedge \omega = \sum_{|K|=\ell-1} \sum_{i\neq j. i,j\notin K}^{n} \varphi_i \omega_{jK} dx_i \wedge dx_j \wedge dx^K.$$

Observe that, for any $a_i, b_i \in \mathbb{R}, i = \overline{1,n}$

$$\sum_{i<j}(a_i b_j - a_j b_i)^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right),$$

we infer

$$\begin{aligned}|d\varphi \wedge \omega|^2 &= \sum_{|K|=\ell-1} \sum_{i<j, i,j\notin K} (\varphi_i \omega_{jK} - \varphi_j \omega_{iK})^2 \\ &\leq \sum_{|K|=\ell-1} \sum_{i<j} (\varphi_i \omega_{jK} - \varphi_j \omega_{iK})^2 \\ &\leq \sum_{|K|=\ell-1} \left(\sum_{i=1}^{n} \varphi_i^2\right)\left(\sum_{j=1}^{n} \omega_{jK}\right) = |d\varphi|^2 |\omega|^2.\end{aligned}$$

The proof is complete. □

3. p-harmonic ℓ-forms in submanifolds of N^{n+m} with flat normal bundle

Theorem 3.1. *Let M^n $(n \geq 3)$ be a complete non-compact immersed submanifold of N^{n+m}. Assume that M has flat normal bundle, N has pure curvature tensor and the $(\ell, n-\ell)$-curvature of $\mathbb{R}^{n+m}$ is not less than $-k$ for $1 \leq \ell \leq n-1$. If one of the following conditions*

1. $$|A|^2 \leq \frac{n^2|H|^2 - 2k}{C_{n,\ell}}, \quad \mathrm{vol}(M) = \infty;$$
2. $$\frac{n^2|H|^2 - 2k}{C_{n,\ell}} < |A|^2 \leq \frac{n^2|H|^2}{C_{n,\ell}} \ \textit{and}\ \lambda_1 > \frac{kQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1))};$$
3. *the total curvature $\|A\|_n$ is bounded by*
$$\|A\|_n^2 < \min\left\{\frac{n^2}{C_{n,\ell}C_S}, \ \frac{2}{C_{n,\ell}C_S}\left[\frac{4(Q-1+(p-1)^2(A_{p,n,\ell}-1))}{Q^2} - \frac{k}{\lambda_1(M)}\right]\right\};$$
4. $\sup_M |A|$ *is bounded and the fundamental tone satisfies*
$$\lambda_1(M) > \frac{Q^2(2k + C_{n,\ell}\sup_M |A|^2)}{8(Q-1+(p-1)^2(A_{p,n,\ell}-1))},$$

holds true, then every p-harmonic ℓ-form with finite L^Q-energy, $(Q \geq 2)$ on M is trivial.

Proof. Let $M_+ := M \setminus \{x \in M, \omega(x) = 0\}$. Let ω be any p-harmonic ℓ-form with finite L^Q norm. Applying the Bochner formula to the form $|\omega|^{p-2}\omega$, we obtain on M_+

$$\begin{aligned}&\frac{1}{2}\Delta|\omega|^{2(p-1)}\\ &= |\nabla(|\omega|^{p-2}\omega)|^2 - \big\langle(\delta d + d\delta)(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\big\rangle + K_\ell(|\omega|^{p-2}\omega, |\omega|^{p-2}\omega)\\ &= |\nabla(|\omega|^{p-2}\omega)|^2 - \big\langle\delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\big\rangle + |\omega|^{2(p-2)}K_\ell(\omega,\omega),\end{aligned}$$

where we used ω is p-harmonic in the second equality. This can be read as

$$\begin{aligned}&|\omega|^{p-1}\Delta|\omega|^{p-1}\\ =\ &\big(|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2\big) - |\omega|^{p-2}\big\langle\delta d(|\omega|^{p-2}\omega), \omega\big\rangle + |\omega|^{2(p-2)}K_\ell(\omega,\omega).\end{aligned}$$

By Kato type inequality, we infer

$$\begin{aligned}&|\omega|\Delta|\omega|^{p-1}\\ &\geq (p-1)^2(A_{p,n,\ell}-1)|\omega|^{p-2}|\nabla|\omega||^2 - \big\langle\delta d(|\omega|^{p-2}\omega), \omega\big\rangle + K_\ell|\omega|^p.\end{aligned}\tag{3.1}$$

Hence, for $q = Q - p$, we have

$$\begin{aligned}&|\omega|^{q+1}\Delta|\omega|^{p-1}\\ &\geq (p-1)^2(A_{p,n,\ell}-1)|\omega|^{p+q-2}|\nabla|\omega||^2 - \big\langle\delta d(|\omega|^{p-2}\omega), |\omega|^q\omega\big\rangle + K_\ell|\omega|^{p+q}.\end{aligned}$$

We choose a cut-off function $\varphi \in \mathcal{C}_0^\infty(M_+)$ then multiplying both sides of the above inequality by φ^2, we obtain

$$\begin{aligned}\int_{M_+}\varphi^2|\omega|^{q+1}\Delta|\omega|^{p-1} \geq\ &(p-1)^2(A_{p,n,\ell}-1)\int_{M_+}\phi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\ &-\int_{M_+}\big\langle\delta d(|\omega|^{p-2}\omega), \varphi^2|\omega|^q\omega\big\rangle + \int_{M_+}K_\ell\varphi^2|\omega|^{p+q}.\end{aligned}$$

By integration by parts, this implies

$$\begin{aligned}&\int_{M_+}\big\langle\nabla(\varphi^2|\omega|^{q+1}), \nabla|\omega|^{p-1}\big\rangle\\ \leq\ &-(p-1)^2(A_{p,n,\ell}-1)\int_{M_+}\phi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\ &+\int_{M_+}\big\langle d(|\omega|^{p-2}\omega), d(\varphi^2|\omega|^q\omega)\big\rangle - \int_{M_+}K_\ell\varphi^2|\omega|^{p+q}.\end{aligned}\tag{3.2}$$

Since the $(\ell, n-\ell)$-curvature of N^{n+m} is not less than $-k$, we have

$$K_\ell \geq \frac{1}{2}(n^2|H|^2 - \max\{\ell, n-\ell\}|A|^2) - k.$$

Obviously,

$$\int_{M_+}\big\langle\nabla(\varphi^2|\omega|^{q+1}), \nabla|\omega|^{p-1}\big\rangle = 2(p-1)\int_{M_+}\varphi|\omega|^{p+q-1}\langle\nabla\varphi, \nabla|\omega|\rangle$$

$$+ (q+1)(p-1)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2. \tag{3.3}$$

Note that for any closed ℓ-form ω and smooth function f, it is proved in Lemma 2.5 that

$$|d(f\wedge\omega)| = |df\wedge\omega| \le |df|\cdot|\omega|.$$

Therefore,

$$\begin{aligned}
\int_{M_+}\left\langle d(|\omega|^{p-2}\omega), d(\varphi^2|\omega|^q\omega)\right\rangle = & \int_{M_+}\left\langle d(|\omega|^{p-2})\wedge\omega, d(\varphi^2|\omega|^q)\wedge\omega\right\rangle\\
\le & \int_{M_+}\left|d(|\omega|^{p-2})\wedge\omega\right|\cdot\left|d(\varphi^2|\omega|^q)\wedge\omega\right|\\
\le & \int_{M_+}\left|\nabla(|\omega|^{p-2})\right||\omega|\cdot\left|\nabla(\varphi^2|\omega|^q)\right||\omega|\\
\le & (p-2)q\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\
& + 2(p-2)\int_{M_+}\varphi|\omega|^{p+q-1}\,|\nabla|\omega||\,|\nabla\varphi|.
\end{aligned} \tag{3.4}$$

1. If $|A|^2 \le \frac{n^2|H|^2-2k}{C_{n,\ell}}$, then $K_\ell \ge 0$. Therefore, from (3.2) we obtain

$$\begin{aligned}
\int_{M_+}\left\langle\nabla(\varphi^2|\omega|^{q+1}),\nabla|\omega|^{p-1}\right\rangle \le & -(p-1)^2(A_{p,n,\ell}-1)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
& + \int_{M_+}\left\langle d(|\omega|^{p-2}\omega), d(\varphi^2|\omega|^q\omega)\right\rangle.
\end{aligned}$$

Thus, by (3.3), (3.4), we have

$$\begin{aligned}
& 2(p-1)\int_{M_+}\varphi|\omega|^{p+q-1}\left\langle\nabla\varphi,\nabla|\omega|\right\rangle + (q+1)(p-1)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
\le & -(p-1)^2(A_{p,n,\ell}-1)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
& + (p-2)q\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2 + 2(p-2)\int_{M_+}\varphi|\omega|^{p+q-1}\,|\nabla|\omega||\,|\nabla\varphi|.
\end{aligned}$$

Hence,

$$\begin{aligned}
& (p+q-1+(p-1)^2(A_{p,n,\ell}-1))\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
\le & \; 2(2p-3)\int_{M_+}\varphi|\omega|^{p+q-1}\,|\nabla|\omega||\,|\nabla\varphi|.
\end{aligned}$$

Using the fundamental inequality $2AB \leq \varepsilon A^2 + \varepsilon^{-1}B^2$, we have that, for every $\varepsilon > 0$,

$$\begin{aligned} &2\int_{M_+} \varphi|\omega|^{p+q-1}\,|\nabla|\omega||\,|\nabla\varphi| \\ &\leq \varepsilon \int_{M_+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 + \frac{1}{\varepsilon}\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2. \end{aligned} \tag{3.5}$$

From the last two inequalities, we obtain

$$\begin{aligned} &(p+q-1+(p-1)^2(A_{p,n,\ell}-1)-\varepsilon(2p-3))\int_{M_+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \\ &\leq \frac{2p-3}{\varepsilon}\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2. \end{aligned}$$

Note that $Q = p+q$, since $Q - 1 + (p-1)^2(A_{p,n,\ell}-1) > 0$, we can choose $\varepsilon > 0$ small enough and a constant $K = K(\varepsilon) > 0$ so that

$$\frac{4}{Q^2}\int_{M_+} \varphi^2\left|\nabla|\omega|^{Q/2}\right|^2 \leq K\int_{M_+} |\omega|^Q|\nabla\varphi|^2 \quad \text{for all } r > 0. \tag{3.6}$$

Applying a variation of the Duzaar-Fuchs cut-off method (see also [12, 25]), we shall show that (3.6) holds for every $\varphi \in C_0^\infty(M)$. Indeed, we define

$$\eta_{\tilde{\epsilon}} = \min\left\{\frac{|\omega|}{\tilde{\epsilon}}, 1\right\}$$

for $\tilde{\epsilon} > 0$. Let $\varphi_{\tilde{\epsilon}} = \psi^2\eta_{\tilde{\epsilon}}$, where $\psi \in \mathcal{C}_0^\infty(M)$. It is easy to see that $\varphi_{\tilde{\epsilon}}$ is a compactly supported continuous function and $\varphi_{\tilde{\epsilon}} = 0$ on $M \setminus M_+$. Now, we replace φ by $\varphi_{\tilde{\epsilon}}$ in (3.6) and get

$$\begin{aligned} &\int_{M_+} \psi^4(\eta_{\tilde{\epsilon}})^2|\omega|^{Q-2}|\nabla|\omega||^2 \\ &\leq 6C\int_{M_+} |\omega|^Q|\nabla\psi|^2\psi^2(\eta_{\tilde{\epsilon}})^2 + 3C\int_{M_+} |\omega|^Q|\nabla\eta_{\tilde{\epsilon}}|^2\psi^4. \end{aligned} \tag{3.7}$$

Observe that

$$\int_{M_+} |\omega|^Q|\nabla\eta_{\tilde{\epsilon}}|^2\psi^4 \leq \tilde{\epsilon}^{Q-2}\int_{M_+} |\nabla|\omega||^2\psi^4\chi_{\{|\omega|\leq\tilde{\epsilon}\}}$$

and the right hand side vanishes by dominated convergence as $\tilde{\epsilon} \to 0$, because $|\nabla|\omega|| \in L^2_{loc}(M)$ and $Q \geq 2$. Letting $\tilde{\epsilon} \to 0$ and applying Fatou lemma to the integral on the left hand side and dominated convergence to the first integral in the right hand side of (3.7), we obtain

$$\int_{M_+} \psi^4|\omega|^{Q-2}|\nabla|\omega||^2 \leq 6C\int_{M_+} |\omega|^Q|\nabla\psi|^2\psi^2, \tag{3.8}$$

where $\psi \in C_0^\infty(M)$. We choose a cut-off function $\psi \in \mathcal{C}_0^\infty(M_+)$ satisfying

$$\psi = \begin{cases} 1, & \text{on } B_r, \\ \in [0,1] \text{ and } |\nabla\varphi| \leq \frac{2}{r}, & \text{on } B_{2r} \setminus B_r, \\ 0, & \text{on } M \setminus B_{2r}, \end{cases}$$

where B_r is the open ball of radius r and center at a fixed point of M.

Letting $r \to \infty$, we conclude that $|\omega|$ is constant on M_+. Since $|\omega| = 0 \in \partial M_+$, it implies that either ω is zero; or $M_+ = \emptyset$. If $M_+ = \emptyset$, then $|\omega|$ is constant on M. Thanks to the assumption $|\omega| \in L^Q(M)$, we infer $\omega = 0$.

2. Assume that

$$\frac{n^2|H|^2 - 2k}{C_{n,\ell}} < |A|^2 \leq \frac{n^2|H|^2}{C_{n,\ell}}$$

then $-k \leq K_\ell < 0$. From this and (3.2) we obtain

$$\begin{aligned} &\int_{M_+} \left\langle \nabla(\varphi^2|\omega|^{q+1}), \nabla|\omega|^{p-1} \right\rangle \\ \leq & -(p-1)^2(A_{p,n,\ell} - 1) \int_{M_+} |\omega|^{p+q-2}|\nabla|\omega||^2 \\ & \int_{M_+} \left\langle d(|\omega|^{p-2}\omega), d(\varphi^2|\omega|^q\omega) \right\rangle + k \int_{M_+} \varphi^2|\omega|^{p+q}. \end{aligned} \tag{3.9}$$

By the definition of $\lambda_1(M)$ and (3.5), we obtain that, for any $\varepsilon > 0$,

$$\begin{aligned} &\lambda_1 \int_{M_+} \varphi^2|\omega|^{p+q} \\ \leq & \int_{M_+} \left|\nabla \left(\varphi|\omega|^{(p+q)/2}\right)\right|^2 \\ = & \frac{(p+q)^2}{4} \int_{M_+} \varphi^2|\omega|^{p+q-2} \left|\nabla|\omega|\right|^2 + \int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2 \\ & + (p+q) \int_{M_+} \varphi|\omega|^{p+q-1} \left\langle \nabla|\omega|, \nabla\varphi \right\rangle \\ \leq & (1+\varepsilon)\frac{(p+q)^2}{4} \int_{M_+} \varphi^2|\omega|^{p+q-2} \left|\nabla|\omega|\right|^2 \\ & + \left(1 + \frac{1}{\varepsilon}\right) \int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2. \end{aligned} \tag{3.10}$$

From this, (3.3), (3.4), and (3.9), we have

$$C_\varepsilon \int_{M_+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \leq D_\varepsilon \int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2,$$

where

$$C_\varepsilon := p+q-1+(p-1)^2(A_{p,n,\ell}-1)-\varepsilon(2p-3)-(1+\varepsilon)\frac{k(p+q)^2}{4\lambda_1(M)}$$
$$= Q-1+(p-1)^2(A_{p,n,\ell}-1)-\varepsilon(2p-3)-(1+\varepsilon)\frac{kQ^2}{4\lambda_1(M)},$$

and

$$D_\varepsilon := \frac{2p-3}{\varepsilon}+\frac{k}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right).$$

Here $Q=p+q\geq 2$. Since

$$\lambda_1(M) > \frac{kQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1))},$$

there are some small enough number $\varepsilon>0$ and constant $K=K(\varepsilon)>0$ such that

$$\frac{4}{Q^2}\int_{M_+}\left|\nabla|\omega|^{Q/2}\right|^2 \leq K\int_{M_+}|\omega|^Q|\nabla\varphi|^2.$$

Arguing similarly as in the proof of the first part, we conclude that $|\omega|$ is constant. Since $\lambda_1(M)>0$, M must have infinite volume, note that $|\omega|\in L^Q(M)$, we have that ω is zero.

3. Due to the previous two cases, we may assume that $|A|^2>\frac{n^2|H|^2}{C_{n,\ell}}$. Then, from (3.2) we have

$$\begin{aligned}
&\int_{M_+}\left\langle\nabla(\varphi^2|\omega|^{q+1}),\nabla|\omega|^{p-1}\right\rangle+\frac{n^2}{2}\int_{M_+}\varphi^2|H|^2|\omega|^{p+q}\\
\leq\ &-(p-1)^2(A_{p,n,\ell}-1)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
&+\int_{M_+}\left\langle d(|\omega|^{p-2}\omega),d(\varphi^2|\omega|^q\omega)\right\rangle+\frac{C_{n,\ell}}{2}\int_{M_+}\varphi^2|A|^2|\omega|^{p+q}\\
&+k\int_{M_+}\varphi^2|\omega|^{p+q}.
\end{aligned}\tag{3.11}$$

By Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}
\int_{M_+}\varphi^2|A|^2|\omega|^{p+q} &\leq \|A\|_n^2\left(\int_{M_+}\left(\varphi|\omega|^{(p+q)/2}\right)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\\
&\leq C_S\|A\|_n^2\left(\int_{M_+}\left|\nabla\left(\varphi|\omega|^{(p+q)/2}\right)\right|^2+\int_{M_+}\varphi^2|H|^2|\omega|^{p+q}\right),
\end{aligned}$$

where C_S is the Sobolev constant depending only on n. From the last inequality and (3.5) we can get that, for any $\varepsilon>0$,

$$\int_{M_+}\varphi^2|A|^2|\omega|^{p+q}$$

$$\leq C_S\|A\|_n^2(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2$$
$$+ C_S\|A\|_n^2\left(1+\frac{1}{\varepsilon}\right)\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2 + C_S\|A\|_n^2\int_{M_+}\varphi^2|H|^2|\omega|^{p+q}.$$

Using this inequality and (3.3), (3.4), (3.11), we have

$$C_\varepsilon\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 + \frac{1}{2}\left(n^2 - C_{n,\ell}C_S\|A\|_n^2\right)\int_{M_+}\varphi^2|H|^2|\omega|^{p+q}$$
$$\leq D_\varepsilon\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2,$$

where, for $Q = p+q \geq 2$

$$\begin{aligned} C_\varepsilon &:= p+q-1+(p-1)^2(A_{p,n,\ell}-1) - \varepsilon(2p-3) \\ &\quad - (1+\varepsilon)\frac{(p+q)^2}{4}\left(\frac{k}{\lambda_1(M)} + \frac{C_{n,\ell}C_S\|A\|_n^2}{2}\right) \\ &= Q-1+(p-1)^2(A_{p,n,\ell}-1) - \varepsilon(2p-3) \\ &\quad - (1+\varepsilon)\frac{Q^2}{4}\left(\frac{k}{\lambda_1(M)} + \frac{C_{n,\ell}C_S\|A\|_n^2}{2}\right), \end{aligned}$$

and

$$D_\varepsilon := \frac{2p-3}{\varepsilon} + \left(1+\frac{1}{\varepsilon}\right)\left(\frac{k}{\lambda_1(M)} + \frac{C_{n,\ell}C_S\|A\|_n^2}{2}\right).$$

Since

$$\|A\|_n^2 < \min\left\{\frac{n^2}{C_{n,\ell}C_S},\ \frac{2}{C_{n,\ell}C_S}\left[\frac{4(Q-1+(p-1)^2(A_{p,n,\ell}-1))}{Q^2} - \frac{k}{\lambda_1(M)}\right]\right\},$$

there are some small enough $\varepsilon > 0$ and constant $K = K(\varepsilon) > 0$ such that

$$\frac{4}{Q^2}\int_{M_+}\left|\nabla|\omega|^{Q/2}\right|^2\varphi^2 \leq K\int_{M_+}|\omega|^Q|\nabla\varphi|^2.$$

Using the same arguments as in the proof of the first and second part, this inequality implies that ω is zero.

4. Suppose that $\sup_M |A|^2 < \infty$. Then using (3.10) we have that, for any $\varepsilon > 0$,

$$\begin{aligned} \int_{M_+}\varphi^2|A|^2|\omega|^{p+q} &\leq \sup_M|A|^2\int_{M_+}\varphi^2|\omega|^{p+q} \\ &\leq \frac{\sup_M|A|^2}{\lambda_1(M)}\int_{M_+}\left|\nabla\left(\varphi|\omega|^{(p+q)/2}\right)\right|^2 \\ &\leq \frac{\sup_M|A|^2}{\lambda_1(M)}(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2 \\ &\quad + \frac{\sup_M|A|^2}{\lambda_1(M)}\left(1+\frac{1}{\varepsilon}\right)\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2. \end{aligned}$$

From this and (3.3), (3.4), (3.11) we obtain that, for any $\varepsilon > 0$,

$$C_\varepsilon \int_{M_+} \varphi^2 |\omega|^{p+q-2} |\nabla|\omega||^2 + \frac{n^2}{2} \int_{M_+} \varphi^2 |H|^2 |\omega|^{p+q} \leq D_\varepsilon \int_{M_+} |\omega|^{p+q} |\nabla \varphi|^2,$$

where

$$\begin{aligned} C_\varepsilon &:= p + q - 1 + (p-1)^2 (A_{p,n,\ell} - 1) - \varepsilon(2p-3) \\ &\quad - (1+\varepsilon) \frac{(p+q)^2}{4\lambda_1(M)} \left(k + \frac{C_{n,\ell} \sup_M |A|^2}{2} \right) \\ &= Q - 1 + (p-1)^2 (A_{p,n,\ell} - 1) - \varepsilon(2p-3) \\ &\quad - (1+\varepsilon) \frac{Q^2}{4\lambda_1(M)} \left(k + \frac{C_{n,\ell} \sup_M |A|^2}{2} \right), \end{aligned}$$

and

$$D_\varepsilon := \frac{2p-3}{\varepsilon} + \left(1 + \frac{1}{\varepsilon}\right) \frac{1}{\lambda_1(M)} \left(k + \frac{C_{n,\ell} \sup_M |A|^2}{2} \right).$$

Since

$$\lambda_1(M) > \frac{Q^2 (2k + C_{n,\ell} \sup_M |A|^2)}{8(Q - 1 + (p-1)^2 (A_{p,n,\ell} - 1))},$$

M must have infinite volume. Moreover, there are some small enough $\varepsilon > 0$ and constant $K = K(\varepsilon) > 0$ such that

$$\frac{4}{Q^2} \int_{M_+} \left| \nabla |\omega|^{Q/2} \right|^2 \varphi^2 \leq K \int_{M_+} |\omega|^Q |\nabla \varphi|^2.$$

Using the same arguments as in the proof of the first and second part, this inequality also implies that ω is zero. $\square$

Now, we will give a proof of Theorem 1.2 which is a geometric application of Theorem 3.1. First, let us recall the following result about the existence of p-harmonic functions on a Riemannian manifold.

Theorem 3.2 ([4]). *Let M be a Riemannian manifold with at least two p-nonparabolic ends. Then, there exists a non-constant, bounded p-harmonic function $u \in \mathcal{C}^{1,\alpha}(M)$ for some $\alpha > 0$ such that $|\nabla u| \in L^p(M)$.*

Note that, it is known that the regularity of (weakly) p-harmonic function u is not better than $\mathcal{C}^{1,\alpha}_{loc}$ (see [30] and the references therein). Moreover $u \in W^{2,2}_{loc}$ if $p \geq 2$; $u \in W^{2,p}_{loc}$ if $1 < p < 2$ (see [30]). In fact, any nontrivial (weakly) p-harmonic function u on M is smooth away from the set $\{\nabla u = 0\}$ which has n-dimensional Hausdorff measure zero.

Proof of Theorem 1.2. The proof follows by applying Theorem 3.1 with $q = 0, l = 1$ and using Theorem 3.2. $\square$

4. p-harmonic ℓ-forms on Riemannian manifolds with a weighted Poincaré inequality

Lemma 4.1. *Let M be a complete Riemannian manifold satisfying a weighted Poincaré inequality with a continuous positive weighted function ρ. Suppose that ω is a closed ℓ-form with finite L^Q norm $(Q \geq 2)$ on M satisfies the following differential inequality*

$$|\omega|\Delta|\omega|^{p-1} \geq B|\omega|^{p-2}\left|\nabla|\omega|\right|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega\right\rangle - a\rho|\omega|^p - b|\omega|^p \tag{4.1}$$

for some constants $0 < a < \frac{4(Q-1+B)}{Q^2}, b > 0$ and $Q \geq 2$. Then the following integral inequality holds

$$\int_{M_+} \left|\nabla|\omega|^{Q/2}\right|^2 \leq \frac{bQ^2}{4(Q-1+B)-aQ^2}\int_{M_+}|\omega|^Q. \tag{4.2}$$

Moreover, if equality holds in (4.2)*, then equality holds in* (4.1)

Proof. (i) Assume that the manifold is compact. Let $q = Q - p$, multiplying inequality (4.1) by $|\omega|^q$ and then integrating by parts, we obtain

$$\begin{aligned}\int_{M_+}|\omega|^{q+1}\Delta|\omega|^{p-1} \geq B\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2 - \int_{M_+}\left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^q\omega\right\rangle\\ - a\int_{M_+}\rho|\omega|^{p+q} - b\int_{M_+}|\omega|^{p+q},\end{aligned}$$

and then,

$$\begin{aligned}[(q+1)(p-1)+B]\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2 \leq &\int_{M_+}\left\langle d(|\omega|^{p-2}\omega), d(|\omega|^q\omega)\right\rangle\\ &+ a\int_{M_+}\rho|\omega|^{p+q} + b\int_{M_+}|\omega|^{p+q}.\end{aligned}$$

Similarly to (3.4), we have

$$\begin{aligned}\int_{M_+}\left\langle d(|\omega|^{p-2}\omega), d(|\omega|^q\omega)\right\rangle &= \int_{M_+}\left\langle d(|\omega|^{p-2})\wedge\omega, d(|\omega|^q)\wedge\omega\right\rangle\\ &\leq \int_{M_+}\left|d(|\omega|^{p-2})\wedge\omega|.|d(|\omega|^q)\wedge\omega\right|\\ &\leq \int_{M_+}\left|\nabla(|\omega|^{p-2})\right||\omega|.\left|\nabla(|\omega|^q)\right||\omega|\\ &= (p-2)q\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2.\end{aligned}$$

By the weighted Poincaré inequality we have that

$$\int_{M_+}\rho|\omega|^{p+q} \leq \int_{M_+}\left|\nabla\left(|\omega|^{(p+q)/2}\right)\right|^2 = \frac{(p+q)^2}{4}\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2.$$

Combining the last three inequalities, we obtain

$$\left[p+q-1+B-a\frac{(p+q)^2}{4}\right]\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2\le b\int_{M_+}|\omega|^{p+q},$$

consequently,

$$\left[\frac{4(p+q-1+B)}{(p+q)^2}-a\right]\int_{M_+}\left|\nabla|\omega|^{(p+q)/2}\right|^2\le b\int_{M_+}|\omega|^{p+q}.$$

Now assume that equality holds in (4.2). Multiplying inequality (4.1) by $|\omega|^q$ where $q=Q-p$, and then integrating by parts and using the above estimates, we obtain

$$\begin{aligned}0\le&\int_{M_+}\Big(|\omega|^{q+1}\Delta|\omega|^{p-1}-B|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2+\left\langle\delta d(|\omega|^{p-2}\omega),|\omega|^q\omega\right\rangle\\&\qquad+a\rho|\omega|^{p+q}+b|\omega|^{p+q}\Big)\\=&-[(q+1)(p-1)+B]\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2+\int_{M_+}\left\langle d(|\omega|^{p-2}\omega),d(|\omega|^q\omega)\right\rangle\\&+a\int_{M_+}\rho|\omega|^{p+q}+b\int_{M_+}|\omega|^{p+q}\\\le&-\left[\frac{4(p+q-1+B)}{(p+q)^2}-a\right]\int_{M_+}\left|\nabla|\omega|^{(p+q)/2}\right|^2+b\int_{M_+}|\omega|^{p+q}\\=&\ 0.\end{aligned}$$

Therefore, we can conclude that equality holds in (4.1) in M_+. Since in $M\setminus M_+$, (4.1) is always true. We complete the proof.

(ii) Assume that the manifold is non-compact. We choose a cut-off function $\varphi\in\mathcal{C}_0^\infty(M_+)$ as in the proof of Theorem 3.1. Multiplying both sides of inequality (4.1) by $\varphi^2|\omega|^q$ and then integrating by parts, we obtain

$$\begin{aligned}&\int_{M_+}\varphi^2|\omega|^{q+1}\Delta|\omega|^{p-1}\\&\ge B\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2-\int_{M_+}\left\langle\delta d(|\omega|^{p-2}\omega),\varphi^2|\omega|^q\omega\right\rangle\\&\quad-a\int_{M_+}\rho\varphi^2|\omega|^{p+q}-b\int_{M_+}\varphi^2|\omega|^{p+q},\end{aligned}$$

and then,

$$\begin{aligned}&\int_{M_+}\left\langle\nabla(\varphi^2|\omega|^{q+1}),\nabla|\omega|^{p-1}\right\rangle+B\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2\\\le&\int_{M_+}\left\langle d(|\omega|^{p-2}\omega),d(\varphi^2|\omega|^q\omega)\right\rangle+a\int_{M_+}\rho\varphi^2|\omega|^{p+q}+b\int_{M_+}\varphi^2|\omega|^{p+q}.\end{aligned}$$

Using the last inequality and (3.3), (3.4), we have

$$\begin{aligned}
&(p+q-1+B)\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\
\leq\ & a\int_{M_+}\rho\varphi^2|\omega|^{p+q}+b\int_{M_+}\varphi^2|\omega|^{p+q}\\
&+2(2p-3)\int_{M_+}\varphi|\omega|^{p+q-1}\,|\nabla|\omega||\,|\nabla\varphi|.
\end{aligned}$$

Similarly to (3.10) and by the weighted Poincaré inequality we have that

$$\begin{aligned}
\int_{M_+}\rho\varphi^2|\omega|^{p+q}\leq\ & \int_{M_+}\left|\nabla\left(\varphi|\omega|^{(p+q)/2}\right)\right|^2\\
&\leq(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\
&\quad+\left(1+\frac{1}{\varepsilon}\right)\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2.
\end{aligned}$$

Combining the last two inequalities and using (3.5), we obtain

$$\begin{aligned}
&\left[p+q-1+B-\varepsilon(2p-3)-(1+\varepsilon)\frac{a(p+q)^2}{4}\right]\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\
\leq\ & b\int_{M_+}\varphi^2|\omega|^{p+q}+\left[\frac{2p-3}{\varepsilon}+a\left(1+\frac{1}{\varepsilon}\right)\right]\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2.
\end{aligned}$$

Since $4(p+q-1+B)-a(p+q)^2>0$,

$$p+q-1+B-(1+\varepsilon)\frac{a(p+q)^2}{4}-\varepsilon(2p-3)>0$$

for all sufficiently small enough $\varepsilon>0$. By the monotone convergence theorem, letting $r\to\infty$, and then $\varepsilon\to0$, we obtain inequality (4.2).

Now suppose that equality in (4.2) holds. Multiplying both sides of inequality (4.1) by $\varphi^2|\omega|^q$ and then integrating by parts and using the above estimates, we obtain

$$\begin{aligned}
0\leq\ & \int_{M_+}\varphi^2\left(|\omega|^{q+1}\Delta|\omega|^{p-1}-B|\omega|^{p+q-2}\,|\nabla|\omega||^2+\left\langle\delta d(|\omega|^{p-2}\omega),|\omega|^q\omega\right\rangle\right.\\
&\left.+a\rho|\omega|^{p+q}+b|\omega|^{p+q}\right)\\
\leq\ & -[p+q-1+B]\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\
&+2(2p-3)\int_{M_+}\varphi|\omega|^{p+q-1}|\nabla|\omega||\;|\nabla\varphi|\\
&+a\int_{M_+}\rho\varphi^2|\omega|^{p+q}+b\int_{M_+}\varphi^2|\omega|^{p+q}
\end{aligned}$$

$$\begin{aligned}&\leq -\left[p+q-1+B-\varepsilon(2p-3)-(1+\varepsilon)\frac{a(p+q)^2}{4}\right]\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2\\&\quad+b\int_{M_+}\varphi^2|\omega|^{p+q}+\left[\frac{2p-3}{\varepsilon}+a\left(1+\frac{1}{\varepsilon}\right)\right]\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2.\end{aligned}$$

Letting $r\to\infty$ in the last inequality and using the monotone convergence theorem, we get

$$\begin{aligned}0&\leq\int_{M_+}\Big(|\omega|^{q+1}\Delta|\omega|^{p-1}-B|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2+\left\langle\delta d(|\omega|^{p-2}\omega),|\omega|^q\omega\right\rangle\\&\qquad+a\rho|\omega|^{p+q}+b|\omega|^{p+q}\Big)\\&\leq-\left[p+q-1+B-\varepsilon(2p-3)-(1+\varepsilon)\frac{a(p+q)^2}{4}\right]\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2\\&\qquad+b\int_{M_+}|\omega|^{p+q}.\end{aligned}$$

And then putting $\varepsilon\to 0$, we obtain

$$\begin{aligned}0&\leq\int_{M_+}\Big(|\omega|^{q+1}\Delta|\omega|^{p-1}-B|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2+\left\langle\delta d(|\omega|^{p-2}\omega),|\omega|^q\omega\right\rangle\\&\qquad+a\rho|\omega|^{p+q}+b|\omega|^{p+q}\Big)\\&\leq-\left[p+q-1+B-\frac{a(p+q)^2}{4}\right]\int_{M_+}|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2+b\int_{M_+}|\omega|^{p+q}\\&=0.\end{aligned}$$

In view of (4.1), we can conclude that equality holds in (4.1). □

In order to derive main results of this section, we need to have the following result which was showed by Vieira in [31].

Lemma 4.2 ([31])**.** *Suppose that u is a smooth function on a complete Riemannian manifold M with finite L^Q norm, for $Q\geq 2$. Then*

$$\lambda_1(M)\int_M|u|^Q\leq\int_M\left|\nabla u^{Q/2}\right|^2.$$

Theorem 4.3. *Let M^n be a complete non-compact Riemannian manifold M^n satisfying a weighted Poincaré inequality with a continuous positive weighted function ρ. Suppose that the curvature operator acting on ℓ-forms has a lower bound*

$$K_\ell\geq -a\rho-b$$

for some constants $b>0, Q\geq 2$ and

$$0<a<\frac{4(Q-1+(p-1)^2(A_{p,n,\ell}-1))}{Q^2}.$$

Assume that the first eigenvalue of the Laplacian has a lower bound

$$\lambda_1(M^n) > \frac{bQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1)) - aQ^2}.$$

Then the space of p-harmonic ℓ-forms with finite L^Q energy on M^n is trivial.

Proof. Let ω be any p-harmonic ℓ-form with finite L^Q norm. From the Bochner formula and the Kato type inequality (see, Lemma 2.2), we obtain (3.1)

$$|\omega|\Delta|\omega|^{p-1} \geq (p-1)^2(A_{p,n,\ell}-1)|\omega|^{p-2}\left|\nabla|\omega|\right|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega\right\rangle + K_\ell|\omega|^p.$$

Hence,

$$|\omega|\Delta|\omega|^{p-1} \geq (p-1)^2(A_{p,n,\ell}-1)|\omega|^{p-2}\left|\nabla|\omega|\right|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega\right\rangle - a\rho|\omega|^p - b|\omega|^p.$$

Applying Lemma 4.1 to $B = (p-1)^2(A_{p,n,\ell}-1)$, we obtain

$$\int_{M_+} \left|\nabla|\omega|^{Q/2}\right|^2 \leq \frac{bQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1)) - aQ^2} \int_{M_+} |\omega|^Q.$$

Since $|\omega| \in L^Q(M)$ this implies $|\omega| \in L^Q(M_+)$. Therefore, by Lemma 4.2, we have

$$\lambda_1(M^n) \int_{M_+} |\omega|^Q \leq \int_{M_+} \left|\nabla\omega^{Q/2}\right|^2.$$

From the last two inequalities, we obtain

$$\lambda_1(M^n) \int_{M_+} |\omega|^Q \leq \frac{bQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1)) - aQ^2} \int_{M_+} |\omega|^Q.$$

If the ℓ-form ω is not identically zero in M_+ (therefore $\omega = 0$ in M), then

$$\lambda_1(M^n) \leq \frac{bQ^2}{4(Q-1+(p-1)^2(A_{p,n,\ell}-1)) - aQ^2},$$

which leads to a contradiction. So ω is identically zero. □

Combining Theorem 3.2 and Theorem 4.3 with $Q = p \geq 2$, $\ell = 1$, we obtain the follows result.

Corollary 4.4. *Let M^n be a complete non-compact Riemannian manifold M^n satisfying a weighted Poincaré inequality with a continuous positive weighted function ρ. Suppose that*

$$Ric_M \geq -a\rho - b \quad \text{for } 0 < a < \frac{4(p-1)(p+n-2)}{p^2(n-1)}$$

and some constant $b > 0$. Assume that the first eigenvalue of the Laplacian has a lower bound

$$\lambda_1(M^n) > \frac{bp^2(n-1)}{4(p-1)(n+p-2) - ap^2(n-1)}.$$

Then the space of L^p p-harmonic 1-forms on M^n is trivial. Therefore, M has at most one p-nonparabolic end.

Let

$$A_{n,\ell} = \begin{cases} \dfrac{n-\ell+1}{n-\ell}, & \text{if } 1 \le \ell \le \frac{n}{2}, \\ \dfrac{\ell+1}{\ell}, & \text{if } \frac{n}{2} \le \ell \le n-1. \end{cases}$$

We conclude this section by the below rigidity property.

Corollary 4.5. *Let M^n be a complete non-compact Riemannian manifold M^n satisfying a weighted Poincaré inequality with a continuous positive weighted function ρ. Suppose that*

$$Ric_M \ge -a\rho - b \quad \textit{for } 0 < a < A_{n,\ell}$$

and some constant $b > 0$. Assume that

$$\lambda_1(M^n) = \frac{b}{A_{n,\ell} - a}.$$

Then either

(1) *The space of L^2 harmonic ℓ-forms on M^n is trivial or*;
(2) *For any L^2 harmonic ℓ-form ω on M, there exixts a 1-form α such that*

$$\nabla\omega = \alpha \otimes \omega - \frac{1}{\sqrt{\ell+1}}\theta_1(\alpha \wedge \omega) + \frac{1}{\sqrt{n+1-\ell}}\theta_2(i_\alpha\omega).$$

Proof. Let $p = 2$, $q = 0$, $Q = 2$, the Bochner formula applying on harmonic ℓ-form ω implies that

$$|\omega|\Delta|\omega| \ge (A_{n,\ell} - 1)|\nabla|\omega||^2 - a\rho|\omega|^2 - b|\omega|^2.$$

If ω is non-trivial, then Lemma 4.1 and Lemma 4.2 imply that

$$\lambda_1(M)\int_{M_+} |\omega|^2 = \int_{M_+} |\nabla|\omega||^2.$$

Therefore,

$$|\omega|\Delta|\omega| = (A_{n,\ell} - 1)|\nabla|\omega||^2 - a\rho|\omega|^2 - b|\omega|^2$$

on M_+, hence it holds true on M. This means that the equality in the Kato type inequality (2.1) holds true. By Lemma 2.2, we are done. □

5. p-harmonic 1-forms on locally conformally flat Riemannian manifolds

In this section we will prove a vanishing theorem for p-harmonic 1-forms with finite L^Q energy ($Q \ge 2$). We will use the following auxiliary lemmas.

It is known that a simply connected, locally conformally flat manifold M^n, $n \ge 3$, has a conformal immersion into $\mathbb{S}^n$, and according to [26], the Yamabe constant of M^n satisfies

$$\mathcal{Y}(M^n) = \mathcal{Y}(\mathbb{S}^n) = \frac{n(n-2)\omega_n^{2/n}}{4},$$

where ω_n is the volume of the unit sphere in $\mathbb{R}^n$. Therefore the following inequality

$$\mathcal{Y}(\mathbb{S}^n)\left(\int_M f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq \int_M |\nabla f|^2 dv + \frac{n-2}{4(n-1)}\int_M Rf^2 dv \tag{5.1}$$

holds for all $f \in C_0^\infty(M)$. By (5.1), it is easy to obtain the following lemma.

Lemma 5.1. *Let (M^n, g), $n \geq 3$, be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$ or $\|R\|_{n/2} < \infty$. Then the following L^2 Sobolev inequality*

$$\left(\int_M f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq S\int_M |\nabla f|^2 dv, \quad \forall f \in C_0^\infty(M),$$

holds for some constant $S > 0$, which is equal to $\mathcal{Y}(\mathbb{S}^n)^{-1}$ in the case of $R \leq 0$. In particular, M has infinite volume.

In [21], the Ricci curvature is estimated as follows.

Lemma 5.2 ([21])**.** *Let (M^n, g) be an n-dimensional complete Riemannian manifold. Then*

$$\mathrm{Ric} \geq -|T|g - \frac{|R|}{\sqrt{n}}g$$

in the sense of quadratic forms. Here T stands for the traceless tensor, namely

$$T = \mathrm{Ric} - \frac{R}{n}g.$$

Now, we introduce the main result of this section.

Theorem 5.3. *Let (M^n, g), $n \geq 3$, be an n-dimensional complete, simply connected, locally conformally flat Riemannian manifold. If one of the following conditions*

1. $$\|T\|_{n/2} + \frac{\|R\|_{n/2}}{\sqrt{n}} < \frac{4(Q-1+\kappa_p)}{SQ^2};$$
2. *the scalar curvature R is nonpositive,*
$$K_{p,Q,n} := \frac{Q-1+\kappa_p}{Q^2} - \frac{n-1}{\sqrt{n}(n-2)} > 0,$$
and
$$\|T\|_{n/2} < \frac{4K_{p,Q,n}}{S} = 4K_{p,Q,n}\mathcal{Y}(\mathbb{S}^n);$$

holds true, then every p-harmonic 1-form with finite $L^Q (Q \geq 2)$ norm on M is trivial. Here

$$\kappa_p = \min\left\{1, \frac{(p-1)^2}{n-1}\right\}.$$

Proof. Since M satisfies a Sobolev inequality, M must have infinite volume. Therefore, as in the previous part, we only need to prove that if ω is any p-harmonic 1-form with finite L^Q-norm, then $\omega = 0$ in M_+. Now, applying the Bochner formula to the form $|\omega|^{p-2}\omega$, we have

$$\begin{aligned}\frac{1}{2}\Delta|\omega|^{2(p-1)} = &\left|\nabla\left(|\omega|^{p-2}\omega\right)\right|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\right\rangle \\ &+ |\omega|^{2(p-2)}\operatorname{Ric}(\omega,\omega).\end{aligned}$$

From this and Lemma 5.2 and the Kato type inequality (see, Lemma 2.2), we obtain

$$|\omega|\Delta|\omega|^{p-1} \geq \kappa_p|\omega|^{p-2}\left|\nabla|\omega|\right|^2 - \left\langle \delta d(|\omega|^{p-2}\omega), \omega\right\rangle - |T||\omega|^p - \frac{|R|}{\sqrt{n}}|\omega|^p.$$

We choose a cut-off function $\varphi \in \mathcal{C}_0^\infty(M_+)$ as in the proof of Theorem 3.1. Multiplying both sides of the last inequality by $\varphi^2|\omega|^q, (q = Q - p)$ and then integrating by parts, we obtain

$$\begin{aligned}&\int_{M_+}\varphi^2|\omega|^{q+1}\Delta|\omega|^{p-1} \\ &\geq \kappa_p\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2 - \int_{M_+}\left\langle \delta d(|\omega|^{p-2}\omega), \varphi^2|\omega|^q\omega\right\rangle \\ &\quad - \int_{M_+}|T|\varphi^2|\omega|^{p+q} - \frac{1}{\sqrt{n}}\int_{M_+}|R|\varphi^2|\omega|^{p+q},\end{aligned}$$

and then,

$$\begin{aligned}&\int_{M_+}\left\langle \nabla(\varphi^2|\omega|^{q+1}), \nabla|\omega|^{p-1}\right\rangle + \kappa_p\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2 \\ &\leq \int_{M_+}\left\langle d(|\omega|^{p-2}\omega), d(\varphi^2|\omega|^q\omega)\right\rangle + \int_{M_+}|T|\varphi^2|\omega|^{p+q} + \frac{1}{\sqrt{n}}\int_{M_+}|R|\varphi^2|\omega|^{p+q}.\end{aligned}$$

By the hypotheses and Lemma 5.1, we obtain that, for some constant $S > 0$ and all functions $f \in C_0^\infty(M)$,

$$\left(\int_{M_+} f^{\frac{2n}{n-2}}dv\right)^{\frac{n-2}{n}} \leq S\int_{M_+}|\nabla f|^2 dv.$$

Using this and (3.10), we have that, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{M_+}|T|\varphi^2|\omega|^{p+q} &\leq \|T\|_{n/2}\left(\int_{M_+}\left(\varphi|\omega|^{(p+q)/2}\right)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \\ &\leq S\|T\|_{n/2}\int_{M_+}\left|\nabla\left(\varphi|\omega|^{(p+q)/2}\right)\right|^2 \\ &\leq S\|T\|_{n/2}(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+}\varphi^2|\omega|^{p+q-2}\left|\nabla|\omega|\right|^2\end{aligned}$$

$$+ S\|T\|_{n/2}\left(1+\frac{1}{\varepsilon}\right)\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2. \tag{5.2}$$

From (3.3), (3.4), (3.5) and (5.2) we obtain that, for any $\varepsilon > 0$,

(5.3)

$$\begin{aligned}&\left[+q-1+\kappa_p-\varepsilon(2p-3)-S\|T\|_{n/2}(1+\varepsilon)\frac{(p+q)^2}{4}\right]\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2\\ \leq&\left[\frac{\kappa_p}{\varepsilon}+S\|T\|_{n/2}\left(1+\frac{1}{\varepsilon}\right)\right]\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2+\frac{1}{\sqrt{n}}\int_{M_+}\varphi^2|R||\omega|^{p+q}.\end{aligned}$$

For the last term in the right hand side of (5.3), we can estimate in the following two ways corresponding to the hypotheses of Theorem 5.3, respectively.

1. Similarly to (5.2), we have that, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{M_+}|R|\varphi^2|\omega|^{p+q} \leq S\|R\|_{n/2}(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+}\varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\ + S\|R\|_{n/2}\left(1+\frac{1}{\varepsilon}\right)\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2.\end{aligned}$$

Consequently, for any $\varepsilon > 0$,

$$C_\varepsilon\int_{M_+}\varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \leq D_\varepsilon\int_{M_+}|\omega|^{p+q}|\nabla\varphi|^2,$$

where

$$\begin{aligned}C_\varepsilon :&= p+q-1+\kappa_p-\varepsilon(2p-3)-S(1+\varepsilon)\frac{(p+q)^2}{4}\left(\|T\|_{n/2}+\frac{\|R\|_{n/2}}{\sqrt{n}}\right)\\ &= Q-1+\kappa_p-\varepsilon(2p-3)-S(1+\varepsilon)\frac{Q^2}{4}\left(\|T\|_{n/2}+\frac{\|R\|_{n/2}}{\sqrt{n}}\right)\end{aligned}$$

and

$$D_\varepsilon := \frac{2p-3}{\varepsilon}+S\left(1+\frac{1}{\varepsilon}\right)\left(\|T\|_{n/2}+\frac{\|R\|_{n/2}}{\sqrt{n}}\right).$$

Since

$$\|T\|_{n/2}+\frac{\|R\|_{n/2}}{\sqrt{n}} < \frac{4(Q-1+\kappa_p}{SQ^2},$$

there are some small enough $\varepsilon > 0$ and constant $K = K(\varepsilon) > 0$ such that

$$\frac{4}{Q^2}\int_{M_+}\left|\nabla|\omega|^{Q/2}\right|^2\varphi^2 \leq K\int_{M_+}|\omega|^Q|\nabla\varphi|^2.$$

Using the same argument as in the proof of the first part of Theorem 3.1, this inequality implies that ω is zero.

2. If the scalar curvature R is nonpositive, then from (5.1) we have that

$$\int_{M_+}|R|f^2dv \leq \frac{4(n-1)}{n-2}\int_{M_+}|\nabla f|^2dv,\ \forall f \in C_0^\infty(M).$$

From this and (3.10), we have that, for any $\varepsilon > 0$,

$$\begin{aligned}\int_{M_+} |R|\varphi^2|\omega|^{p+q} &\leq \frac{4(n-1)}{n-2}\int_{M_+} |\nabla(\varphi|\omega|^{(p+q)/2})|^2\\ &\leq \frac{4(n-1)}{n-2}(1+\varepsilon)\frac{(p+q)^2}{4}\int_{M_+} \varphi^2|\omega|^{p+q-2}\,|\nabla|\omega||^2\\ &\quad + \frac{4(n-1)}{n-2}\left(1+\frac{1}{\varepsilon}\right)\int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2.\end{aligned}$$

Substituting this inequality into (5.3), we yield that, for any $\varepsilon > 0$,

$$C_\varepsilon \int_{M_+} \varphi^2|\omega|^{p+q-2}|\nabla|\omega||^2 \leq D_\varepsilon \int_{M_+} |\omega|^{p+q}|\nabla\varphi|^2,$$

where

$$\begin{aligned}C_\varepsilon :&= p+q-1+\kappa_p - \varepsilon(2p-3) - (1+\varepsilon)\frac{(p+q)^2}{4}\left(S\|T\|_{n/2} + \frac{4(n-1)}{(n-2)\sqrt{n}}\right)\\ &= Q-1+\kappa_p-\varepsilon(2p-3)-(1+\varepsilon)\frac{Q^2}{4}\left(S\|T\|_{n/2}+\frac{4(n-1)}{(n-2)\sqrt{n}}\right),\end{aligned}$$

and

$$D_\varepsilon := \frac{2p-3}{\varepsilon} + \left(1+\frac{1}{\varepsilon}\right)\left(S\|T\|_{n/2} + \frac{4(n-1)}{(n-2)\sqrt{n}}\right).$$

Since

$$K_{p,Q,n} := \frac{Q-1+\kappa_p}{Q^2} - \frac{n-1}{\sqrt{n}(n-2)} > 0$$

and

$$\|T\|_{n/2} < \frac{4K_{p,Q,n}}{S},$$

there are some small enough $\varepsilon > 0$ and constant $K = K(\varepsilon) > 0$ such that

$$\frac{4}{Q^2}\int_{M_+} \left|\nabla|\omega|^{Q/2}\right|^2 \varphi^2 \leq K\int_{M_+} |\omega|^Q|\nabla\varphi|^2.$$

As in previous part, we infer that $\omega = 0$. The proof is complete. □

Note that when $q = 0$, we recover Theorem 1.3 and Theorem 1.4 in [21]. Hence, Theorem 5.3 can be considered as a generalization of Lin's results to the nonlinear setting. Now, we give a simple proof of Theorem 1.3.

Proof of Theorem 1.3. The proof follows by combining Theorem 3.2 and Theorem 5.3 with $Q = p$. □

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