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ON CARLESON'S INEQUALITY II

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Abstract. We present a new and simple proof of improved Carleson's inequality.

1. Introduction

The classical Hardy's inequality reads:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \;\le\; \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

where f is a nonnegative measurable function on $(0,\infty)$ and $p>1$. A weighted modification of (1.1) was proved also by Hardy [4] as

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \;\le\; \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha\,dx, \tag{1.2}$$

where f is a nonnegative measurable function on $(0,\infty)$, $p>1$, and $\alpha<p-1$. The constant

$$\left(\frac{p}{p-1-\alpha}\right)^p$$

is the best possible.

The importance and the usefulness of these inequalities could never have been overestimated. As a limit versions of Hardy's inequality, the following inequality appeared in the literature. See [5] and [9].

Theorem A (Polya-Knopp inequality).

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\} dx \le e\int_0^\infty f(x)dx \tag{1.3}$$

for any measurable $f\ge 0$.

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The discrete version of inequality (1.3) is the famous Carleman's inequality [2]. Being a limiting form of Hardy's inequality (1.1) and a continuous form of Carleman's inequality, (1.3) has attracted a lot of attention and its extensions have been investigated frequently. See for example [3] and the references therein. Among extensions of (1.3) is the following result of L. Carleson.

THEOREM B (Carleson's inequality). Let $F(x)$ be a convex function for $x \geq 0$, satisfying $F(0) = 0$. If $-1 < \alpha < \infty$, the following inequality holds true:

$$\int_0^\infty x^\alpha e^{-F(x)/x} dx \leq e^{\alpha+1} \int_0^\infty x^\alpha e^{-F'(x)} dx. \tag{1.4}$$

See [1]. Theorem B gives, for any measurable $f \geq 0$, that

$$\int_0^\infty x^\alpha \exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\} dx \leq e^{\alpha+1} \int_0^\infty f(x)\; x^\alpha dx, \tag{1.5}$$

whence Theorem B extends Theorem A. See [7]. So, (1.3) is a special case of (1.5), and (1.5) can be regarded as a special case of (1.4) as well as a limiting case of (1.2).

The goal of this note is to present a new and simple proof of the following improved version of Carleson's inequality.

THEOREM 1.1. *Let $0 < b \leq \infty$. Let $F(x)$ be a real valued function, differentiable almost everywhere on $(0, b)$, continuous on $[0, b)$ with $F(0) = 0$. If $-1 < \alpha < \infty$, then the following inequality holds true:*

$$\int_0^b x^\alpha e^{F(x)/x} dx \leq e^{\alpha+1} \int_0^b \left(1 - \frac{x}{b}\right) x^\alpha e^{F'(x)} dx. \tag{1.6}$$

As was shown in [1] and [9], the proofs of Carleson's inequality and its L^p version inequality there depended heavily on the convexity of F. The most striking fact of Theorem 1.1 may be that the convexity assumption on F in Theorem B is removed. Note that the convexity of F implies the continuity of F and the existence of F' almost everywhere. Instead, our proof depends on the following inequality, called the continuous form of Hölder's inequality (see [6, 8]).

THEOREM C (Continuous form of Hölder's inequality). Let $X = (X, \mu)$ be a probability measure space and $Y = (Y, \nu)$ be a positive σ-finite measure space. Let $f(x, y)$ be a positive measurable function on

$X \times Y$. Then

$$\begin{aligned} &\int_Y \exp\left(\int_X \ln f(x,y)\ d\mu(x)\right) d\nu(y) \\ &\qquad \le \exp\left\{\int_X \ln\left(\int_Y f(x,y)\ d\nu(y)\right) d\mu(x)\right\}. \end{aligned} \tag{1.7}$$

Equality holds in (1.7) as a nonzero finite value if and if only if $f(x,y) = g(x)h(y)$ almost everywhere $\mu \times \nu$ for a positive μ-measurable function g with $-\infty < \int_X \ln g\ d\mu < \infty$ and a positive ν-measurable h with $\int_Y h d\nu = 1$.

2. Proof of Theorem 1.1

Note that

$$\frac{F(x)}{x} = \int_0^1 F'(xy)\ dy, \quad x \in (0,b). \tag{2.1}$$

For simplicity, let $G = \exp F'$. Then by (2.1)

$$\begin{aligned} \int_0^b x^\alpha \exp\left\{\frac{F(x)}{x}\right\} dx &= \int_0^b x^\alpha \exp\left\{\int_0^1 F'(xy)\ dy\right\} dx \\ &= \int_0^b x^\alpha \exp\left\{\int_0^1 \ln G(xy)\ dy\right\} dx. \end{aligned} \tag{2.2}$$

Applying (1.7) we obtain

$$\int_0^b x^\alpha \exp\left\{\int_0^1 \ln G(xy)\ dy\right\} dx \le \exp\left[\int_0^1 \ln\left(\int_0^b x^\alpha G(xy)dx\right) dy\right]. \tag{2.3}$$

On the other hand, a straightforward calculation gives

$$\begin{aligned} &\exp\left[\int_0^1 \ln\left(\int_0^b x^\alpha G(xy)dx\right) dy\right] \\ &= \exp\left[\int_0^1 \ln\left(\frac{1}{y^{\alpha+1}}\int_0^{by} x^\alpha G(x)dx\right) dy\right] \\ &= e^{\alpha+1}\cdot \exp\left[\int_0^1 \ln\left(\int_0^{by} x^\alpha G(x)dx\right) dy\right]. \end{aligned} \tag{2.4}$$

By the convexity of the exponential function and by changing the order of the integration,

$$
\begin{aligned}
&\exp\left[\int_0^1 \ln\left(\int_0^{by} x^\alpha G(x)dx\right)dy\right] \\
&\le \int_0^1 \left(\int_0^{by} x^\alpha G(x)dx\right)dy \\
&= \int_0^b x^\alpha G(x)\int_{x/b}^1 dy\ dx \\
&= \int_0^b \left(1-\frac{x}{b}\right) x^\alpha G(x)dx.
\end{aligned} \tag{2.5}
$$

Inequality (1.6) now follow from (2.1)∼(2.5). The proof is complete.

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