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ADDITIVE ρ-FUNCTIONAL EQUATIONS IN NON-ARCHIMEDEAN BANACH SPACE

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Abstract. In this paper, we solve the additive ρ-functional equations

(0.1) $f(x+y)+f(x-y)-2f(x)=\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right),$

where ρ is a fixed non-Archimedean number with $|\rho|<1$, and

(0.2) $2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)=\rho(f(x+y)+f(x-y)-2f(x)),$

where ρ is a fixed non-Archimedean number with $|\rho|<|2|$.

Furthermore, we prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

1. Introduction and Preliminaries

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs|=|r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s|\le|r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s|\le\max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1|=|-1|=1$ and $|n|\le1$ for all $n\in\mathbb{N}$. A trivial

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example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1 ([7])**.** Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space.*

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [10] concerning the stability of group homomorphisms. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f(x + y) + f(x - y) = 2f(x)$ is called the *Jensen type additive equation.*

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [9] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [4] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 3]).

In this paper, we solve the additive ρ-functional equations (0.1) and (0.2) and prove the Hyers-Ulam stability of the additive ρ-functional equations (0.1) and (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. Additive ρ-functional Equation (0.1) in Non-Archimedean Normed Spaces

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

In this section, we solve the additive ρ-functional equation (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *If a mapping $f: X \to Y$ satisfies $f(0)=0$ and*

$$f(x+y)+f(x-y)-2f(x)=\rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right) \tag{2.1}$$

for all $x, y \in X$, then $f: X \to Y$ is additive.

Proof. Assume that $f: X \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $f(2x)-2f(x)=0$ and so $f(2x)=2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x) \tag{2.2}$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} f(x+y)+f(x-y)-2f(x) &= \rho\left(2f\left(\frac{x+y}{2}\right)+f(x-y)-2f(x)\right) \\ &= \rho(f(x+y)+f(x-y)-2f(x)) \end{aligned}$$

and so $f(x+y)+f(x-y)=2f(x)$ for all $x,y\in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $r<1$ and θ be nonnegative real numbers and let $f:X\to Y$ be a mapping satisfying $f(0)=0$ and*

$$\begin{aligned}&\left\|f(x+y)+f(x-y)-2f(x)-\rho\left(2f\left(\frac{x+y}{2}\right)+f\left(x-y\right)-2f(x)\right)\right\| \\ &\leq \theta(\|x\|^r+\|y\|^r) \qquad (2.3)\end{aligned}$$

for all $x,y\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(x)-A(x)\|\leq \frac{2\theta}{|2|^r}\|x\|^r \qquad (2.4)$$

for all $x\in X$.

Proof. Letting $y=x$ in (2.3), we get

$$\|f(2x)-2f(x)\|\leq 2\theta\|x\|^r \qquad (2.5)$$

for all $x\in X$. So $\left\|f(x)-2f\left(\frac{x}{2}\right)\right\|\leq \frac{2}{|2|^r}\theta\|x\|^r$ for all $x\in X$. Hence

$$\begin{aligned}(2.6)\quad &\left\|2^l f\left(\frac{x}{2^l}\right)-2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right)-2^{l+1}f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,\left\|2^{m-1}f\left(\frac{x}{2^{m-1}}\right)-2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &=\max\left\{|2|^l\left\|f\left(\frac{x}{2^l}\right)-2f\left(\frac{x}{2^{l+1}}\right)\right\|,\cdots,|2|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right)-2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{\frac{|2|^l}{|2|^{rl+r}},\cdots,\frac{|2|^{m-1}}{|2|^{r(m-1)+r}}\right\}2\theta\|x\|^r=\frac{2\theta}{|2|^{(r-1)l+r}}\|x\|^r\end{aligned}$$

for all nonnegative integers m and l with $m>l$ and all $x\in X$. It follows from (2.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x\in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A:X\to Y$ by

$$A(x):=\lim_{n\to\infty}2^n f(\frac{x}{2^n})$$

for all $x\in X$. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$\begin{aligned}
&\left\|A(x+y)+A(x-y)-2A(x)-\rho\left(2A\left(\frac{x+y}{2}\right)+A\left(x-y\right)-2A(x)\right)\right\| \\
&= \lim_{n\to\infty}|2|^n\left\|f\left(\frac{x+y}{2^n}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right. \\
&\qquad\left.-\rho\left(2f\left(\frac{x+y}{2^{n+1}}\right)+f\left(\frac{x-y}{2^n}\right)-2f\left(\frac{x}{2^n}\right)\right)\right\| \\
&\le \lim_{n\to\infty}\frac{|2|^n\theta}{|2|^{nr}}(\|x\|^r+\|y\|^r)=0
\end{aligned}$$

for all $x, y \in X$. So

$$A(x+y)+A(x-y)-2A(x)=\rho\left(2A\left(\frac{x+y}{2}\right)+A\left(x-y\right)-2A(x)\right)$$

for all $x, y \in X$. By Lemma 2.1, the mapping $A : X \to Y$ is additive .

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have

$$\begin{aligned}
\|A(x)-T(x)\| &= \left\|2^qA\left(\frac{x}{2^q}\right)-2^qT\left(\frac{x}{2^q}\right)\right\| \\
&\le \max\left\{\left\|2^qA\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|,\left\|2^qT\left(\frac{x}{2^q}\right)-2^qf\left(\frac{x}{2^q}\right)\right\|\right\} \\
&\le \frac{2\theta}{|2|^{(r-1)q+r}}\|x\|^r,
\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $A : X \to Y$ is a unique additive mapping satisfying (2.4). □

Theorem 2.3. *Let $r > 1$ and θ be nonnegative real numbers and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(x)-A(x)\| \le \frac{2\theta}{|2|}\|x\|^r$$

for all $x \in X$.

Proof. It follows from (2.5) that

$$\left\|f(x)-\frac{1}{2}f(2x)\right\| \le \frac{2}{|2|}\theta\|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\| \\
&\quad \le \max\left\{ \left\| \frac{1}{2^l} f\left(2^l x\right) - \frac{1}{2^{l+1}} f\left(2^{l+1} x\right) \right\|, \cdots, \left\| \frac{1}{2^{m-1}} f\left(2^{m-1} x\right) - \frac{1}{2^m} f\left(2^m x\right) \right\| \right\} \\
&\quad = \max\left\{ \frac{1}{|2|^l} \left\| f\left(2^l x\right) - \frac{1}{2} f\left(2^{l+1} x\right) \right\|, \cdots, \frac{1}{|2|^{m-1}} \left\| f\left(2^{m-1} x\right) - \frac{1}{2} f\left(2^m x\right) \right\| \right\} \\
&\quad \le \max\left\{ \frac{|2|^{lr}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(1-r)l+1}} \|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

3. ADDITIVE ρ-FUNCTIONAL EQUATION (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |2|$.

In this section, we solve the additive ρ-functional equation (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *If a mapping $f : X \to Y$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) = \rho(f(x+y) + f(x-y) - 2f(x)) \tag{3.1}$$

for all $x, y \in X$, then $f : X \to Y$ is additive.

Proof. Assume that $f : X \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$2f\left(\frac{x}{2}\right) = f(x) \tag{3.2}$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned}
f(x+y) + f(x-y) - 2f(x) &= 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \\
&= \rho(f(x+y) + f(x-y) - 2f(x))
\end{aligned}$$

and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in X$. It is easy to show that f is additive. □

We prove the Hyers-Ulam stability of the additive ρ-functional equation (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) - \rho(f(x+y) + f(x-y) - 2f(x))\right\| \leq \theta(\|x\|^r + \|y\|^r) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \theta\|x\|^r \tag{3.4}$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.3), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \tag{3.5}$$

for all $x \in X$. So

$$\begin{aligned} &\left\|2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\| \\ &\leq \max\left\{\left\|2^l f\left(\frac{x}{2^l}\right) - 2^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|2^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &= \max\left\{|2|^l \left\|f\left(\frac{x}{2^l}\right) - 2f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |2|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\leq \max\left\{\frac{|2|^l}{|2|^{rl}}, \cdots, \frac{|2|^{m-1}}{|2|^{r(m-1)}}\right\}\theta\|x\|^r = \frac{\theta}{|2|^{(r-1)l}}\|x\|^r \end{aligned} \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. □

Theorem 3.3. *Let $r > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.3). *Then there exists a unique additive mapping $A : X \to Y$*

such that

$$\|f(x) - A(x)\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r \tag{3.7}$$

for all $x \in X$.

Proof. It follows from (3.5) that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{|2|^r\theta}{|2|}\|x\|^r$$

for all $x \in X$. Hence

(3.8)

$$\begin{aligned}
&\left\|\frac{1}{2^l}f(2^lx) - \frac{1}{2^m}f(2^mx)\right\| \\
&\quad \leq \max\left\{\left\|\frac{1}{2^l}f\left(2^lx\right) - \frac{1}{2^{l+1}}f\left(2^{l+1}x\right)\right\|, \cdots, \left\|\frac{1}{2^{m-1}}f\left(2^{m-1}x\right) - \frac{1}{2^m}f\left(2^mx\right)\right\|\right\} \\
&\quad = \max\left\{\frac{1}{|2|^l}\left\|f\left(2^lx\right) - \frac{1}{2}f\left(2^{l+1}x\right)\right\|, \cdots, \frac{1}{|2|^{m-1}}\left\|f\left(2^{m-1}x\right) - \frac{1}{2}f\left(2^mx\right)\right\|\right\} \\
&\quad \leq \max\left\{\frac{|2|^{rl}}{|2|^{l+1}}, \cdots, \frac{|2|^{r(m-1)}}{|2|^{(m-1)+1}}\right\}|2|^r\theta\|x\|^r = \frac{|2|^r\theta}{|2|^{(1-r)l+1}}\|x\|^r
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{2^n}f(2^nx)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^nx)\}$ converges. So one can define the mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty}\frac{1}{n}f(2^nx)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proofs of Theorems 2.2 and 3.2. □

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