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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH AN (ℓ, m)-TYPE CONNECTION

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Abstract. We define a new connection on semi-Riemannian manifolds, which is a non-symmetric and non-metric connection. We say that this connection is an (ℓ, m)-*type connection*. Semi-symmetric non-metric connection and non-metric ϕ-symmetric connection are two important examples of this connection such that (ℓ, m) = (1, 0) and (ℓ, m) = (0, 1), respectively. In this paper, we study lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection.

1. Introduction

We define a new connection on semi-Riemannian manifolds $(\bar{M}, \bar{g})$ as follow: A linear connection $\bar{\nabla}$ on $\bar{M}$ is called a *non-symmetric non-metric connection of type* (ℓ, m), and abbreviate it to (ℓ, m)-*type connection*, if there exist two smooth functions ℓ and m such that $\bar{\nabla}$ itself and its torsion tensor $\bar{T}$ satisfy

$$(1.1)\qquad (\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\},$$

$$(1.2)\qquad \bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\},$$

where J is a tensor field of type $(1, 1)$ and θ is a 1-form associated with a smooth unit vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Throughout this paper, we set $(\ell, m) \neq (0, 0)$ and we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

Two special cases are important for both the mathematical study and the applications to physics: (1) In case (ℓ, m) = (1, 0): The above connection $\bar{\nabla}$ becomes a semi-symmetric non-metric connection. The notion of semi-symmetric non-metric connection on a Riemannian manifold was introduced by Ageshe-Chafle [1,2] and later, studied by several authors [12,14]. (2) In case (ℓ, m) = (0, 1): The above connection $\bar{\nabla}$ becomes a non-metric ϕ-symmetric

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connection such that $\phi(\bar{X},\bar{Y}) = \bar{g}(J\bar{X},\bar{Y})$. The notion of the non-metric ϕ-symmetric connection was introduced by Jin [11,13,15].

Furthermore, (3) in case $(\ell,\, m) = (1,\, 0)$ in (1.1) and $(\ell,\, m) = (0,\, 1)$ in (1.2): The above connection $\bar{\nabla}$ becomes a quarter-symmetric non-metric connection. The notion of quarter-symmetric non-metric connection was introduced by Golab [7] and then, studied by Sengupta-Biswas [17] and Ahmad-Haseeb [3]. (4) In case $(\ell,\, m) = (0,\, 0)$ in (1.1) and $(\ell,\, m) = (0,\, 1)$ in (1.2): The above connection $\bar{\nabla}$ becomes a quarter-symmetric metric connection. The notion of quarter-symmetric metric connection was introduced Yano-Imai [18]. (5) In case $(\ell,\, m) = (0,\, 0)$ in (1.1) and $(\ell,\, m) = (1,\, 0)$ in (1.2): The above connection $\bar{\nabla}$ becomes a semi-symmetric metric connection. The notion of semi-symmetric metric connection was introduced Hayden [8].

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M},\bar{g})$ with respect to $\bar{g}$. By directed calculations, we see that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is an $(\ell,\, m)$-type connection if and only if $\bar{\nabla}$ satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{\ell\bar{X} + mJ\bar{X}\}. \tag{1.3}$$

The subject of study in this paper is lightlike hypersurfaces of an indefinite trans-Sasakian manifold $M = (\bar{M},\zeta,\theta,J,\bar{g})$ endowed with an $(\ell,\, m)$-type connection subject to the following two conditions that (1) the tensor field J and the 1-form θ, defined by (1.1) and (1.2), are identical with the indefinite trans-Sasakian structure tensor J and the structure 1-form θ of $\bar{M}$, respectively, and (2) the structure vector field ζ of $\bar{M}$ is tangent to M.

2. Lightlike hypersurfaces

An odd-dimensional semi-Riemannian manifold $(\bar{M},\bar{g})$ is called an *indefinite almost contact metric manifold*, and denoted by $\bar{M} = (\bar{M},\, J,\, \zeta,\, \theta,\, \bar{g})$, if there exists a set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$, where J is a tensor field of type (1, 1), ζ is a vector field which is called the *structure vector field* of $\bar{M}$, θ is a 1-form associated with ζ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta,\ \bar{g}(J\bar{X},\, J\bar{Y}) = \bar{g}(\bar{X},\, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}),\ \theta(\zeta) = 1, \tag{2.1}$$

where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike, respectively. The set $\{J,\, \zeta,\, \theta,\, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

From (2.1), we show that

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X},\zeta), \quad \bar{g}(J\bar{X},\bar{Y}) = -\bar{g}(\bar{X},J\bar{Y}).$$

In the entire discussion of this article, we shall assume that the structure vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Definition. An indefinite almost contact metric manifold $\bar{M}$ is said to be an *indefinite trans-Sasakian manifold* if, for the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$, there exist two smooth functions α and β such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X},\bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

Then $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure, of type* (α, β).

The notion of indefinite trans-Sasakian manifold was introduced by Oubina [16]. Indefinite Sasakian, Kenmotsu and cosymplectic manifolds are three important kinds of this indefinite trans-Sasakian manifold such that

$$\alpha = 1,\ \ \beta = 0; \quad \alpha = 0,\ \ \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}$$

By directed calculation from (1.3), (2.1) and $\theta(JY) = 0$, we obtain

$$(\bar{\nabla}_{\bar{X}} J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\} - \theta(\bar{Y})\{\ell J\bar{X} - m\bar{X} + m\theta(\bar{X})\zeta\}. \tag{2.2}$$

Replacing $\bar{Y}$ by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_X\zeta) = \ell\theta(X)$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta = (m - \alpha)J\bar{X} + (\ell + \beta)\bar{X} - \beta\theta(\bar{X})\zeta. \tag{2.3}$$

Let (M, g) be a lightlike hypersurface of $\bar{M}$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of the three equations in (2.1). We use same notations for any others. It is known [6] that the normal bundle $TM^\perp$ of M is a vector subbundle of the tangent bundle TM, of rank 1, and coincides with the radical distribution $Rad(TM) = TM \cap TM^\perp$. A complementary vector bundle $S(TM)$ of $TM^\perp$ in TM is non-degenerate distribution on M, which is called a *screen distribution* on M, such that

$$TM = TM^\perp \oplus_{orth} S(TM),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. For any null section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^\perp$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen distribution $S(TM)$ respectively. The tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus_{orth} S(TM).$$

In the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingartan formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.6}$$

$$\nabla_X \xi = -A^*_\xi X - \sigma(X)\xi, \tag{2.7}$$

where ∇ and ∇^* are the induced linear connections on M and $S(TM)$, B and C are the local second fundamental forms on M and $S(TM)$, respectively, A_N and A^*_ξ are the shape operators, and τ and σ are 1-forms on M.

Due to Jin [9], it is known that, for any lightlike hypersurface M of an indefinite almost contact manifold $\bar{M}$, $J(TM^{\perp})$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1. In the following, we shall assume that ζ is tangent to M. Călin [5] proved that if ζ is tangent to M, then it belongs to $S(TM)$. In this case, there exists two non-degenerate almost complex distributions D_o and D with respect to J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^{\perp}) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = TM^{\perp} \oplus_{orth} J(TM^{\perp}) \oplus_{orth} D_o.$$

In this case, the tangent bundle TM of M is decomposed as follow:

$$TM = D \oplus J(tr(TM)).$$

Consider two null vector fields U and V and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{2.8}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{2.9}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $FX = JSX$. Applying J to (2.9) and using $(2.1)_1$ and (2.8), we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \tag{2.10}$$

As $u(U) = \theta(\zeta) = 1$ and $FU = F\zeta = 0$, (F, u, θ, U, ζ) defines an well-known indefinite (f, g, u, v, λ) structure on M such that $\lambda = 0$ and F is called the *structure tensor field* of M and U is called the *structure vector field* of M.

3. (ℓ, m)-type connections

Using (1.1), (1.2), (1.3), (2.4) and (2.9), we obtain

$$\begin{aligned}(\nabla_X g)(Y, Z) = {} & B(X, Y)\eta(Z) + B(X, Z)\eta(Y) \\ & - \ell\{\theta(Y)g(X, Z) + \theta(Z)g(X, Y)\} \\ & - m\{\theta(Y)\bar{g}(JX, Z) + \theta(Z)\bar{g}(JX, Y)\},\end{aligned} \tag{3.1}$$

$$T(X, Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{3.2}$$

$$B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \tag{3.3}$$

where T is the torsion tensor with respect to the connection ∇ on M and η is a 1-form such that $\eta(X) = \bar{g}(X, N)$.

Theorem 3.1. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ is tangent to M. Then B is symmetric if and only if $m = 0$.*

Proof. If $m = 0$, then B is symmetric by (3.3). Conversely, if B is symmetric, then, taking $X = \zeta$ and $Y = U$ to (3.3), we have $m = 0$. □

From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and satisfies

$$B(X,\xi) = 0, \qquad B(\xi, X) = 0. \tag{3.4}$$

The local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y) + mu(X)\theta(Y), \tag{3.5}$$

$$C(X,PY) = g(A_N X, PY) + \{\ell\eta(X) + mv(X)\}\theta(PY), \tag{3.6}$$

$$\bar{g}(A^*_\xi X, N) = 0, \qquad \bar{g}(A_N X, N) = 0, \qquad \sigma = \tau. \tag{3.7}$$

As $S(TM)$ is non-degenerate, taking $X = \xi$ to (3.5) and using $(3.4)_2$, we get

$$A^*_\xi \xi = 0, \qquad \bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{3.8}$$

by (2.4), (2.7), $(3.4)_1$ and $(3.7)_3$. Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta,\xi) = 0$ and $\bar{g}(\zeta, N) = 0$ by turns and using (1.1), (2.3), (2.5), (3.5), (3.6) and $(3.8)_2$, we have

$$g(A^*_\xi X, \zeta) = -\alpha u(X), \qquad B(X,\zeta) = (m-\alpha)u(X), \tag{3.9}$$

$$\begin{aligned} &g(A_N X, \zeta) = -\alpha v(X) + \beta\eta(X), \\ &C(X,\zeta) = (\ell+\beta)\eta(X) + (m-\alpha)v(X). \end{aligned} \tag{3.10}$$

Substituting (2.9) into (2.3) and using (2.4), we have

$$\nabla_X \zeta = (m-\alpha)FX + (\ell+\beta)X - \beta\theta(X)\zeta. \tag{3.11}$$

Applying $\bar{\nabla}_X$ to (2.8) and (2.9) and using (2.2), (2.4), (2.5), (2.9), (2.10), (3.1), (3.6), $(3.8)_2$ and the facts that $\theta(V) = \theta(U) = 0$, we have

$$B(X,U) = C(X,V), \tag{3.12}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.13}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \beta u(X)\zeta, \tag{3.14}$$

$$\begin{aligned} (\nabla_X F)(Y) = {} & u(Y)A_N X - B(X,Y)U \\ & + \{\alpha g(X,Y) + \beta\bar{g}(JX,Y) - m\theta(X)\theta(Y)\}\zeta \\ & + (m-\alpha)\theta(Y)X - (\ell+\beta)\theta(Y)FX, \end{aligned} \tag{3.15}$$

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X,FY) - (\ell+\beta)\theta(Y)u(X), \tag{3.16}$$

$$\begin{aligned} (\nabla_X v)(Y) = {} & v(Y)\tau(X) - g(A_N X, FY) \\ & - (\ell+\beta)\theta(Y)v(X) + (m-\alpha)\theta(Y)\eta(X). \end{aligned} \tag{3.17}$$

4. Some results

Definition. The structure tensor field F of M is said to be *recurrent* [10] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y = \varpi(X)FY.$$

Theorem 4.1. *There exist no lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F is recurrent.*

Proof. If M is recurrent, then, from the above definition and (3.12), we get

$$\begin{aligned}\varpi(X)FY &= u(Y)A_N X - B(X,Y)U \\ &\quad + \{\alpha g(X,Y) + \beta \bar{g}(JX,Y) - m\theta(X)\theta(Y)\}\zeta \\ &\quad + (m-\alpha)\theta(Y)X - (\ell+\beta)\theta(Y)FX.\end{aligned} \tag{4.1}$$

Replacing Y by ξ to (4.1) and using $(3.4)_1$ and the fact that $F\xi = -V$, we get

$$\varpi(X)V + \beta u(X)\zeta = 0.$$

Taking the scalar product with U and ζ to this equation, we obtain

$$\varpi = 0, \qquad \beta = 0.$$

As $\varpi = 0$, we see that F is parallel with respect to the connection ∇.

Taking $Y = \zeta$ to (4.1) and using $(3.9)_2$, we get

$$(m-\alpha)\{X - u(X)U - \theta(X)\zeta\} = \ell FX.$$

Taking $X = V$ to this, we get $(m-\alpha)V = \ell\xi$. It follows that $m = \alpha$ and $\ell = 0$. Taking the scalar product with ζ to (4.1) and using $(3.10)_1$, we get

$$\alpha\{g(X,Y) - \theta(X)\theta(Y) - v(X)u(Y)\} = 0.$$

Taking the skew-symmetric part of this equation, we obtain

$$\alpha\{u(X)v(Y) - u(Y)v(X)\} = 0.$$

Taking $X = U$ and $Y = V$ to this equation, we have $\alpha = 0$. Therefore $m = 0$. It is a contradiction to $(\ell, m) \neq (0, 0)$. Thus we have our theorem. □

Corollary 4.2. *There exist no lightlike hypersurfaces of an indefinite trans-Sasakian manifold with an (ℓ, m)-type connection subject such that ζ is tangent to M and F is parallel with respect to the connection ∇ of M.*

Definition. The structure tensor field F of M is said to be *Lie recurrent* [10] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY,$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X,Y].$$

The structure tensor field F is called *Lie parallel* if $\mathcal{L}_X F = 0$.

Theorem 4.3. *Let M be a lightlike hypersurface of an indefinite Kaehler manifold $\bar{M}$ with an (ℓ, m)-type connection subject such that ζ is tangent to M and F is Lie recurrent. Then*

1. *F is Lie parallel,*
2. *the function α satisfies $\alpha = 0$,*
3. *the 1-form τ satisfies $\tau = 0$, and*
4. *the shape operator A^*_ξ satisfies $A^*_\xi U = A^*_\xi V = 0$.*

Proof. (1) Using the above definition, (2.10), (3.2) and (3.15), we get

$$\begin{aligned}\vartheta(X)FY = &-\nabla_{FY}X + F\nabla_Y X \\ &+ u(Y)A_N X - \{B(X,Y) - m\theta(Y)u(X)\}U \\ &- \theta(Y)\{\alpha X + \beta FX\} + \{\alpha g(X,Y) + \beta\bar{g}(JX,Y)\}\zeta.\end{aligned} \tag{4.2}$$

Taking $Y = \xi$ to (4.2) and using $(3.4)_1$, we have

$$-\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \beta u(X)\zeta. \tag{4.3}$$

Taking the scalar product with V and ζ to (4.3) by turns, we have

$$u(\nabla_V X) = 0, \qquad \theta(\nabla_V X) = -\beta u(X). \tag{4.4}$$

Replacing Y by V to (4.2) and using the fact that $\theta(V) = 0$, we have

$$\vartheta(X)\xi = -\nabla_\xi X + F\nabla_V X - B(X,V)U + \alpha u(X)\zeta. \tag{4.5}$$

Applying F to this equation and using (2.10) and (4.4), we obtain

$$\vartheta(X)V = \nabla_V X + F\nabla_\xi X + \beta u(X)\zeta.$$

Comparing this equation with (4.3), we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with ζ to (4.5), we have $g(\nabla_\xi X, \zeta) = \alpha u(X)$. Taking $X = U$ to this result and using (3.13), we obtain $\alpha = 0$.

(3) Taking the scalar product with N to (4.2) and using $(3.7)_2$, we have

$$-\bar{g}(\nabla_{FY}X, N) + \bar{g}(\nabla_Y X, U) = 0. \tag{4.6}$$

Replacing X by ξ to (4.6) and using (2.7), and (3.5), we have

$$B(X,U) = \tau(FX). \tag{4.7}$$

Replacing X by U to (4.7) and using (3.12) and the fact that $FU = 0$, we get

$$C(U,V) = B(U,U) = 0. \tag{4.8}$$

Replacing X by V to (4.6) and using (3.5) and (3.14), we have

$$B(FY,U) = -\tau(Y).$$

Taking $Y = U$ and $Y = \zeta$ and using the fact that $FU = F\zeta = 0$, we obtain

$$\tau(U) = 0, \qquad \tau(\zeta) = 0. \tag{4.9}$$

Taking $X = U$ to (4.2) and using (3.3) $(3.10)_1$, (3.12) and (3.13), we get

$$u(Y)A_N U - F(A_N FY) - A_N Y - \tau(FY)U + \beta\eta(Y)\zeta = 0.$$

Taking the scalar product with V and using (3.6), (3.12) and (4.8), we get

$$B(X,U) = -\tau(FX).$$

Comparing this with (4.7), we obtain $\tau(FX) = 0$. Replacing X by FY to this result and using (2.10) and (4.9), we have $\tau = 0$.

(4) Replacing Y by U to (3.3) and using (4.7) and $\tau = 0$, we have

$$B(U,X) = m\theta(X). \tag{4.10}$$

Taking $X = U$ to (3.5) and using (4.10), we have $g(A^*_\xi U, X) = 0$, As $S(TM)$ is non-degenerate, we get $A^*_\xi U = 0$. Replacing X by ξ to (4.3) and using $(3.8)_1$ and the fact that $\tau = 0$, we obtain $A^*_\xi V = 0$. □

Theorem 4.4. *Let M be a lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type connection such that ζ is tangent to M. If U or V is parallel with respect to the connection ∇ on M, then $\tau = 0$ and $\alpha = \beta = 0$, i.e., $\bar{M}$ is an indefinite cosymplectic manifold.*

Proof. (1) If U is parallel with respect to ∇, then, taking the scalar product with V and ζ to (3.13) such that $\nabla_X U = 0$ by turns, we obtain $\tau = 0$ and $\alpha = \beta = 0$, respectively. Applying F to (3.13): $F(A_{_N} X) = 0$ and using (2.10), $(3.10)_1$ and the fact that $\alpha = \beta = 0$, we obtain

$$A_{_N} X = u(A_{_N} X)U. \tag{4.11}$$

(2) If V is parallel with respect to ∇, then, taking the scalar product with U and ζ to (3.14) such that $\nabla_X V = 0$ by turns, we have $\tau = 0$ and $\beta = 0$. Applying F to (3.14): $F(A^*_\xi X) = 0$ and using (2.10) and $(3.9)_1$, we obtain

$$A^*_\xi X = -\alpha u(X)\zeta + u(A^*_\xi X)U.$$

Taking the scalar product with U and using (3.5), we have $B(X, U) = 0$. Thus $B(\zeta, U) = 0$. Taking $X = U$ and $Y = \zeta$ to (3.3), we get $B(U, \zeta) = m$. On the other hand, replacing X by U to $(3.9)_2$, we have $B(U, \zeta) = m - \alpha$. From the above two results, we get $\alpha = 0$ and

$$A^*_\xi X = u(A^*_\xi X)U. \tag{4.12}$$

As $\alpha = \beta = 0$ in (1) and (2), $\bar{M}$ is an indefinite cosymplectic manifold. □

5. Indefinite generalized Sasakian space forms

Denote by $\bar{R}$, R and R^* the curvature tensors of the (ℓ, m)-type connection $\bar{\nabla}$ on $\bar{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain Gauss equations for M and $S(TM)$, respectively, such that

$$\begin{aligned} \bar{R}(X,Y)Z \;=\; & R(X,Y)Z + B(X,Z)A_{_N}Y - B(Y,Z)A_{_N}X \\ & + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \\ & - \tau(Y)B(X,Z) + B(T(X,Y),Z)\}N, \end{aligned} \tag{5.1}$$

$$\begin{aligned} R(X,Y)PZ = & R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ & + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) - \sigma(X)C(Y,PZ) \\ & + \sigma(Y)C(X,PZ) + C(T(X,Y),PZ)\}\xi. \end{aligned} \tag{5.2}$$

Definition. An indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form*, denote it by $\bar{M}(f_1, f_2, f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\widetilde{R}(\bar{X},\bar{Y})\bar{Z} \;=\; f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \tag{5.3}$$

$$+ f_2\{\bar{g}(\bar{X}, J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y}, J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X}, J\bar{Y})J\bar{Z}\}$$
$$+ f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X}$$
$$+ \bar{g}(\bar{X}, \bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y}, \bar{Z})\theta(\bar{X})\zeta\},$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\bar{M}$.

The generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ was introduced by Alegre *et. al.* [4]. Sasakian, Kenmotsu and cosymplectic space forms are important kinds of generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4},\ f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4},\ f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4},$$

respectively, where c is a constant J-sectional curvature of each space forms.

By directed calculations from (1.2), (1.3) and (2.2), we see that

$$\begin{aligned}\bar{R}(\bar{X}, \bar{Y})\bar{Z} = {} & \widetilde{R}(\bar{X}, \bar{Y})\bar{Z} \\ & + (\bar{\nabla}_{\bar{X}}\theta)(\bar{Z})\{\ell\bar{Y} + mJ\bar{Y}\} - (\bar{\nabla}_{\bar{Y}}\theta)(\bar{Z})\{\ell\bar{X} + mJ\bar{X}\} \\ & + \theta(\bar{Z})\{(\bar{X}\ell)\bar{Y} - (\bar{Y}\ell)\bar{X} + (\bar{X}m)J\bar{Y} - (\bar{Y}m)J\bar{X} \\ & \quad - m\alpha[\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}] - m\beta[\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}] \\ & \quad - 2m\beta\bar{g}(\bar{X}, J\bar{Y})\zeta\}.\end{aligned} \tag{5.4}$$

Taking the scalar product with ξ and N to (5.4) by turns and then, substituting (5.1), (3.2) and (5.3) and using (5.2) and $(3.7)_2$, we get

$$\begin{aligned}& (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ & + \{\tau(X) - \ell\theta(X)\}B(Y, Z) - \{\tau(Y) - \ell\theta(Y)\}B(X, Z) \\ & - m\{\theta(X)B(FY, Z) - \theta(Y)B(FX, Z)\} \\ & - m\{(\bar{\nabla}_X\theta)(Z)u(Y) - (\bar{\nabla}_Y\theta)(Z)u(X)\} \\ & - \theta(Z)\{[Xm + m\beta\theta(X)]u(Y) - [Ym + m\beta\theta(Y)]u(X)\} \\ = {} & f_2\{u(Y)\bar{g}(X, JZ) - u(X)\bar{g}(Y, JZ) + 2u(Z)\bar{g}(X, JY)\},\end{aligned} \tag{5.5}$$

$$\begin{aligned}& (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ & - \{\tau(X) + \ell\theta(X)\}C(Y, PZ) + \{\tau(Y) + \ell\theta(Y)\}C(X, PZ) \\ & - m\{\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)\} \\ & - (\bar{\nabla}_X\theta)(PZ)\{\ell\eta(Y) + mv(Y)\} + (\bar{\nabla}_Y\theta)(PZ)\{\ell\eta(X) + mv(X)\} \\ & - \theta(PZ)\{[X\ell + m\alpha\theta(X)]\eta(Y) - [Y\ell + m\alpha\theta(Y)]\eta(X) \\ & \qquad + [Xm + m\beta\theta(X)]v(Y) - [Ym + m\beta\theta(Y)]v(X)\} \\ = {} & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ & + f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\} \\ & + f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).\end{aligned} \tag{5.6}$$

Theorem 5.1. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type connection subject such that the structure vector field ζ is tangent to M. Then*

(1) *α is a constant on M,*
(2) *$\alpha\beta = 0$, and*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

Proof. Applying $\bar{\nabla}_X$ to $\theta(V) = 0$ and $\theta(U) = 0$ by turns and using (2.4), (3.13), (3.14) and the facts that $\theta \circ J = \theta \circ F = \theta(N) = 0$, we have

$$(\bar{\nabla}_X\theta)(V) = \beta u(X), \qquad (\bar{\nabla}_X\theta)(U) = \alpha\eta(X) + \beta v(X). \tag{5.7}$$

Applying ∇_X to (3.12): $B(Y,U) = C(Y,V)$ and using (2.9), (3.5), (3.6), (3.7), $(3.9)_2$, $(3.10)_2$, (3.13) and (3.14), we obtain

$$\begin{aligned}(\nabla_X B)(Y,U) = &(\nabla_X C)(Y,V) - 2\tau(X)C(Y,V)\\ &+ \beta(m-\alpha)\{u(Y)v(X) - u(X)v(Y)\}\\ &+ \alpha(m-\alpha)u(Y)\eta(X) - \beta(\ell+\beta)u(X)\eta(Y)\\ &- g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this and (3.12) into (5.6) with $Z = U$ and using $(5.7)_2$, we get

$$\begin{aligned}&(\nabla_X C)(Y,V) - (\nabla_Y C)(X,V)\\ &- \{\tau(X) + \ell\theta(X)\}C(Y,V) + \{\tau(Y) + \ell\theta(Y)\}C(X,V)\\ &- m\{\theta(X)C(FY,V) + \theta(Y)C(FX,V)\}\\ &+ \beta(m-2\alpha)\{u(Y)v(X) - u(X)v(Y)\}\\ &+ (\ell\beta - \alpha^2 + \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ = \;& f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X,JY)\}.\end{aligned}$$

Comparing this with (5.6) such that $PZ = V$ and using $(5.7)_1$, we obtain

$$\begin{aligned}&\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u(Y)\eta(X) - u(X)\eta(Y)\}\\ &= 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\}.\end{aligned}$$

Taking $Y = U$, $X = \xi$ and $Y = U$, $X = V$ to this by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar{\nabla}_X$ to $\eta(Y) = \bar{g}(Y,N)$ and using (1.1) and (2.5), we have

$$(\nabla_X\eta)(Y) = -g(A_N X, Y) + \tau(X)\eta(Y) - \{\ell\eta(X) + mv(X)\}\theta(Y). \tag{5.8}$$

Applying $\bar{\nabla}_X$ to $\theta(\zeta) = 1$ and using (2.3), we obtain

$$(\bar{\nabla}_X\theta)(\zeta) = -\ell\theta(X). \tag{5.9}$$

Applying ∇_Y to $(3.10)_2$ and using (3.11), (3.17) and (5.8), we have

$$\begin{aligned}&(\nabla_X C)(Y,\zeta)\\ &= X(\ell+\beta)\eta(Y) + X(m-\alpha)v(Y)\\ &\quad + (\ell+\beta)\{-g(A_N X,Y) - g(A_N Y,X)\\ &\qquad\qquad + \tau(X)\eta(Y) - \ell[\theta(Y)\eta(X) + \theta(X)\eta(Y)]\\ &\qquad\qquad + \beta\theta(X)\eta(Y) - m[\theta(Y)v(X) + \theta(X)v(Y)]\}\end{aligned}$$

$$+ (m-\alpha)\{-g(A_N X, FY) - g(A_N Y, FX) + v(Y)\tau(X) + (m-\alpha)\theta(Y)\eta(X) + \beta\theta(X)v(Y) - (\ell+\beta)\theta(Y)v(X)\}.$$

Substituting this equation and $(3.10)_2$ into (5.6) such that $PZ = \zeta$ and using (5.9) and the fact that $\alpha\beta = 0$, we obtain

$$\{X\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(X)\}\eta(Y) - \{Y\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(Y)\}\eta(X) = (X\alpha)v(Y) - (Y\alpha)v(X).$$

Taking $X = \zeta$, $Y = \xi$ and $X = U$, $Y = V$ to this by turns, we have

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U\alpha = 0.$$

Applying ∇_Y to $(3.9)_2$ and using (3.11) and (3.16), we have

$$(\nabla_X B)(Y,\zeta) = X(m-\alpha)u(Y) - (\ell+\beta)B(Y,X) - (m-\alpha)\{B(X,FY) + B(Y,FX) + u(Y)\tau(X) + \ell\theta(Y)u(X) + \beta[\theta(Y)u(X) - \theta(X)u(Y)]\}.$$

Substituting this into (5.5) such that $Z = \zeta$ and using (3.3) and (5.9), we have

$$(X\alpha)u(Y) = (Y\alpha)u(X).$$

Taking $Y = U$ to this result and using the fact that $U\alpha = 0$, we have $X\alpha = 0$. Therefore α is a constant. This completes the proof of the theorem. □

Definition. (1) A lightlike hypersurface M is called *totally umbilical* [6] if there exists a smooth function ρ on a coordinate neighborhood $\mathcal{U}$ such that

$$B(X,Y) = \rho g(X,Y).$$

In case $\rho = 0$, we say that M is *totally geodesic.*

(2) A screen distribution $S(TM)$ is called *totally umbilical* [6] in M if there exists a smooth function γ on a coordinate neighborhood $\mathcal{U}$ such that

$$C(X,PY) = \gamma g(X,Y).$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(3) A lightlike hypersurface M is called *screen conformal* [9] if there exists a non-vanishing smooth function φ on $\mathcal{U}$ such that

$$C(X,PY) = \varphi B(X,PY). \tag{5.10}$$

Theorem 5.2. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type connection subject such that ζ is tangent to M. If one of the following four conditions*

(1) *M is Lie recurrent,*
(2) *M is totally umbilical,*
(3) *$S(TM)$ is totally umbilical, and*
(4) *M is screen conformal,*

is satisfied, then

$$\alpha = 0, \qquad f_1 = -\beta^2, \qquad f_2 = 0, \qquad f_3 = -\zeta\beta.$$

In case (3) *and* (4), $m = \alpha = 0$, $\ell = -\beta \neq 0$ *and* $\bar{M}(f_1, f_2, f_3)$ *is an indefinite* β*-Kenmotsu manifold with a semi-symmetric non-metric connection.*

Proof. (1) By Theorem 4.2, we have (4.7), (4.10) and $\alpha = \tau = 0$. Thus

$$B(X, U) = 0, \qquad B(U, X) = m\theta(X). \tag{5.11}$$

Applying ∇_Y to $(5.11)_1$ and using $(3.9)_2$, (3.13) and the result: $\alpha = 0$, we get

$$(\nabla_X B)(Y, U) = -B(Y, F(A_N X)) + m\beta u(Y)v(X).$$

Substituting this into (5.5) such that $Z = U$ and using $(5.7)_2$, we have

$$B(X, F(A_N Y)) - B(Y, F(A_N X)) = f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Taking $X = \xi$ and $Y = U$ to this and using $(3.4)_2$, $(5.11)_2$ and $\theta \circ F = 0$, we obtain $f_2 = 0$. Therefore $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$ by Theorem 5.1.

(2) If M is totally umbilical, then B is symmetric. Thus $m = 0$ by Theorem 3.1. In this case, the equation $(3.9)_2$ is reduced to

$$\rho\theta(X) = -\alpha u(X).$$

Taking $X = \zeta$ and $X = U$ to this equation by turns, we have $\rho = 0$ and $\alpha = 0$, respectively. As $\rho = 0$, M is totally geodesic. Taking $Z = U$ to (5.5) and using the facts that $B = 0$ and $m = 0$, we have

$$f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\} = 0.$$

Taking $X = \xi$ and $Y = U$ to this equation, we get $f_2 = 0$. Thus we also have $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$ by Theorem 5.1.

(3) If $S(TM)$ is totally umbilical, then $(3.10)_2$ is reduced to

$$\gamma\theta(X) = (\ell + \beta)\eta(X) + (m - \alpha)v(X).$$

Taking $X = \zeta$, $X = \xi$ and $X = V$ to this equation by turns, we have

$$\gamma = 0, \qquad \ell = -\beta, \qquad m = \alpha,$$

respectively. As $\gamma = 0$, $C(X, V) = 0$. From (3.12), $B(X, U) = 0$. Replacing Y by U to (3.3) and using the facts that $B(X, U) = 0$ and $m = \alpha$, we obtain

$$B(U, X) = \alpha\theta(X).$$

Taking $X = \zeta$ to this and using $(3.9)_2$ with $m = \alpha$, we have $\alpha = 0$.

As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $\bar{M}$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection and $f_1 + \beta^2 = f_2$ by Theorem 5.1. Taking $PZ = V$ to (5.6) and using $(5.7)_1$ and the fact that $C = 0$, we have

$$(f_1 + \beta^2)\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2f_2\, \bar{g}(X, JY) = 0.$$

Taking $X = \xi$ and $Y = U$, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$.

(4) If M is screen conformal, then, from $(3.9)_2$, $(3.10)_2$, and (5.10), we have

$$(\ell + \beta)\eta(X) + (m - \alpha)v(X) = \varphi(m - \alpha)u(X).$$

Taking $X = \xi$ and $X = V$ to this equation by turns, we have

$$\ell = -\beta, \qquad m = \alpha,$$

respectively. As $\alpha\beta = 0$, it follow that

$$\ell m = \ell\alpha = m\beta = 0, \qquad \ell\beta = -\beta^2, \qquad m\alpha = \alpha^2. \tag{5.12}$$

Let $\mu = U - \varphi V$. Then, from (3.12), we obtain

$$B(X, \mu) = 0, \qquad g(\mu, \mu) = -2\varphi, \qquad J\mu = N - \varphi\xi. \tag{5.13}$$

Substituting (3.5) and (3.6) into (5.10) and using the facts that $A_N X - \varphi A^*_\xi X$, $\zeta \in \Gamma(S(TM))$ and $S(TM)$ is non-degenerate, we obtain

$$A_N X - \varphi A^*_\xi X = -\{\ell\eta(X) + mv(X) - \varphi mu(X)\}\zeta. \tag{5.14}$$

Applying ∇_X to $\mu = U - \varphi V$ and using (3.13), (3.14), (5.14) and the facts that F is a linear operator and $F\zeta = 0$, we have

$$\begin{aligned} \nabla_X\mu = \tau(X)U - \{X\varphi - \varphi\tau(X)\}V \\ - \{\alpha\eta(X) + \beta v(X) - \varphi\beta u(X)\}\zeta. \end{aligned} \tag{5.15}$$

Applying $\bar{\nabla}_X$ to $\theta(\mu) = 0$ and using (5.15), we obtain

$$(\bar{\nabla}_X\theta)(\mu) = \alpha\eta(X) + \beta v(X) - \varphi\beta u(X). \tag{5.16}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this equation into (5.6) and using (5.5) and $m\beta = 0$, we have

$$\begin{aligned} &\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ) \\ &- (\bar{\nabla}_X\theta)(PZ)\{\ell\eta(Y) + mv(Y) - \varphi mu(Y)\} \\ &+ (\bar{\nabla}_Y\theta)(PZ)\{\ell\eta(X) + mv(X) - \varphi mu(X)\} \\ &- \theta(PZ)\{[X\ell + m\alpha\theta(X)]\eta(Y) - [Y\ell + m\alpha\theta(Y)]\eta(X) \\ &\qquad + (Xm)g(\mu, Y) - (Ym)g(\mu, X)\} \\ = \;& f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\} \\ &+ f_2\{g(\mu, Y)\bar{g}(X, JPZ) - g(\mu, X)\bar{g}(Y, JPZ) + 2g(\mu, PZ)\bar{g}(X, JY)\} \\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ). \end{aligned} \tag{5.17}$$

Replacing PZ by μ to this and using (5.12), (5.13) and (5.16), we obtain

$$\begin{aligned} &(\alpha^2 + \beta^2)\{g(\mu, X)\eta(Y) - g(\mu, Y)\eta(X)\} \\ &= (f_1 + f_2)\{g(\mu, Y)\eta(X) - g(\mu, X)\eta(Y)\} - 4\varphi f_2\bar{g}(X, JY). \end{aligned}$$

Taking $X = \xi$ and $Y = V$ to this equation and using $g(\mu, V) = 1$, we get

$$f_1 + f_2 = -(\alpha^2 + \beta^2).$$

From this result and Theorem 5.1, we see that $\alpha = 0$. As $\alpha = m = 0$ and $\beta = -\ell \neq 0$, $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu manifold with a semi-symmetric non-metric connection.

Applying $\bar{\nabla}_X$ to $\theta(\xi) = 0$ and using $(3.8)_2$ and $(3.9)_1$, we obtain

$$(\bar{\nabla}_X \theta)(\xi) = -\alpha u(X) = 0. \tag{5.18}$$

Replacing Y by ξ to (5.17) and using (5.18), we obtain

$$\begin{aligned} &\{\xi\varphi - 2\varphi\tau(\xi)\}B(X, PZ) \\ &= f_1 g(X, PZ) + f_2\{g(\mu, X)u(PZ) + 2f_2 g(\mu, PZ)u(X)\} - f_3\theta(X)\theta(PZ). \end{aligned}$$

Taking $X = V,\ PZ = U$ and then, $X = U,\ PZ = V$ by turns, we have

$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(V, U) = f_1 + f_2,$$
$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(U, V) = f_1 + 2f_2,$$

respectively. As $B(U, V) = B(V, U)$ by (3.3), from the last two equations we show that $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = -\zeta\beta$. □

Theorem 5.3. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type connection such that ζ is tangent to M. If U or V is parallel with respect to ∇, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure such that*

$$\alpha = \beta = 0, \qquad f_1 = f_2 = f_3 = 0.$$

Proof. (1) If U is parallel with respect to ∇, then, by (1) of Theorem 4.4, we have (4.11) and the results: $\tau = \alpha = \beta = 0$. Thus $f_1 = f_2 = f_3$ by Theorem 5.1. Taking the scalar product with U to (4.11) and using (3.6), we obtain

$$C(X, U) = 0.$$

Applying ∇_X to $C(Y, U) = 0$ and using the fact that U is parallel, we get

$$(\nabla_X C)(Y, U) = 0.$$

Substituting the last two equations into (5.6) such that $PZ = U$ and using $(5.7)_2$ such that $\alpha = \beta = 0$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$, we obtain $f_1 + f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$.

(2) If V is parallel with respect to ∇, then, by (2) of Theorem 4.4, we have (4.12) and the results: $\tau = \alpha = \beta = 0$. Thus $f_1 = f_2 = f_3$ by Theorem 5.1. Taking the scalar product with U to (4.12) and using (3.5) and (3.12), we get

$$C(X, V) = 0.$$

Applying ∇_X to $C(Y, V) = 0$ and using the fact that V is parallel, we obtain

$$(\nabla_X C)(Y, V) = 0.$$

Substituting the last two equations into (5.6) such that $PZ = V$ and using $(5.7)_1$ such that $\beta = 0$, we have

$$f_1\{u(Y)\eta(X) - u(X)\eta(Y)\} + 2f_2\bar{g}(X, JY) = 0.$$

Taking $X = \xi$ and $Y = U$, we obtain $f_1 + 2f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$. □

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