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ON THE MINUS PARTS OF CLASSICAL POINCARÉ SERIES

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Abstract. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$. We show that $S_k(N)$ is the direct sum of subspaces $S_k^+(N)$ and $S_k^-(N)$. Where $S_k^+(N)$ is the vector space of cusp forms of weight k for the group $\Gamma_0^+(N)$ generated by $\Gamma_0(N)$ and W_N and $S_k^-(N)$ is the subspace consisting of elements f in $S_k(N)$ satisfying $f|_k W_N = -f$. We find generators spanning the space $S_k^-(N)$ from Poincaré series and give all linear relations among such generators.

1. Introduction and statement of results

Let N be a positive integer and k be a positive even integer greater that 2. Let $S_k(N)$ be the space of cusp forms of weight k for $\Gamma_0(N)$ and $S_k^+(N)$ be the subspace of $S_k(N)$ consisting of cusp forms f with $f|_k W_N = f$, i.e.,

$$S_k^+(N) := \{f \in S_k(N)|\ f|_k W_N = f\},$$

where $W_N = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}$ is the Fricke involution. We define the subspace of $S_k(N)$ as:

$$S_k^-(N) := \{f \in S_k(N)|\ f|_k W_N = -f\}.$$

For each $f \in S_k(N)$, we see $f = \frac{f+f|_k W_N}{2} + \frac{f-f|_k W_N}{2}$. We notice that $\frac{f+f|_k W_N}{2} \in S_k^+(N)$ and $\frac{f-f|_k W_N}{2} \in S_k^-(N)$. We call the cusp form $\frac{f+f|_k W_N}{2}$ the plus part of f and the cusp form $\frac{f-f|_k W_N}{2}$ the minus part of f. We can show that $S_k(N)$ is the direct sum of subspaces $S_k^+(N)$ and $S_k^-(N)$, that is,

$$S_k(N) = S_k^+(N) \bigoplus S_k^-(N).$$

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Here, in fact $S_k^+(N)$ is the vector space of cusp forms of weight k for the group $\Gamma_0^+(N)$ generated by $\Gamma_0(N)$ and W_N.

For each positive integer m, we define the classical cuspidal Poincaré series $P(m,k,N;z)$ as follows:

$$P(m,k,N;z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \frac{e^{2\pi i m\gamma z}}{(cz+d)^k},$$

where $z \in \mathbb{H}$ and $\mathbb{H}$ is the complex upper half plane, and $\Gamma_\infty = \{\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$. It is well known that the Poincaré series are cusp forms in $S_k(N)$ and the set $\{P(m,k,N;z) \mid m \geq 1\}$ spans the cusp form space $S_k(N)$. In [6], Iwanic raise an open problem that ask to find all linear relations among the Poincaré series. In [7] it is shown that linear relations among the Poincaré series $P(m,k,N;z)$ are given by certain weakly holomorphic modular forms. The Poincaré series $P(m,k,N;z)$ can be splitted into

$$P(m,k,N;z) = P^+(m,k,N;z) + P^-(m,k,N;z),$$

where $P^+(m,k,N;z) := \frac{P(m,k,N;z)+P(m,k,N;z)|_k W_N}{2}$ and $P^-(m,k,N;z) := \frac{P(m,k,N;z)-P(m,k,N;z)|_k W_N}{2}$. Indeed, $P^+(m,k,N;z)$ is just the Poincaré series related to the group $\Gamma_0^+(N)$ as follows:

$$P^+(m,k,N;z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0^+(N)} \frac{e^{2\pi i m\gamma z}}{(cz+d)^k}.$$

In [4] the author and Kim considered the Poincaré series $P^+(m,k,N;z)$ that is the plus part of $P(m,k,N;z)$ and found all linear relations among $P^+(m,k,N;z)$ by giving certain weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0^+(N)$. For given $g \in S_k^-(N)$, we have $g(z) = \sum_m a_m P(m,k,N;z)$ for some $a_m \in \mathbb{C}$. Thus we have

$$g(z) = \sum_m a_m P^-(m,k,N;z)$$

which implies that the set $\{P^-(m,k,N;z) \mid m \geq 1\}$ spans the space $S_k^-(N)$. In this paper, following the argument in [4, 7] we find all linear relations among the series $P^-(m,k,N;z)$. With the notations above we can state the result.

Theorem 1.1. *Let I be a nonempty finite subset of $\mathbb{N}$. Then the following are equivalent.*

(1) $\sum_{m\in I} a_m P^-(m,k,N;z) \equiv 0$ *for some* $a_m \in \mathbb{C}$.

(2) *There exists a weakly holomorphic modular form g of weight $2-k$ for $\Gamma_0(N)$ with the principal part at ∞*

$$\sum_{m\in I} \frac{\overline{a_m}}{m^{k-1}} q^{-m}$$

and zero principal part at all cusps except ∞.

This paper is organized as follows. In Section 2 we give basic properties of Harmonic weak Maass forms. In Section 3 we give the proof of Theorem 1.1.

2. Harmonic weak Maass forms

A smooth function $f : \mathbb{H} \to \mathbb{C}$ is called a harmonic weak Maass form of weight $2-k$ for $\Gamma_0(N)$ if it satisfies:

(1) For all $\gamma \in \Gamma_0(N)$ we have $f|_{2-k}\gamma = f$.
(2) $\triangle_{2-k} f = 0$, where $\triangle_{2-k}$ is the weight $2-k$ hyperbolic Laplace operator.
(3) There is Fourier polynomial $P_f(z) = \sum_{-\infty n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) = P_f(z) + O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. And analogous conditions are required at all cusps. Here $q = e^{2\pi i z}$ as usual.

We denote $H_{2-k}(N)$ the space of harmonic weak Maass forms of weight k for $\Gamma_0(N)$. Let $H_{2-k}^+(N)$ be the subspace of $H_{2-k}(N)$ consisting of elements G satisfying $G|_{2-k}W_N = G$ and $H_{2-k}^-(N)$ be the subspace of $H_{2-k}(N)$ consisting of elements G satisfying $G|_{2-k}W_N = -G$. Then we can show that $H_{2-k}(N)$ is the direct sum of $H_{2-k}^+(N)$ and $H_{2-k}^-(N)$.

Harmonic weak Maass forms are related to cusp forms in terms of differential operator. We define a differential operator

$$\xi_v := 2iy^v \frac{\overline{\partial}}{\partial \overline{z}}.$$

Then we have

$$\xi_{2-k}(G|_{2-k}\gamma) = (\xi_{2-k}G)|_k\gamma \qquad \text{for all } \gamma \in SL_2(\mathbb{R}). \tag{2.1}$$

Proposition 2.1. *The differential operator ξ_{2-k} define a surjective map from $H_{2-k}^-(N)$ to $S^-(N)$ and the kernel of ξ_{2-k} is the space of weakly holomorphic modular forms f satisfying $f|_{2-k}W_N = -f$.*

Proof. Given $g \in S_k^-(N)$, take $G \in H_{2-k}(N)$ such that $\xi_{2-k}G = g$. Then $2g = \xi_{2-k}G - (\xi_{2-k}G)|_k W_N = \xi_{2-k}(G - G|_{2-k}W_N)$ and $G - G|_k W_N \in H_{2-k}^-(N)$. So ξ_{2-k} is surjective. It is well known that $\ker \xi_{2-k}$ is the space of weakly holomorphic modular forms of weight $2-k$ for $\Gamma_0(N)$. Thus $\ker \xi_{2-k} \cap H_{2-k}^-$ is the space of weakly holomorphic modular form G of weight $2-k$ for $\Gamma_0(N)$ satisfying $G|_{2-k}W_N == G$ since G satisfies $G|_{2-k}W_N = -G$ for $G \in H_{2-k}^-(N)$. □

Proposition 2.2. *If $f \in H_{2-k}(N)$ is not weakly holomorphic modular form then the principal part of f is non-constant for at least one cusp.*

Proof. It follows form the work of Bruinier and Funke [1]. □

For $s \in \mathbb{C}$, let

$$\mathcal{M}_{s,\kappa}(y) := |y|^{-\frac{\kappa}{2}} M_{\frac{\kappa}{2}sgn(y), s-\frac{1}{2}}(|y|),$$

where $M_{\mu,\nu}(z)$ is the usual M-Whittaker function. We define

$$Q(-m,k,N;z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_{-m}^*|_{2-k}\gamma)(z),$$

where $\varphi_{-m}^*(z) = \mathcal{M}_{\frac{k}{2},2-k}(-4\pi m y)e^{-2\pi i m x}$ and $z = x+iy \in \mathbb{H}$. We let

$$Q^-(-m,k,N;z) := Q(-m,k,N;z) - Q(-m,k,N;z)|_{2-k}W_N.$$

Proposition 2.3. *With the notations above we have the following:*

(*i*) *$Q^-(-m,k,N;z) \in H_{2-k}^-(N)$ and its principal part at ∞ equals to $\Gamma(k)q^{-m}$ and its principal part at the cusps other than ∞ are constant.*

(*ii*) $\xi_{2-k}(Q^-(-m,k,N;z)) = (4\pi m)^{k-1}(k-1)P^-(-m,k,N;z)$.

Proof. It follows from (2.1) and the results in [2]. □

3. Proof of Theorem 1.1

We assume that

$$\sum_{m \in I} a_m P^-(m,k,N;z) \equiv 0.$$

Define $f \in H_{k-1}^-(N)$ by

$$f = \sum_{m \in I} \frac{a_m}{m^{k-1}} Q^-(-m,k,N;z).$$

Then we have from Proposition 2.3 that $\xi_{2-k} f = 0$. So by Proposition 2.2, Proposition 2.1 and Proposition 2.3 we see that $f/\Gamma(k)$ is weakly holomorphic modular form of weight $2-k$ for $\Gamma_0(N)$ that we desire.

Conversely we assume that f is such a weakly holomorphic modular form. Define $\widetilde{f}$ by

$$\widetilde{f} := -f + \frac{1}{\Gamma(k)} \sum_{m \in I} \frac{a_m}{m^{k-1}} Q^-(-m, k, N; z).$$

Then $\widetilde{f} \in H^-_{2-k}(N)$ has constant principal parts at all cusps. It then follows from Proposition 2.2 that $\xi_{2-k}(\widetilde{f}) = 0$ and hence by Proposition 2.3 we have the desired relation.

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