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THE MODULUS MULTIPLICATION TRANSFORM OF BOUNDED LINEAR OPERATORS

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Abstract. In this paper, we study which transform preserves the k-hyponormality of weighted shifts. For this, we introduce a new transform, the modulus multiplication transform, and then examine various properties of it.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and T be a bounded linear operator defined on $\mathcal{H}$ whose polar decomposition is $T = U|T|$. The *Aluthge transform* of T is the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform was first studied in [1] and has received much attention in recent years. One reason the Aluthge transform is interesting is in relation to the invariant subspace problem. We recall that the *Duggal transform* $\widetilde{T}^D = |T|U$ of T, which is first referred in [9]. Clearly, the spectrum of $\widetilde{T}$ $\left(\text{resp. } \widetilde{T}^D\right)$ equals that of T. For $\alpha \equiv \{\alpha_k\}_{k=0}^{\infty}$ a bounded sequence of positive real numbers (called *weights*), let $W_\alpha \equiv shift(\alpha_0, \alpha_1, \cdots) : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ be the associated *unilateral weighted shift*, defined by $W_\alpha e_k := \alpha_k e_{k+1}$ (all $k \geq 0$), where $\{e_k\}_{k=0}^{\infty}$ is the canonical orthonormal basis in $\ell^2(\mathbb{Z}_+)$. The *moments* of W_α are given as

$$\gamma_n \equiv \gamma_n(W_\alpha) := \begin{cases} 1, & \text{if } n = 0 \\ \alpha_0^2 \cdot \ldots \cdot \alpha_{n-1}^2, & \text{if } n > 0 \end{cases} . \tag{1.1}$$

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For a shift W_α, we let $\widetilde{W}_\alpha$ be the Aluthge transform of W_α. Then we can see that $\widetilde{W}_\alpha = shift(\sqrt{\alpha_0\alpha_1}, \sqrt{\alpha_1\alpha_2}, \cdots) =: shift(\widetilde{\alpha}_0, \widetilde{\alpha}_1, \cdots)$ (called the shift of *the geometric mean* of a sequence). In [11] we study some properties of the mean transform $\widehat{T} := \frac{1}{2}(U|T| + |T|U) = \frac{1}{2}\left(U|T| + \widetilde{T}^D\right)$. Let $\widehat{W}_\alpha$ be the Mean transform of W_α. Then we have that $\widehat{W}_\alpha = shift\left(\frac{\alpha_0+\alpha_1}{2}, \frac{\alpha_1+\alpha_2}{2}, \cdots\right) =: shift(\widehat{\alpha}_0, \widehat{\alpha}_1, \cdots)$ (called the shift of *the arithmetic mean* of a sequence). Thus, based on the arithmetic and geometric means of sequences just given above, it is natural to consider hamonic and quadratic means of sequences. For a weighted shift W_α, we let $\widetilde{W}_\alpha^H := shift\left(\frac{2\alpha_0\alpha_1}{\alpha_0+\alpha_1}, \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2}, \cdots\right)$ be the *hamonic mean transform* of W_α and $\widetilde{W}_\alpha^Q := shift\left(\sqrt{\frac{\alpha_0^2+\alpha_1^2}{2}}, \sqrt{\frac{\alpha_1^2+\alpha_2^2}{2}}, \cdots\right)$ be the *quadratic mean transform* of W_α, respectively. We call the arithmetic, geometric and hamonic means *Pythagorean* means.

We say that $T \in \mathcal{B}(\mathcal{H})$ is *normal* if $T^*T = TT^*$, *subnormal* if $T = N|_\mathcal{H}$, where N is normal and $N(\mathcal{H}) \subseteq \mathcal{H}$, and *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for some $p \in (0,\infty)$. If $p = 1$, T is called *hyponormal* and if $p = \frac{1}{2}$, T is called *semi-hyponormal.* It is well known that q-hyponormal operators are p-hyponormal operators for $p < q$ ([1]). It is called that $T \in \mathcal{B}(\mathcal{H})$ is *quasinormal* if T commutes with T^*T. It is well known that normal $\Longrightarrow$ quasinormal $\Longrightarrow$ subnormal $\Longrightarrow$ hyponormal.

For $k \geq 1$, $T \in \mathcal{B}(\mathcal{H})$ is called *k-hyponormal* if

$$\begin{pmatrix} I & T^* & {T^*}^2 & \cdots & {T^*}^k \\ T & T^*T & {T^*}^2T & \cdots & {T^*}^kT \\ T^2 & T^*T^2 & {T^*}^2T^2 & \cdots & {T^*}^kT^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ T^k & {T^*}^2T^k & {T^*}^2T^k & \cdots & {T^*}^kT^k \end{pmatrix}_{(k+1)\times(k+1)} \geq 0.$$

The Bram-Halmos characterization of subnormality ([3, III.1.9]) can be paraphrased as follow: T is subnormal if and only if T is k-hyponormal for every $k \geq 1$ ([4, Proposition 1.9]).

In this paper, we study which transform preserves the k-hyponormality of weighted shifts. For this, we recall: for any $s, t \geq 0$, let $T(s,t) := |T|^sU|T|^t$ [14], then the Aluthge transform $\widetilde{T}$ of T is $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = T\left(\frac{1}{2}, \frac{1}{2}\right)$. Now we define a new transform (called *the modulus multiplication transform*): if $T = U|T|$ is the polar decomposition of T, then we define

$$\widetilde{T}^M := T\,(1,1) = |T|U|T|$$

and then examine various properties of it. We first recall:

Lemma 1.1. (cf. [14]) *Let T be p-hyponormal for some $p > 0$. Then for any $s, t \geq 0$ such that $\max(s,t) \leq p$, we have*

$$T\,(s,t)\,T\,(s,t)^* \leq |T|^{2(s+t)} \leq T\,(s,t)^*\,T\,(s,t)$$

and for $p < \max(s,t)$, we have

$$\{T\,(s,t)\,T\,(s,t)^*\}^{\frac{p+\min(s,t)}{s+t}} \leq |T|^{2\{p+\min(s,t)\}} \leq \{T\,(s,t)^*\,T\,(s,t)\}^{\frac{p+\min(s,t)}{s+t}}.$$

Then, we have:

Theorem 1.2. *Let $T = U|T|$ be hyponormal. Then the modulus multiplication transform $\widetilde{T}^M$ of T is hyponormal.*

Proof. Since T is hyponormal, by Lemma 1.1, for $p, s, t = 1$, we have that

$$\begin{aligned} & T\,(1,1)\,T\,(1,1)^* \leq |T|^4 \leq T\,(1,1)^*\,T\,(1,1) \\ & \Longleftrightarrow |T|U^*|T|^2U|T| \leq |T|^4 \leq |T|U^*|T|^2U|T| \end{aligned} \tag{1.2}$$

Thus, by (1.2), we can see that

$$\left(\widetilde{T}^M\right)^*\widetilde{T}^M = |T|U^*|T|^2U|T| \geq |T|U^*|T|^2U|T| = \widetilde{T}^M\left(\widetilde{T}^M\right)^*,$$

so, the modulus multiplication transform $\widetilde{T}^M$ is hyponormal, as desired. This completes the proof. □

For the polar decomposition $T = U|T|$ of $T \in \mathcal{B}(\mathcal{H})$, we can easily check that $U|T| = |T|U$ if and only if T is quasinormal. If instead $U^2|T| = |T|U^2$, then T will be said to be *in the δ-class*, denoted by $T \in \delta(\mathcal{H})$. We now have:

Theorem 1.3. *Let $T = U|T| \in \delta(\mathcal{H})$ be p-hyponormal for $\frac{1}{2} \leq p < 1$. Then $\widetilde{T}^M$ is hyponormal.*

Proof. Since any p-hyponormal operator is semi-hyponormal, we have that $(T^*T)^{\frac{1}{2}} \geq (TT^*)^{\frac{1}{2}}$, that is, $U^*|T|U \geq U|T|U^*$ which implies

$$(U^*U - UU^*)\,|T^*| \geq 0,$$

because $T \in \delta(\mathcal{H})$. By the functional calculus, we can observe that

$$T \in \delta(\mathcal{H}) \Longrightarrow U|T|^q = |T|^qU \text{ for } q > 0. \tag{1.3}$$

Thus, by (1.3), we have that

$$\begin{aligned}
&\left(|T^*|^{\frac{1}{2}}|T|\right)^* ((U^*U - UU^*)\,|T^*|)\left(|T^*|^{\frac{1}{2}}|T|\right) \geq 0 \\
&\Longrightarrow |T||T^*|^{\frac{1}{2}}\,(U^*U - UU^*)\,|T^*|\left(|T^*|^{\frac{1}{2}}|T|\right) \geq 0 \\
&\Longrightarrow |T|\left(U^*|T|^{\frac{1}{2}}U|T^*|^{\frac{3}{2}} - U|T|^{\frac{1}{2}}U^*|T^*|^{\frac{3}{2}}\right)|T| \geq 0 \\
&\Longrightarrow |T|U^*|T|^2U|T| - |T|U|T|^2U^*|T| \geq 0 \\
&\Longrightarrow \left(\widetilde{T}^M\right)^*\left(\widetilde{T}^M\right) - \left(\widetilde{T}^M\right)\left(\widetilde{T}^M\right)^* \geq 0.
\end{aligned}$$

Therefore, $\widetilde{T}^M$ is hyponormal, as desired. □

For the hyponormality of the modulus multiplication transform for the p-hyponormality of $T = U|T|$ for $\frac{1}{2} \leq p < 1$, we recall the following result.

Lemma 1.4. (cf. [1]) *If A and B are bounded self-adjoint operators such that $A \geq B \geq 0$. Then for each $r \geq 0$,*

$$(B^rA^pB^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$$

and

$$A^{\frac{p+2r}{q}} \geq (A^rB^pA^r)^{\frac{1}{q}}$$

hold for each p and q such that $p \geq 0$, $q \geq 1$, and $\frac{1+2r}{q} \geq p + 2r$.

Theorem 1.5. *Let $T = U|T|$ be p-hyponormal for $0 < p < \frac{1}{2}$. Then $\widetilde{T}^M$ is $\left(\frac{1+p}{2}\right)$-hyponormal.*

Proof. From the p-hyponormality of T, we have that $(T^*T)^p \geq (TT^*)^p$, that is,

$$U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*.$$

Let

$$A := U^*|T|^{2p}U,\ B := |T|^{2p}, \text{ and } C := U|T|^{2p}U^*.$$

By Lemma 1.4, we then have

$$\left(\left(\widetilde{T}^M\right)^*\left(\widetilde{T}^M\right)\right)^{\frac{1+p}{2}} = \left(|T|U^*|T|^2U|T|\right)^{\frac{1+p}{2}} = \left(B^{\frac{1}{2p}}A^{\frac{1}{p}}B^{\frac{1}{2p}}\right)^{\frac{1+p}{2}}$$

$$\geq \left(B^{\left(\frac{1}{p}+\frac{1}{p}\right)}\right)^{\frac{1+p}{2}} = B^{\frac{p+1}{p}} \geq \left(B^{\frac{1}{2p}}C^{\frac{1}{p}}B^{\frac{1}{2p}}\right)^{\frac{1+p}{2}} = \left(|T|U|T|^2U^*|T|\right)^{\frac{1+p}{2}}$$

$$= \left(\left(\widetilde{T}^M\right)\left(\widetilde{T}^M\right)^*\right)^{\frac{1+p}{2}},$$

because

$$\tfrac{2}{p} = \tfrac{2}{p} \Longleftrightarrow \left(1+\tfrac{1}{p}\right)\left(\tfrac{1+p}{2}\right) = \tfrac{2}{p} \Longleftrightarrow \left(1+2\tfrac{1}{2p}\right)\left(\tfrac{1+p}{2}\right) = \tfrac{1}{p}+\tfrac{1}{p}.$$

Therefore, $\widetilde{T}^M$ is $\left(\frac{1+p}{2}\right)$-hyponormal, as desired. □

REMARK 1.6. From Theorem 1.3, we may ask that for $\frac{1}{2} \leq p < 1$, if T is p-hyponormal, does it follow that the modulus multiplication transform $\widetilde{T}^M$ is hyponormal?

Note that Aluthge, mean, hamonic and quadratic transforms of weighted shifts need not preserve the k-hyponormality. In contrast to those transforms, the modulus multiplication transform $\widetilde{W}_\alpha^M$ of W_α preserves the k-hyponormality of W_α. For this, recall that for matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*, i.e., $(A \circ B)_{ij} := A_{ij}B_{ij}$ $(1 \leq i, j \leq n)$. The following result is well known: If $A \geq 0$ and $B \geq 0$, then $A \circ B \geq 0$ ([12]). For matrices $A, B \in M_n(\mathbb{C})$, we let $A \circ B$ denote their *Schur product*. For $\alpha \equiv \{\alpha_n\}_{n=0}^\infty$ and $\beta \equiv \{\beta_n\}_{n=0}^\infty$, the Schur product of α and β is defined by $\alpha \circ \beta := \{\alpha_n\beta_n\}_{n=0}^\infty$. Thus, for given two 1-variable subnormal weighted shifts W_α and W_β, their Schur product $W_\alpha \circ W_\beta$, which we denote by $W_{\alpha\beta}$, is subnormal. That is, if W_α and W_β are k-hyponormal $(k \geq 1)$ 1-variable weighted shifts, then the Schur product

(1.4) $W_{\alpha\beta} \equiv W_\alpha \circ W_\beta$ is a k-hyponormal 1-variable weighted shift [5].

Now we have:

THEOREM 1.7. *Let W_α is k-hyponormal for $k \geq 1$. Then the modulus multiplication transform $\widetilde{W}_\alpha^M$ of W_α is also k-hyponormal.*

Proof. Note that the polar decomposition of W_α is U_+D_α, where $D_\alpha := diag(\alpha_0, \alpha_1, \cdots)$. Hence, we have that $\widetilde{W}_\alpha^M = D_\alpha U_+ D_\alpha$. For

$n \geq 0$ and the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\ell^2(\mathbb{Z}_+)$, we can see that

$$D_\alpha U_+ D_\alpha (e_n) = \alpha_n D_\alpha U_+ (e_n) = \alpha_n D_\alpha (e_{n+1}) = \alpha_n \alpha_{n+1} e_{n+1}$$

Therefore, we get that

$$\widetilde{W}_\alpha^M (e_n) = D_\alpha U_+ D_\alpha (e_n) = (\alpha_n \alpha_{n+1}) e_{n+1},$$

that is,

$$\widetilde{W}_\alpha^M = shift(\alpha_0\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_3, \cdots).$$

Assume that W_α is k-hyponormal. Let $\mathcal{L}_n := \bigvee\{e_h : h \geq n\}$ denote the invariant subspace obtained by removing the first n vectors in the canonical orthonormal basis of $\ell^2(\mathbb{Z}_+)$. For $n \geq 0$, we also let $shift(\alpha_0, \alpha_1, \alpha_2, \cdots)|_{\mathcal{L}_n} := shift(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \cdots)$. Then $W_\alpha|_{\mathcal{L}_1}$ is also k-hyponormal. Thus by (1.4), $\widetilde{W}_\alpha^M$ is k-hyponormal, as desired. $\square$

By the Bram-Halmos criterion for subnormality and Theorem 1.7, we have:

Corollary 1.8. *If W_α is subnormal, then $\widetilde{W}_\alpha^M$ is also subnormal.*

From ([7], [11]), we recall that the Aluthge transform map $T \to \widetilde{T}$ is $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$ and the Duggal transform map $T \to \widetilde{T}^D$ and the mean transform map $T \to \widehat{T}$ are both $(\|\cdot\|, SOT)-$ continuous on $\mathcal{B}(\mathcal{H})$, respectively. Similarly, we have the following.

Theorem 1.9. *The modulus multiplication transform map $T \to \widetilde{T}^M$ is $(\|\cdot\|, \|\cdot\|)-$ continuous on $\mathcal{B}(\mathcal{H})$.*

Proof. Let T_0 be arbitrary in $\mathcal{B}(\mathcal{H})$ and suppose that a sequence $\{T_n = U_n|T_n|\}$ converges in norm to $T_0 = U_0|T_0|$. Since the mappings $T \to T^*$ and $(S, T) \to ST$ are norm continuous, it follows that

$$\| |T_n| - |T_0| \| \to 0. \tag{1.5}$$

By (1.5), we can observe that

$$\begin{aligned}
\left\| \widetilde{T}_n^M - \widetilde{T}_0^M \right\| &= \| |T_n|U_n|T_n| - |T_0|U_0|T_0| \| \\
&\leq \| |T_n|U_n|T_n| - |T_0|U_n|T_n| \| + \| |T_0|U_n|T_n| - |T_0|U_0|T_0| \| \\
&\leq \| |T_n| - |T_0| \| \, \|U_n|T_n|\| + \| |T_0| \| \, \|U_n|T_n| - U_0|T_0|\| \to 0
\end{aligned}$$

Thus, we have that $\left\{\widetilde{T}_n^M\right\}$ converges in norm to $\widetilde{T}_0^M$, as desired. $\square$

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