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UPPER BOUNDS OF SECOND HANKEL DETERMINANT FOR UNIVERSALLY PRESTARLIKE FUNCTIONS

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Abstract. In [12,13] the researchers introduced universally convex, universally starlike and universally prestarlike functions in the slit domain $\mathbb{C}\backslash[1,\infty)$. These papers extended the corresponding notions from the unit disc to other discs and half-planes containing the origin. In this paper, we introduce universally prestarlike generalized functions of order α with $\alpha \leq 1$ and we obtain upper bounds of the second Hankel determinant $|a_2a_4 - a_3^2|$ for such functions.

1. Introduction

Let $\mathcal{H}(\Omega)$ denote the set of all analytic functions in a domain Ω. Suppose Ω contains the origin and $\mathcal{H}_0(\Omega)$ stands for the set of all functions $f \in \mathcal{H}(\Omega)$ with $f(0) = 1$ and also let

$$\mathcal{H}_1(\Omega) = \{zf : f \in \mathcal{H}_0(\Omega)\}.$$

If $\Omega = \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, we write $\mathcal{H} \equiv \mathcal{H}(\mathbb{U}), \mathcal{H}_0 \equiv \mathcal{H}_0(\mathbb{U})$ and $\mathcal{H}_1 \equiv \mathcal{H}_1(\mathbb{U})$. Let the Hadamard (or convolution) product of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \ z \in \mathbb{U}$$

in $\mathcal{H}_0(\Omega)$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in \mathcal{H}_1$ is called a starlike function of order α $(0 \leq \alpha \leq 1)$ if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U})$$

and the set of such functions is denoted by $\mathcal{S}_\alpha$.

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Due to Ruscheweyh [10], for $f \in \mathcal{H}_1$, let us denote by $\mathcal{R}_\alpha$, the set of all prestarlike functions of order α ($\alpha \le 1$) in $\mathbb{U}$ satisfying the criteria

$$\begin{cases} \frac{z}{(1-z)^{2-2\alpha}} * f \in \mathcal{S}_\alpha, & \alpha < 1, \\ \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2}, & \alpha = 1,\ z \in \mathbb{U}, \end{cases}$$

where

$$\frac{z}{(1-z)^{2-2\alpha}} = z + \sum_{n=2}^{\infty} \mathcal{C}(\alpha, n) z^n$$

is a well-known extremal function in $\mathcal{S}_\alpha$ and

$$\mathcal{C}(\alpha, n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!}; \quad (n \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \dots\}).$$

Note that $\mathcal{C}(\alpha, n)$ is a decreasing function of α with

$$\lim_{n\to\infty} \mathcal{C}(\alpha, n) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2}, \\ 1 & \text{if } \alpha = \frac{1}{2}, \\ 0 & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

While working with prestarlike functions and convolutions, the following notation turned out to be useful:

$$(D^n f)(z) = \frac{z}{(1-z)^n} * f,$$

where $n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ and therefore we have $D^{n+1} f = \frac{z}{n!}(z^{n-1} f)^{(n)}$ for $n \in \mathbb{N}_0$. Using this operator we find that a function $f \in \mathcal{H}_1$ is prestarlike of order $\alpha \le 1$ if and only if

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{P},$$

where

$$\mathcal{P} = \{g \in \mathcal{H}_0 : \Re(g(z)) > \frac{1}{2},\ z \in \mathbb{U}\}$$

or, equivalently, by Herglotz formula,

$$g \in \mathcal{P} \Leftrightarrow g(z) = \int_0^1 \frac{d\mu(t)}{1 - e^{-it} z},$$

where μ is a probability measure on $[0,2\pi]$.

The notion of prestarlike functions of order α has recently been extended from the unit disc $\mathbb{U}$ to other discs and half-planes containing the origin (see [11–13]). Define one such disc $\Omega_{\gamma,\rho}$ by

$$\Omega_{\gamma,\rho} = \{\omega_{\gamma,\rho}(z) : z \in \mathbb{U}\},$$

where $\gamma \in \mathbb{C}\backslash\{0\}$ and $\rho \in [0,1]$ are two unique parameters and $\omega_{\gamma,\rho}(z) = \frac{\gamma z}{1-\rho z}$. Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma+\rho| \leq 1$. For $\alpha \leq 1$, and for some admissible pair (γ, ρ), we define

$$\mathcal{R}_\alpha(\Omega_{\gamma,\rho}) = \{f \in \mathcal{H}_1(\Omega_{\gamma,\rho}) : \frac{1}{\gamma} f(\omega_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha\},$$

where $\mathcal{H}_1(\Omega_{\gamma,\rho}) = \{zf : f \in \mathcal{H}_0(\Omega_{\gamma,\rho}) \text{ with } f(0) = 1\}$. A function f in $\mathcal{R}_\alpha(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ (see [12]).

Definition 1 ([13])**.** Let $\alpha \leq 1$ and $\Lambda = \mathbb{C}\backslash[1,\infty)$. A function $f \in \mathcal{H}_1(\Lambda)$ is called universally prestarlike of order α in Λ if and only if f is prestarlike of order α in all sets $\omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. Denote the set of all universally prestarlike functions in Λ by $\mathcal{R}^u_\alpha$.

Due to Ma-Minda [8] we state the following subordination principle:

Definition 2. Suppose ϕ is an analytic function such that

(1) $\Re\ (\phi) > 0$ in $\mathbb{U}$,
(2) $\phi(0) = 1$, $\phi'(0) > 0$,
(3) ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

For $\alpha \leq 1$ and a function $f \in \mathcal{H}_1(\Lambda)$, we let $\mathcal{R}^u_\alpha(\phi)$ be the generalized class of universally prestarlike functions satisfying the condition

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \phi(z), \tag{1}$$

where $\prec$ denotes the subordination and ϕ is an analytic function given by Definition 2. Note that for different choices of ϕ, the class $\mathcal{R}^u_\alpha(\phi)$ gives rise to several known and unknown classes of universally prestarlike functions of order α as given in the following example.

Example 1.1. If $\alpha \leq 1$, and $f \in \mathcal{H}_1(\Lambda)$, then

$$f \in \mathcal{R}^u_\alpha(A, B) \Longleftrightarrow \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1) \tag{2}$$

and

$$f \in \mathcal{R}^u_\alpha(\beta) \Longleftrightarrow \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \prec \frac{1 + (1 - 2\beta)z}{1 - z}, \quad (0 \leq \beta < 1). \tag{3}$$

In particular $\mathcal{R}^u_{\frac{1}{2}}(1, -1) = \mathcal{S}^*$ is the class of starlike univalent functions.

Recall that the Hankel determinants $H_q(n)$ $(n = 1, 2, 3, \ldots; q = 1, 2, \ldots)$ of the functions $f(z) = \sum_{n=1}^{\infty} a_n z^n,\ a_1 = 1$ are defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In particular,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_1a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

For more details on Hankel determinants, one may refer to the papers [4–7,9,17].

Though there has been an increasing interest to study the functional $H_2(1)$ (that is, $a_1a_3 - a_2^2$) for certain classes of universally prestarlike functions (see [14–16]) and in particular, the Fekete and Szegö estimates of $|a_3 - \mu a_2^2|$ (see [2]), the study of the functional $H_2(2)$ (that is, $a_2a_4 - a_3^2$) for universally prestarlike functions is not yet known. The main purpose of this paper is to obtain the upper bounds of Hankel determinant $|a_2a_4 - a_3^2|$ for functions $f \in \mathcal{R}_\alpha^\mu(\phi)$.

1.1. Preliminary results

To prove our main results, we state the following lemmas.

Lemma 1.2 (see [1, p. 41]). *Let* $\mathbf{P}$ *be the class of all analytic functions* p *of the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{4}$$

satisfying $\Re\,(p(z)) > 0$ $(z \in \mathbb{U})$ *and* $p(0) = 1$. *Then*

$$|p_n| \leq 2 \ (n = 1, 2, 3, \ldots).$$

This inequality is sharp for each n. *In particular, equality holds for all* n *for the function*

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n.$$

Lemma 1.3 (see [6]). *If the function* $p \in \mathbf{P}$ *is given by* (4)*, then*

$$2p_2 = p_1^2 + x(4 - p_1^2), \tag{5}$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z, \tag{6}$$

for some x, z *with* $|x| \leq 1$, $|z| \leq 1$ *and* $p_1 \in [0, 2]$.

Lemma 1.4 ([3]). *The power series for a function* p *given in* (4) *converges in* $\mathbb{U}$ *to a function in* $\mathbf{P}$ *if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \ldots \tag{7}$$

and $p_{-k} = \overline{p_k}$, *are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k z}),\ \rho_k > 0,\ t_k \text{ real}$$

and $t_k \neq t_j$ *for* $k \neq j$; *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \geq m$.

This necessary and sufficient condition is due to Caratheodory and Toeplitz and can be found in [3].

2. Coefficient bounds for the function class $\mathcal{R}_\alpha^u(\phi)$

In this section we obtain the upper bounds of the Hankel determinant

$$|a_2a_4 - a_3^2|$$

for $f \in \mathcal{R}_\alpha^u(\phi)$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1-tz},$$

where

$$a_k = \int_0^1 t^k d\mu(t),$$

$\mu(t)$ is a probability measure on $[0,1]$.

Theorem 2.1. *Let* $f \in \mathcal{R}_\alpha^u(\phi)$ *be given by*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad (a_0 = 0 \text{ and } a_1 = 1) \tag{8}$$

and suppose ϕ, *defined by Definition 2, is of the form*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{9}$$

(i) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$(2-2\alpha)|B_2| \leq B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$\begin{aligned}&\left|2(3-2\alpha)B_1B_3 - (2-2\alpha)^2B_1^4 - (4-2\alpha)B_2^2\right| \\ &\quad + (2-2\alpha)(1-2\alpha)B_1^2|B_2| - (4-2\alpha)B_1^2 \leq 0,\end{aligned}$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(3-2\alpha)^2}.$$

(ii) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$(2-2\alpha)|B_2| \geq B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$\begin{aligned}&\left|2(3-2\alpha)B_1B_3 - (2-2\alpha)^2B_1^4 - (4-2\alpha)B_2^2\right| - (2-2\alpha)B_1|B_2| \\ &\quad - (3-2\alpha)B_1^2 + (1-\alpha)(1-2\alpha)\{2B_1^2|B_2| + B_1^3\} \geq 0,\end{aligned}$$

(*or*) *the conditions*

$$(2-2\alpha)|B_2| \leq B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$\left|(3-2\alpha)B_1B_3 - 2(1-\alpha)^2B_1^4 - (2-\alpha)B_2^2\right| + (1-\alpha)(1-2\alpha)B_1^2|B_2| - (2-\alpha)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{(3-2\alpha)^2(4-2\alpha)}[2(1-\alpha)^2B_1^4 + (2-\alpha)|B_2|^2 - (3-2\alpha)B_1|B_3| - (1-\alpha)(1-2\alpha)B_1^2|B_2|]. \tag{10}$$

(iii) *If B_1, B_2 and B_3 satisfy the conditions*

$$(2-2\alpha)|B_2| > B_1 - (1-\alpha)(1-2\alpha)B_1^2,$$

$$\left|4(3-2\alpha)B_1B_3 - 2(2-2\alpha)^2B_1^4 - 2(4-2\alpha)B_2^2\right| - 2(2-2\alpha)B_1|B_2| - 2(3-2\alpha)B_1^2 + (2-2\alpha)(1-2\alpha)B_1^2(2|B_2| + B_1) \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(3-2\alpha)^2(4-2\alpha)}\left(\frac{M}{N}\right),$$

where

$$\begin{aligned} M &= 8(3-2\alpha)B_1[(2-2\alpha)B_2 - (4-2\alpha)|B_3|] - 4(2-2\alpha)(1-2\alpha)B_1^2[2|B_2| \\ &\quad + (3-2\alpha)B_1] + 16|B_2|^2(3-2\alpha) - 4B_1^2(4\alpha^2 - 12\alpha + 9) \\ &\quad + (2-2\alpha)^2B_1^4(15 - 8\alpha - 4\alpha^2), \\ N &= 4(2-2\alpha)B_1^2[(2-2\alpha)B_1^2 - 1] - 8B_1[(3-2\alpha)|B_3| + (2-2\alpha)|B_2|] \\ &\quad - 4(2-2\alpha)(1-2\alpha)B_1^2(|B_2| - B_1) - 4(4-2\alpha)|B_2|^2. \end{aligned}$$

Proof. Since $f \in \mathcal{R}_\alpha^u(\phi)$, there exists a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$ such that

$$\frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \phi(\omega(z)). \tag{11}$$

Define the function P_1 by

$$P_1(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots.$$

Since ω is a Schwarz function, we see that $\Re(P_1(z)) \geq 0$ and $P_1(0) = 1$ and therefore $P_1 \in \mathbf{P}$. It follows that

$$\omega(z) = \frac{P_1(z)-1}{P_1(z)+1} = \frac{1}{2}\left[p_1z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)z^3 + \cdots\right]. \tag{12}$$

Then, by a simple computation we get

$$\phi(\omega(z)) = 1 + \frac{B_1p_1}{2}z + \left[\frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2\right]z^2$$

$$+\left[\frac{B_1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{B_2p_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3p_1^3}{8}\right]z^3 + \cdots$$

$$\equiv 1 + b_1z + b_2z^2 + b_3z^3 + \cdots \tag{13}$$

and therefore

$$b_1 = \frac{B_1p_1}{2}, \tag{14}$$

$$b_2 = \frac{B_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2, \tag{15}$$

$$b_3 = \frac{B_1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right) + \frac{B_2p_1}{2}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{B_3p_1^3}{8}. \tag{16}$$

On the other hand, in view of (11) and (13), we have

$$1 + \sum_{n=1}^{\infty} b_nz^n = \frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} = \frac{z + \sum_{n=2}^{\infty} C_2(\alpha,n)a_nz^n}{z + \sum_{n=2}^{\infty} C_1(\alpha,n)a_nz^n}, \tag{17}$$

where

$$\mathcal{C}_1(\alpha,n) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!},\ \mathcal{C}_2(\alpha,n) = \frac{\prod_{k=2}^{n}(k+1-2\alpha)}{(n-1)!}.$$

Equating the coefficients of z, z^2 and z^3 in (17), we obtain

$$b_1 = [\mathcal{C}_2(\alpha,2) - \mathcal{C}_1(\alpha,2)]a_2, \tag{18}$$

$$b_2 = [\mathcal{C}_2(\alpha,3) - \mathcal{C}_1(\alpha,3)]a_3 + [\mathcal{C}_1(\alpha,2)a_2]^2 - [\mathcal{C}_1(\alpha,2)\mathcal{C}_2(\alpha,2)]{a_2}^2, \tag{19}$$

and

$$\begin{aligned} b_3 = {} & [\mathcal{C}_2(\alpha,4) - \mathcal{C}_1(\alpha,4)]a_4 \\ & + [2\mathcal{C}_1(\alpha,2)\mathcal{C}_1(\alpha,3) - \mathcal{C}_2(\alpha,3)\mathcal{C}_1(\alpha,2) - \mathcal{C}_2(\alpha,2)\mathcal{C}_1(\alpha,3)]a_2a_3 \\ & + \mathcal{C}_2(\alpha,2)[\mathcal{C}_1(\alpha,2)a_2]^2a_2 - [\mathcal{C}_1(\alpha,2)a_2]^3. \end{aligned} \tag{20}$$

Simplifying (18), (19) and (20) we have

$$a_2 = b_1, \qquad a_3 = \frac{b_2 + (2-2\alpha){b_1}^2}{(3-2\alpha)}, \tag{21}$$

and

$$a_4 = \frac{2b_3}{(3-2\alpha)(4-2\alpha)} + \frac{3(2-2\alpha)b_1b_2}{(3-2\alpha)(4-2\alpha)} - \frac{(2-2\alpha)^2{b_1}^3}{(3-2\alpha)(4-2\alpha)}. \tag{22}$$

Using the equations (14), (15) and (16) in (21) and (22), it follows that

$$\begin{aligned} a_2 &= \frac{B_1p_1}{2}, \\ a_3 &= \frac{1}{(3-2\alpha)}\left[\frac{B_1}{2}\left(p_2 - \frac{{p_1}^2}{2}\right) + \frac{B_2{p_1}^2}{4} + (2-2\alpha)\frac{{B_1}^2{p_1}^2}{4}\right], \\ a_4 &= \frac{1}{8(3-2\alpha)(4-2\alpha)}\Big[\,8B_1p_3 - 2\left\{4(B_1-B_2) - 3(2-2\alpha){B_1}^2\right\}p_1p_2 \end{aligned}$$

$$+ \left\{2(B_1 - 2B_2 + B_3) - 3(2-2\alpha){B_1}^2 + 3(2-2\alpha)B_1B_2 \right.$$
$$\left. + \ (2-2\alpha)^2{B_1}^3\right\}{p_1}^3 \Big] .$$

Thus we establish that the estimate of the second Hankel determinant is given by

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{\mathcal{D}(\alpha)}\Big[\ & \left\{(2-2\alpha)B_1(B_1-2B_2) - (2-2\alpha)(1-2\alpha){B_1}^2(B_1-B_2)\right. \\ & \left. -(2-2\alpha)^2{B_1}^4 + 2(3-2\alpha)B_1B_3 - (4-2\alpha){B_2}^2\right\}{p_1}^4 \\ & - 2(2-2\alpha)\left\{2B_1(B_1-B_2) - (1-2\alpha){B_1}^3\right\}{p_1}^2p_2 \\ & - 4(4-2\alpha){B_1}^2{p_2}^2 + 8(3-2\alpha){B_1}^2p_1p_3\Big], \end{aligned} \tag{23}$$

where

$$\mathcal{D}(\alpha) = 16(3-2\alpha)^2(4-2\alpha).$$

Using Lemma 1.3 in (23), we have

$$\begin{aligned} |a_2a_4 - a_3^2| = \frac{1}{\mathcal{D}(\alpha)}\Big| \ & \left[2(3-2\alpha)B_1B_3 + (2-2\alpha)(1-2\alpha){B_1}^2B_2\right. \\ & \left. - (2-2\alpha)^2{B_1}^4 - (4-2\alpha){B_2}^2\right]{p_1}^4 \\ & + (2-2\alpha)\left[2B_1B_2 + (1-2\alpha){B_1}^3\right](4-{p_1}^2){p_1}^2x \\ & - \left\{(2-2\alpha)p_1^2 + 4(4-2\alpha)\right\}{B_1}^2(4-{p_1}^2)x^2 \\ & + 4(3-2\alpha){B_1}^2(4-{p_1}^2)p_1(1-|x|^2)z\Big|. \end{aligned} \tag{24}$$

Letting $|p_1| = \xi$ and in view of Lemma 1.2, we may assume without restriction that $\xi \in [0,2]$. Thus, applying the triangle inequality in (24) with $\delta = |x| \le 1$ and $|z| \le 1$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| \le \frac{1}{\mathcal{D}(\alpha)}\Big[\ & \big|-(2-2\alpha)^2{B_1}^4 + 2(3-2\alpha)B_1B_3 - (4-2\alpha){B_2}^2 \\ & +(2-2\alpha)(1-2\alpha){B_1}^2B_2\big|\,\xi^4 \\ & + (2-2\alpha)\left|2B_1B_2 + (1-2\alpha){B_1}^3\right|(4-\xi^2)\xi^2\delta \\ & + \{(2-2\alpha)\xi^2 + 4(4-2\alpha)\}{B_1}^2(4-\xi^2)\delta^2 \\ & + 4(3-2\alpha){B_1}^2\xi(4-\xi^2)(1-\delta^2)\Big] = \mathcal{F}(\xi,\delta). \end{aligned}$$

Or equivalently

$$\begin{aligned} |a_2a_4 - a_3^2| \le \frac{B_1}{\mathcal{D}(\alpha)}\Big[\Big| \ & - (2-2\alpha)^2{B_1}^3 + 2(3-2\alpha)B_3 - (4-2\alpha)\frac{{B_2}^2}{B_1} \\ & + (2-2\alpha)(1-2\alpha)B_1B_2\Big|\xi^4 \\ & + (2-2\alpha)\left|2B_2 + (1-2\alpha){B_1}^2\right|(4-\xi^2)\xi^2\delta \\ & + \{(2-2\alpha)\xi^2 + 4(4-2\alpha)\}B_1(4-\xi^2)\delta^2 \end{aligned}$$

$$+ 4(3-2\alpha)B_1\xi(4-\xi^2)(1-\delta^2)\Big] = \mathcal{F}(\xi,\delta).$$

Note that for $(\xi,\delta) \in [0,2) \times [0,1]$, differentiating $\mathcal{F}(\xi,\delta)$, partially with respect to δ yields

$$\frac{\partial \mathcal{F}}{\partial \delta} = \frac{B_1}{\mathcal{D}(\alpha)}\Big[\,(2-2\alpha)\left|2B_2+(1-2\alpha){B_1}^2\right|(4-\xi^2)\xi^2$$

$$+2\left\{(2-2\alpha)\xi^2-4(3-2\alpha)\xi+4(4-2\alpha)\right\}B_1(4-\xi^2)\delta\,\Big]. \tag{25}$$

Equivalently

$$\frac{\partial \mathcal{F}}{\partial \delta} = \frac{B_1}{\mathcal{D}(\alpha)}\Big[\,(2-2\alpha)\left|2B_2+(1-2\alpha){B_1}^2\right|(4-\xi^2)\xi^2$$

$$+2\left\{2(2-\xi)[2+(1-\alpha)(2-\xi)]\right\}B_1(4-\xi^2)\delta\,\Big].$$

It is obvious that $2(2-\xi)[2+(1-\alpha)(2-\xi)]$, the coefficient term of δ in (25) is always a positive real number for all $(\xi,\delta) \in [0,2) \times [0,1]$. Hence it follows that the expression (25) is always positive for $\delta > 0$ and $\alpha \le 1$, which implies that $\mathcal{F}(\xi,\delta)$ is an increasing function of δ. Therefore, there exists no point of maximum in the interior of the closed region $[0,2) \times [0,1]$. Moreover for fixed $\xi \in [0,2)$, we have

$$\max \mathcal{F}(\xi,\delta) = \mathcal{F}(\xi,1) = \mathcal{G}(\xi).$$

On simplification we find that

$$\mathcal{F}(\xi,1) = \mathcal{G}(\xi) = \frac{B_1}{\mathcal{D}(\alpha)}[Pt^2+Qt+R], \tag{26}$$

where

$$P = \Bigg|-(2-2\alpha)^2{B_1}^3+2(3-2\alpha)B_3-(4-2\alpha)\frac{{B_2}^2}{B_1}$$

$$+(2-2\alpha)(1-2\alpha)B_1B2|$$

$$-(2-2\alpha)|2B_2+(1-2\alpha){B_1}^2\Big|-(2-2\alpha)B_1, \tag{27}$$

$$Q = 4(2-2\alpha)|2B_2+(1-2\alpha){B_1}^2|-8B_1, \tag{28}$$

$$R = 16(4-2\alpha)B_1, \tag{29}$$

and $t=\xi^2$. Since

$$\max_{0\le t\le 4}(Pt^2+Qt+R) = \begin{cases} R, & Q \le 0,\ P \le -\frac{Q}{4}, \\ 16P+4Q+R, & Q \ge 0,\ P \ge -\frac{Q}{8} \text{ or } Q \le 0,\ P \ge -\frac{Q}{4}, \\ \frac{4PR-Q^2}{4P}, & Q > 0,\ P \le -\frac{Q}{8}, \end{cases}$$

we have

$$|a_2a_4 - a_3^2| \le \begin{cases} R, & Q \le 0,\ P \le -\frac{Q}{4}, \\ 16P + 4Q + R, & Q \ge 0,\ P \ge -\frac{Q}{8} \text{ or } Q \le 0,\ P \ge -\frac{Q}{4}, \\ \frac{4PR-Q^2}{4P}, & Q > 0,\ P \le -\frac{Q}{8}, \end{cases}$$

where P, Q and R are given in (27), (28) and (29). This completes the proof of the theorem. □

Remark 2.2. We note that by taking $\alpha = 1/2$ in Theorem 2.1 we obtain the corresponding results in [5].

2.1. Concluding remarks

As a special case of Theorem 2.1, let ϕ be

$$\phi(z) = \frac{1 + Az}{1 + Bz}, \quad (-1 \le B < A \le 1).$$

This gives

$$\phi(z) = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \cdots,$$

so that $B_1 = (A - B)$, $B_2 = -B(A - B)$ and $B_3 = B^2(A - B)$ one can state the Hankel determinant inequality for the subclasses defined in Example 1.1.

Letting $A = 1 - 2\beta$ and $B = -1$ in (2.1), we have

$$\begin{aligned} \phi(z) &= \frac{1 + (1 - 2\beta)z}{1 - z} \\ &= 1 + 2(1 - \beta)z + 2(1 - \beta)z^2 + 2(1 - \beta)z^3 + \cdots \quad (0 \le \beta < 1). \end{aligned} \tag{30}$$

Comparing with (9) we have $B_1 = B_2 = B_3 = 2(1 - \beta)$. Thus Theorem 2.1 yields the Hankel inequality for $f \in \mathcal{R}^u_\alpha(\beta)$.

Further, by taking $\beta = 0$, in (30), we let

$$\phi(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + 2z^3 + \cdots.$$

Thus by comparing with (9) we note that, $B_1 = B_2 = B_3 = 2$ and making use of Theorem 2.1 one can easily state the Hankel inequality for $f \in \mathcal{R}^u_\alpha(\frac{1+z}{1-z})$.

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