Commun. Korean Math. Soc. **33** (2018), No. 3, pp. 1025–1037
https://doi.org/10.4134/CKMS.c170258
pISSN: 1225-1763 / eISSN: 2234-3024

STABLE APPROXIMATION OF THE HEAT FLUX IN AN INVERSE HEAT CONDUCTION PROBLEM

Leïla Alem and Lahcène Chorfi

Abstract. We consider an ill-posed problem for the heat equation $u_{xx} = u_t$ in the quarter plane $\{x > 0,\ t > 0\}$. We propose a new method to compute the heat flux $h(t) = u_x(1,t)$ from the boundary temperature $g(t) = u(1,t)$. The operator $g \mapsto h = Hg$ is unbounded in $L^2(\mathbb{R})$, so we approximate $h(t)$ by $h_\delta(t) = u_x(1+\delta, t)$, $\delta \to 0$. When noise is present, the data is g_ϵ leading to a corresponding heat $h_{\delta,\epsilon}$. We obtain an estimate of the error $\|h - h_{\delta,\epsilon}\|$, as well as the error when $h_{\delta,\epsilon}$ is approximated by the trapezoidal rule. With an a priori choice rule $\delta = \delta(\epsilon)$ and $\tau = \tau(\epsilon)$, the step size of the trapezoidal rule, the main theorem gives the error of the heat flux as a function of noise level ϵ. Numerical examples show that the proposed method is effective and stable.

1. Introduction

In this article, we consider the problem of finding a function $u(x,t)$ from the given data $u(1,t) = g(t)$ in the following sideways heat equation

$$(1) \qquad \begin{cases} u_{xx} = u_t, & x > 0, & t > 0, \\ u(1,t) = g(t), & \lim\limits_{x\to+\infty} u(x,t) = 0, & t \geq 0, \\ u(x,0) = 0, & x \geq 0. \end{cases}$$

We decompose this problem into two problems.

Problem 1. We consider the well-posed problem: given g, determine the heat flux $h(t) = u_x(1,t)$, where u is the solution of the heat problem

$$(2) \qquad \begin{cases} u_{xx}(x,t) = u_t(x,t), \quad x > 1, \quad t > 0, \\ u(1,t) = g(t), \quad \lim\limits_{x\to+\infty} u(x,t) = 0, \quad t \geq 0, \\ u(x,0) = 0, \quad x \geq 1. \end{cases}$$

Received June 20, 2017; Revised January 3, 2018; Accepted March 9, 2018.
2010 *Mathematics Subject Classification.* Primary 35K05, 65M32.
Key words and phrases. inverse problems, ill-posed problem, stable approximation, error estimate.

Problem 2. We solve the Cauchy problem

$$
\left\{
\begin{array}{l}
u_{xx}(x,t) = u_t(x,t), \quad 0 < x < 1, \quad t > 0, \\
u(1,t) = g(t), \qquad u_x(1,t) = h(t), \quad t \geq 0, \\
u(x,0) = 0, \quad 0 \leq x \leq 1.
\end{array}
\right. \tag{3}
$$

The aim of this paper consists of two steps. In the first step we compute the heat flow $h(t) = u_x(1,t)$ in a stable way from perturbed data g_ϵ, $\|g - g_\epsilon\| \leq \epsilon$. In the next step, we test our approximation by solving the Cauchy problem (system (3)) with difference scheme.

Problem 2 is severely ill-posed [10]; i.e., the solution (if it exists) does not depend continuously on the given data. This problem has been considered by many authors with different methods. Mollification method has been used by Murio [9] in the regularization of the Cauchy problem (system (3)). In [8], Hào computed the heat flux $h(t) = u_x(1,t)$ by solving Problem 1 in the exterior domain $[1,+\infty[\times[0,T]$ by finite differences schemes. Recently, in [1], the flux has been regularized by the Fourier method (as in [2, 4]). In this paper, we use a new method based on the formula $h(t) = u_x(1,t)$ where $u(x,t)$ is an explicit solution of Problem 1. The mapping $H : g \to h$ is an unbounded linear operator in L^2. To evaluate Hg in a stable way, we approximate h by a method of truncation, more precisely, we set $h_\delta(t) = u_x(1+\delta,t)$, $\delta > 0$, where $u(x,t)$ is the solution of (2). To our knowledge, this representation has not been used yet in the literature. Our aim is to provide an error estimate with respect to the cut off parameter when $\delta \to 0$ and the level noise ϵ in the data $g(t) = u(1,t)$. Also, the error due to the numerical integration is investigated, giving an a priori choice rule $\tau = \tau(\delta)$ of the parameter of discretization τ which leads to the convergence as $\delta \to 0$.

This paper is organized as follows. In Section 2, we obtain the formulation of the heat flux h from the solution of Problem 1 and we propose its approximation h_δ. In Section 3, the method of choosing the regularization parameter $\delta = \delta(\epsilon)$ when ϵ is the level noise and the error estimates are provided. In Section 4, we study the error due to the discretization of the integral by trapezoidal formula, giving an a priori choice rule of the parameter $\tau = \tau(\delta)$. In Section 5, we discretize the Cauchy problem by finite differences schemes. In Section 6, two numerical examples are given, and compared with two methods: the Fourier regularization method and an approximation of the inverse Abel transform.

Notations. 1) Let $\widehat{v}$ denote the Fourier transform of function $v(t) \in L^2(\mathbb{R})$ defined by

$$\widehat{v}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v(t) e^{-i\xi t} dt, \quad \xi \in \mathbb{R},$$

and $\|\cdot\|_s$ denote the norm in Sobolev space $H^s(\mathbb{R})$, $s \geq 0$, defined by

$$\|v\|_s := \left(\int_{-\infty}^{+\infty} (1+\xi^2)^s |\widehat{v}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

When $s = 0$, $\|\cdot\|_0 := \|\cdot\|$ denotes the $L^2(\mathbb{R})$ norm.

2) For $f \in C^m([0,T],\mathbb{R})$ we define the norm $\|f\|_{\infty,m}$ as follows:

$$\|f\|_{\infty,m} = \max_{0\le k\le m}\{\sup_{t\in[0,T]} |f^{(k)}(t)|\}.$$

When $m = 0$, $\|f\|_{\infty,0} = \|f\|_\infty$ is the uniform norm.

2. Approximation of the heat flux

Assume that $g \in L^2(\mathbb{R}_+)$, then the solution of (2) is given by the integral (see [1])

$$u(x,t) = \int_0^t \frac{x-1}{t-s} k(x-1, t-s) g(s) ds, \quad x \ge 1, \quad t \ge 0, \tag{4}$$

where $k(x,t)$ is the heat kernel

$$k(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(\frac{-x^2}{4t}\right),$$

then the flux at $x = 1$ is written as

$$h(t) := \lim_{x\to 1} u_x(x,t) = \lim_{x\to 1} \int_0^t (1 - \frac{(x-1)^2}{2(t-s)}) \frac{1}{t-s} k(x-1, t-s) g(s) ds. \tag{5}$$

Under some restriction on g, this limit exists in L^2 (see Theorem 3.2).

As we can not interchange the limit and the integral in (5), we approximate h by h_δ, $\delta > 0$, by setting

$$h_\delta(t) := u_x(1+\delta, t) = \int_0^t (1 - \frac{\delta^2}{2(t-s)}) \frac{1}{t-s} k(\delta, t-s) g(s) ds. \tag{6}$$

Now we compute the Fourier transform of h and h_δ. For this we extend the functions $u(x,t)$ and $g(t)$ to the whole real t-axis by defining them to be zero for $t < 0$. Taking the Fourier transform of (1) with respect to t, we obtain the solution u in frequency domain (see also [1,3]):

$$\widehat{u}(x,\xi) = e^{\sqrt{i\xi}(1-x)} \widehat{g}(\xi), \tag{7}$$

where

$$\sqrt{i\xi} = \begin{cases} (1+i)\sqrt{\frac{|\xi|}{2}}, & \xi \ge 0, \\ (1-i)\sqrt{\frac{|\xi|}{2}}, & \xi < 0, \end{cases}$$

which leads to

$$\widehat{h}(\xi) = \lim_{x\to 1} \widehat{u}_x(x,\xi) = -\sqrt{i\xi}\,\widehat{g}(\xi) \quad \text{a.e. } \xi \in \mathbb{R} \tag{8}$$

and

$$\widehat{h}_\delta(\xi) = -\sqrt{i\xi}\, e^{-\delta\sqrt{i\xi}} \widehat{g}(\xi). \tag{9}$$

Remark 2.1. Using Fourier transform analysis, we show in the appendix that equation (8), in ξ, is equivalent to the integro-differential equation in time

$$h(t) = -\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{g(s)}{\sqrt{t-s}}ds, \tag{10}$$

which is the inverse of Abel transform

$$g(t) = -\sqrt{2\pi}\int_0^t \frac{h(s)}{\sqrt{t-s}}ds. \tag{11}$$

Murio proposed, in his book ([9, Chapter 2]), four methods for the approximation of the solution of Abel equation (11). In the third one, he approximated h by the formula

$$h_\gamma(t) = -\sqrt{\frac{2}{\pi}}\left\{\gamma^{-3/2}\int_{t-\gamma}^t g(s)ds - \frac{1}{2}\int_0^{t-\gamma} g(s)(t-s)^{-3/2}ds\right\}, \tag{12}$$

with $0 < \gamma < 1$. If g is continuously differentiable and $g(0) = 0$, another representation is proposed, namely

$$h(t) := Lg = -\sqrt{\frac{2}{\pi}}\int_0^t \frac{g'(s)}{\sqrt{t-s}}ds. \tag{13}$$

This formula is unstable. More precisely, the operator L with domain $D(L) = \{g \in H^1(0,1), g(0) = 0\}$ is unbounded in $L^2(0,1)$. To show the instability, we construct the functions g_n defined as follows:

$$g_n(t) = \begin{cases} nt & \text{if} \quad 0 \le t \le 1/n, \\ 1 & \text{if} \quad \frac{1}{n} \le t \le 1. \end{cases}$$

The sequence $g_n \in D(T)$ and satisfies the properties:

$$g_n \to g = 1 \text{ in } L^2(0,1) \text{ and } \|Lg_n\| \to +\infty \text{ as } n \to +\infty.$$

Groetsch, in [6,7], approximates the values of L (in a stable way) by Tikhonov-like method.

In the numerical tests, we shall compare our method (formulas (6)) with (12). It seems that our method is comparable to the mollification method (the first method introduced by Murio in his book). Indeed, the heat kernel $k(\delta, t)$ in the expression (6) plays the role of the δ-mollifier.

3. Convergence and error estimates

In order to give an error estimate for the regularized flux h_δ, we need the following lemma.

Lemma 3.1. *We have*

(i) $|1 - e^{-\delta\sqrt{i\xi}}| \le 2$ *for* $\xi \in \mathbb{R}$, (ii) $|1 - e^{-\delta\sqrt{i\xi}}| \le \delta\sqrt{A}e^{\delta\sqrt{A}}$ *for* $|\xi| \le A$.

Proof. (i) follows from $\forall\xi$, $|e^{-\delta\sqrt{i\xi}}| \le e^{-\delta\sqrt{\frac{|\xi|}{2}}} \le 1$.

(ii) Using the series $e^X = 1 + X + \cdots + \frac{X^n}{n!} + \cdots$, we obtain the upper bound $|1 - e^X| \le |X|e^{|X|}$, $\forall X$, which leads to the inequality. $\square$

Theorem 3.2. *Assume that* $g \in H^s(\mathbb{R})$, $s > 1$, *and* $\|g\|_s \le M$. *Then we obtain the error bound*

$$\|h - h_\delta\| \le \sqrt{4 + e^2}\ \ \delta^{1-\frac{1}{s}} M. \tag{14}$$

Proof. Using Parseval formula for the Fourier transform together with equations (8) and (9), we obtain

$$\|h - h_\delta\|^2 = \int_{\mathbb{R}} |1 - e^{-\delta\sqrt{i\xi}}|^2 |\xi| |\widehat{g}(\xi)|^2 d\xi = \int_{|\xi|\le A} + \int_{|\xi|\ge A}.$$

From Lemma 3.1 it follows that for all $A \ge 1$

$$\|h - h_\delta\|^2 \le A\delta^2 e^{2\delta\sqrt{A}} \|g\|^2_{\frac{1}{2}} + \frac{4}{A^{s-1}} \int_{|\xi|\ge A} |\xi|^s |\widehat{g}(\xi)|^2 d\xi.$$

We choose A such that $A\delta^2 = \frac{1}{A^{s-1}} \Leftrightarrow A = (\frac{1}{\delta})^{\frac{2}{s}}$, then

$$\|h - h_\delta\|^2 \le A^{1-s} e^2 \|g\|^2_{1/2} + 4A^{1-s}\|g\|_s^2,$$

hence

$$\|h - h_\delta\| \le \sqrt{e^2 + 4}\|g\|_s\ \ \delta^{1-\frac{1}{s}}.$$ $\square$

Theorem 3.3. *Suppose that* $g \in H^s(\mathbb{R})$, $s > 1$, *and* $g_\epsilon \in L^2(\mathbb{R})$ *satisfying* $\|g - g_\epsilon\| \le \epsilon$. *Then we get the error bound*

$$\|h_\delta - h_{\delta,\epsilon}\| \le \frac{\sqrt{2}}{e}\frac{\epsilon}{\delta}, \tag{15}$$

where $h_{\delta,\epsilon}$ *is defined by its transform* $\widehat{h}_{\delta,\epsilon}(\xi) = -\sqrt{i\xi}e^{-\delta\sqrt{i\xi}}\widehat{g}_\epsilon(\xi)$.

Proof. We have

$$\begin{aligned}\|h_\delta - h_{\delta,\epsilon}\|^2 = \int_{\mathbb{R}} |\xi| e^{-\delta\sqrt{2|\xi|}} |\widehat{g}(\xi) - \widehat{g}_\epsilon(\xi)|^2 d\xi &\le (\sup_\xi |\xi| e^{-\delta\sqrt{2|\xi|}}) \|\widehat{g} - \widehat{g}_\epsilon\|^2 \\ &\le \frac{2}{e^2}(\frac{\epsilon}{\delta})^2.\end{aligned}$$ $\square$

Theorem 3.4. *Under the conditions of Theorem 3.2 and Theorem 3.3, we get the error estimate*

$$\|h - h_{\delta,\epsilon}\| \le M\sqrt{4 + e^2}\ \ \delta^{1-\frac{1}{s}} + \frac{\sqrt{2}}{e}\frac{\epsilon}{\delta}. \tag{16}$$

Proof. The estimate follows from Theorem 3.2, Theorem 3.3 and the triangle inequality. $\square$

The error estimate is minimized by choosing δ such that $\frac{\epsilon}{\delta} = \delta^{1-\frac{1}{s}} \Leftrightarrow \delta^{1-\frac{1}{s}} = \epsilon^{\frac{s-1}{2s-1}}$, and gives convergence as $\epsilon \to 0$ with rate $O(\epsilon^{\beta})$, $\beta = \frac{s-1}{2s-1}$. The order of convergence β is less than $\frac{1}{2}$. In particular $\beta = \frac{2}{5}$ for $s = 3$. Hence we have the corollary.

Corollary 3.5. *Under the conditions of Theorem 3.4, if we choose* $\delta = \epsilon^{\frac{s}{2s-1}}$ *we obtain the error bound*

$$\|h - h_{\delta,\epsilon}\| \le [M\sqrt{4+e^2} + \frac{\sqrt{2}}{e}]\epsilon^{\beta} \quad \textit{with } \beta = \frac{s-1}{2s-1}. \tag{17}$$

4. Discretization of the integral

In this section we approximate the integral (6) by the trapezoidal rule and we give an estimate of the discrete error. In the following, the generic constant C does not depend on δ and g. The integral (6) is of the form

$$I(\delta, g, t) = \int_0^t k_1(\delta, t-s) g(s) ds = \int_0^t k_1(\delta, s) g(t-s) ds$$

with $k_1(\delta, t) = \frac{1}{2\sqrt{\pi}}(t^{-3/2} - \frac{\delta^2}{2} t^{-5/2}) \exp(-\frac{\delta^2}{4t})$.

Denoting $f(\delta, s) = k_1(\delta, s) g(t-s)$, the trapezoidal formula is written as follows

$$I(\delta, g, t_n) \sim T_\tau(\delta, g, t_n) := \tau \sum_{j=1}^{n} \omega_j f_j, \quad (\omega_0 = \omega_n = 0.5,\ \omega_j = 1,\ j = 1, n-1)$$

where $f_j = f(\delta, s_j)$, $T = N\tau$, $s_j = j\tau$, $n \le N$. If g is of class C^2, we have the error bound

$$|I(\delta, g, t_n) - T_\tau(\delta, g, t_n)| \le \frac{T}{12} \tau^2 M_2(\delta),$$

where $M_2(\delta) = \sup_{0<s<T} |f''(\delta, s)|$. We can see that $M_2 \le \|k_1\|_{\infty,2} \|g\|_{\infty,2}$ and

$$|k_1''(\delta, t)| \le C\delta^6 t^{-\frac{13}{2}} \exp(-\frac{\delta^2}{4t}) \le C\delta^{-7},$$

which leads to

$$|I(\delta, g, t_n) - T_\tau(\delta, g, t_n)| \le CT\frac{\tau^2}{\delta^7} \|g\|_{\infty,2}.$$

Comparing this estimate with (14), for $s = 3$, we select τ such that

$$\frac{\tau^2}{\delta^7} \le \delta^{\frac{2}{3}} \Leftrightarrow \tau \le \delta^{\frac{23}{6}},$$

this occurs if $\tau = \delta^4$. Therefore

$$\|I(\delta, g) - T_\tau(\delta, g)\|_{l^2} \le CT\delta^{\frac{2}{3}} \|g\|_{\infty,2}.$$

Now we compare $T_\tau(\delta, g)$ and $T_\tau(\delta, g_\epsilon)$ under the condition $\|g - g_\epsilon\| \leq \epsilon$. We have

$$T_\tau(\delta, g, t_n) - T_\tau(\delta, g_\epsilon, t_n) = \tau \sum_{j=1}^{n} \omega_j k_1(\delta, s_j)(g(s_j) - g_\epsilon(s_j)),$$

then

$$\|T_\tau(\delta, g) - T_\tau(\delta, g_\epsilon)\|_{l^2} \leq \sqrt{\tau}\|k_1(\delta)\|_\infty \|g - g_\epsilon\|_{l^2} \leq C\sqrt{\tau}\frac{\epsilon}{\delta^3},$$

where $\|g\|_{l^2}^2 = \tau \sum_{j=1}^{n} |g(s_j)|^2$ is the discrete $L^2(0,T)$ norm.

With $\delta = \epsilon^{\frac{1}{2}}$ and $\tau = \delta^4 = \epsilon^2$ we get

$$\|T_\tau(\delta, g) - T_\tau(\delta, g_\epsilon)\|_{l^2} \leq C\epsilon^{\frac{1}{2}}$$

and

$$\|I(\delta, g) - T_\tau(\delta, g)\|_{l^2} \leq CT\epsilon^{\frac{1}{3}}\|g\|_{\infty,2}.$$

We summarize these results in the following theorem.

Theorem 4.1 (Discrete Error Estimate)**.** *Suppose that* $g \in H^3(\mathbb{R})$ *and* $\|g - g_\epsilon\| \leq \epsilon$. *Then, by choosing* $\delta = \sqrt{\epsilon}$ *and* $\tau = \epsilon^2$ *we get*

$$\|I(\delta, g) - T_\tau(\delta, g_\epsilon)\|_{l_2} \leq CT\epsilon^{\frac{1}{3}}\|g\|_{\infty,2} + O(\epsilon^{\frac{1}{2}}). \tag{18}$$

Denote by $h_\epsilon = T_\tau(\delta, g_\epsilon)$ the approximate flux with the choice $\delta = \sqrt{\epsilon}$ and $\tau = \epsilon^2$. By adding the different errors we get the global error.

Theorem 4.2 (Global Error)**.** *Suppose that* $g \in H^3(\mathbb{R})$ *and* $\|g - g_\epsilon\| \leq \epsilon$. *Then*

$$\|h - h_\epsilon\| \leq CT\epsilon^{\frac{1}{3}}\|g\|_3 + O(\sqrt{\epsilon}). \tag{19}$$

Proof. The proof follows from the estimate (17) with $s = 3$ and the estimate (18), observing that $\|g\|_{\infty,2} \leq C\|g\|_3$, according to the Sobolev embedding $H^3(\mathbb{R}) \subset C^2$. $\square$

Remark 4.3. If $g(t) = u(1,t)$ is an exact data, then $\widehat{g}(\xi) = e^{-\sqrt{i\xi}}\widehat{f}(\xi)$ where $f = u(0,\cdot) \in L^2(\mathbb{R})$. From [1, Theorem 2.1], we see that $g \in H^s(\mathbb{R})$ for all $s > 0$ and $\|g\|_s \leq C_s\|f\|$. Then the estimate (19) takes the form

$$\|h - h_\epsilon\| \leq C_s TE\epsilon^{\frac{1}{3}} + O(\sqrt{\epsilon}),$$

where E is an priori bound on f, that is, $\|f\| \leq E$.

5. Discretization of the Cauchy problem

The Cauchy problem (3) is approximated by central finite differences schemes. Letting $w := u_x$, the system (3) can be rewritten as

$$\begin{cases} u_x(x,t) = w(x,t), \quad w_x(x,t) = u_t(x,t), \quad 0 < x < 1, \quad t > 0, \\ u(1,t) = g(t), \quad w(1,t) = h(t), \qquad t > 0, \\ u(x,0) = 0, \quad 0 < x < 1. \end{cases} \tag{20}$$

Now we define the discrete solution (u_n^m, w_n^m) on the grid $\mathcal{G} = \{x_m = mk;\ t_n = n\tau :\ m = 0, M;\ n = 0, N\}$, then this problem is discretized by

$$
(21)\qquad \begin{cases}
\dfrac{u_m^{n+1} - u_m^n}{k} = w_m^{n+1}, \quad n = 1, \dots, N, \quad m = 1, \dots, M+1, \\
\dfrac{w_m^{n+1} - w_m^n}{k} = \dfrac{u_{m+1}^n - u_{m-1}^n}{2\tau}, \quad n = 1, \dots, N, \quad m = 1, \dots, M, \\
u_m^{N+1} = g_m, \quad w_m^{N+1} = h_m, \quad m = 1, \dots, M+1, \\
u_m^1 = 0, \quad m = 1, \dots, M+1.
\end{cases}
$$

This scheme is unconditionally stable (see [8, Theorem 3.2] and the references therein). In this article, Hào use the mollification method, developed by Murio in [9], which consists of mollifying g_ϵ and h_ϵ by convolution with the Dirichlet kernel $D_\nu(t) = \frac{\sin(\nu t)}{2\pi t}$, $\nu \to \infty$. He proved the stability estimate

$$\|u^n\|_{l^2} \le e^{1+\nu}(\|g\|_2 + \|h\|_2).$$

We shall not establish such an estimate for our case, but we are satisfied with a numerical validation.

6. Numerical results and comparisons

6.1. Algorithm

We give some numerical examples to show the validity of the proposed numerical schemes given in Sections 4 and 5. Also we compare the performance of our method (Method I) with two methods.

- Method II: the Fourier method (see [1], formula (3.6)) which can be implemented numerically by Fast Fourier Transform (FFT).
- Method III: Abel inversion formula (10) approximated by (12) due to Murio ([9]).

Our algorithm is as follows:

(1) To simulate a data, we compute $g = Af$ for given f, where A is the forward operator defined in [1] by the formula (2.6) with M points in $[0, T]$.
(2) Compute the exact flux $h = Hg$ given by the expression (2.4) of [1] with M points.
(3) Introduce random error of amplitude ϵ leading to the function $g_\epsilon(t) = g(t) + \epsilon * \sigma(t)$ where σ is the Gaussian random function.
(4) Compute the approximate flux $h_{\delta,\epsilon}$ by the quadratic formula $T_\tau(\delta, g_\epsilon)$, given in Section 4, with a priori choice $\delta = \tau^{\frac{1}{4}} = (\frac{T}{N})^{\frac{1}{4}}$ according to Theorem 4.1. In practice we take $N = 10 \times M$ points in the interval $[0, T]$ in order to get δ close to zero. Here we are limited by the memory allowed for array by *Matlab 2011*, $N \le 6.10^3$.
(5) To reconstruct f, we solve Cauchy problem (system (3)), with perturbed data $(g_\epsilon, h_{\delta,\epsilon})$ ($h_{\delta,\epsilon}$ is computed in the step 4), by applying

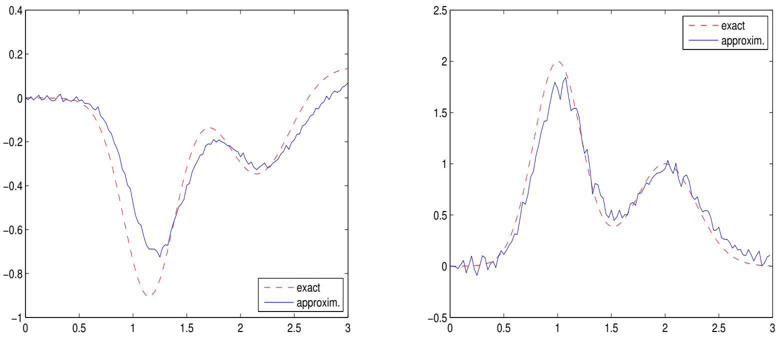

FIGURE 1. Example 1-Method I: (left) Exact and approximate heat flux at $x = 1$; with $\delta = 0.224$, $\epsilon = 0.01$, $N = 2000$; (right): Exact and approximate solution at $x = 0$ with $M = 200$.

the central difference scheme described in Section 5. We use a grid of $M \times N$ points in the rectangle $[0, 1] \times [0, T]$.

(6) Compare our method (Method I) with the Methods II and III.

6.2. Examples

- **Example 1.** As the first example we consider the function

$$u(0, t) = f(t) = 2 \exp(-10(t - 1)^2) + \exp(-6(t - 2)^2), \quad 0 \leq t \leq T = 5,$$

the exact solution of the problem (1). The function f belongs to $H^s(\mathbb{R})$ for all $s \geq 0$. We can assume (numerically) that $f(t) = 0$ for $t \notin [0, T]$.

- **Example 2.** The exact solution is defined as follows

$$f(t) = \begin{cases} 1 & \text{if } 1 \leq t \leq 2, \\ 0 & \text{elsewhere.} \end{cases}$$

The function f belongs to $L^2(\mathbb{R})$, but the exact data function $g = Af$ satisfies the necessary hypothesis ($g \in H^3(\mathbb{R})$) of Theorem 4.2 for the convergence of the heat flux $h_{\delta,\epsilon}$ (see Remark 4.3).

6.3. Discussion

The computational results show that our method (Method I) is convergent and stable for $\epsilon \leq 10^{-2}$, however the parameter N must be large enough ($N \geq 10^3$) to ensure a good precision. Method I provides an automatic choice rule to select $\delta = \tau^{\frac{1}{4}} = (\frac{T}{N})^{\frac{1}{4}}$. In Method III, the regularization parameter $\gamma = \gamma(N)$ is obtained by numerical experimentation (see Figure 4, where $\gamma = 0.085$ if $N = 500$). Moreover, we observe that the Method III is sensitive to the variation of N and is stable only if the level noise $\epsilon \leq 10^{-3}$. The Method II

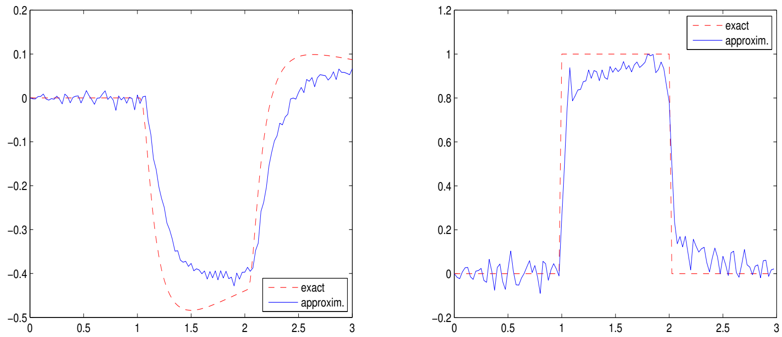

FIGURE 2. Example 2-Method I: (left) Exact and approximate heat flux at $x = 1$; with $\delta = 0.224$, $\epsilon = 0.01$, $N = 2000$; (right): Exact and approximate solution at $x = 0$ with $M = 200$.

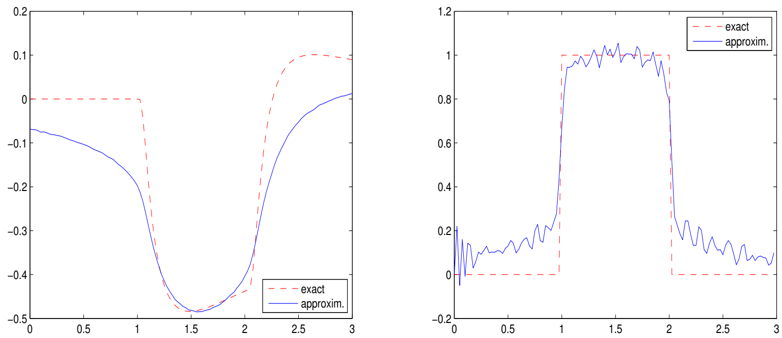

FIGURE 3. Example 2-Method II: (left) Exact and approximate heat flux at $x = 1$; with $\epsilon = 0.001$, $N = 200$; (right): Exact and approximate solution at $x = 0$ with $M = 200$.

is the easiest to implement, it is stable for $\epsilon \leq 10^{-3}$ and not well precise in the reconstruction of f. Finally remark that we solve the ill-posed Cauchy problem by finite difference schemes without additional regularization. Indeed in our case the heat kernel $k(\delta, t)$ in the formula (6) plays the role of the δ-mollifier of the heat flux h_ϵ, by comparison with the mollification method used by Hào et al. in [8]. We can see in Figures 1 and 2 that h_ϵ is enough smooth.

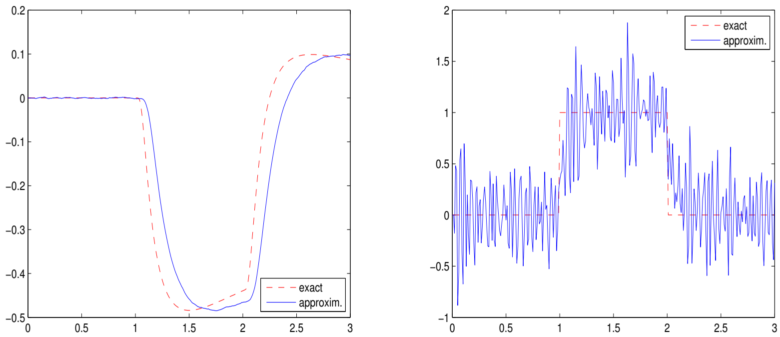

FIGURE 4. Example 2-Method III: (left) Exact and approximate heat flux at $x = 1$; with $\epsilon = 0.001$, $\gamma = 0.085$, $N = 500$; (right): Exact and approximate solution at $x = 0$ with $M = 200$.

7. Conclusion

We have proposed a new method (Method I) to compute the heat flux from the measured temperature in an inverse heat conduction problem. We have proved that our method is stable and given an error estimate. Our method is based on a truncation procedure, where the parameter of regularization is chosen by discrepancy principle (the discrete error must be of the same order of the truncation error). The numerical tests confirm the efficiency of the method. Our method has been compared with two methods: Fourier method and inverse Abel formula, the numerical results show that Method I is slightly better in term of stability.

Appendix A. Abel equation

Assume that h is the solution of Abel equation, i.e., h is given by the equation (12). In this section we show that $\widehat{h}(\xi) = -\sqrt{i\xi}\widehat{g}(\xi)$. For this we introduce the function K such that $K(t) = \frac{1}{\sqrt{t}}$ for $t > 0$ and $K(t) = 0$ for $t \leq 0$. If we assume that $g(t)$ is causal i.e., $g(t) = 0$ for $t \leq 0$, then we have $(K * g)(t) = \int_0^t K(t-s)g(s)ds$ and we can write $h(t) = -\sqrt{\frac{2}{\pi}}\frac{d}{dt}(K * g)(t)$. Applying the Fourier transform, we obtain

$$\widehat{h}(\xi) = -\sqrt{\frac{2}{\pi}} i\xi \widehat{K}(\xi)\widehat{g}(\xi)$$

with $\widehat{K}(\xi) = \int_0^\infty e^{-i\xi t} \frac{dt}{\sqrt{t}}$. By using the change of variable $x^2 = |\xi| t$, we get

$$\widehat{K}(\xi) = \frac{2}{\sqrt{|\xi|}} \int_0^\infty e^{-i\sigma x^2} dx, \quad (\sigma = \text{sgn } \xi).$$

Now we use the Fresnel integrals $\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ to have

$$\widehat{K}(\xi) = \sqrt{\frac{\pi}{2|\xi|}}(1 - i\sigma). \tag{22}$$

Using the definition $\sqrt{i\xi} = (1 + i\sigma)\sqrt{\frac{|\xi|}{2}}$ and $\xi = \sigma|\xi|$, this leads to

$$\widehat{h}(\xi) = -\sqrt{i\xi}\widehat{g}(\xi).$$

The formula (22) can be rigorously justified in the space $\mathcal{S}'(\mathbb{R})$ of the generalized functions (see [11]). Indeed, $K \in L^1_{loc}(\mathbb{R})$ and $\lim_{|t|\to\infty} K(t) = 0$, hence $K \in \mathcal{S}'(\mathbb{R})$, moreover, we have

$$\widehat{K}(\xi) = \lim_{R\to+\infty} \int_0^R e^{-i\xi t} \frac{dt}{\sqrt{t}} \quad \text{in } \mathcal{S}'(\mathbb{R}),$$

since $K_R(t) =: \chi_{[0,R]}K(t) \to K(t)$ in $\mathcal{S}'(\mathbb{R})$ as $R \to \infty$. Hence $\widehat{K} \in \mathcal{S}'(\mathbb{R})$. Finally we remark that if $g \in H^s(\mathbb{R})$, $s \geq 1$, then $\widehat{h} = -\sqrt{i\xi}\widehat{g} \in L^2_{s-\frac{1}{2}}(\mathbb{R})$ and hence $h(t) = -\sqrt{\frac{2}{\pi}} \frac{d}{dt}(K * g)$ belongs to $H^{s-\frac{1}{2}}(\mathbb{R})$. For more details we can see the book [5].

Acknowledgments. The authors would like to thanks the referee for his valuable comments and suggestions that improve the quality of our paper.

References

[1] L. Chorfi and L. Alem, *Stable algorithm for identifying a source in the heat equation*, Electron. J. Differential Equations **2015** (2015), No. 267, 14 pp.

[2] L. Elden, *Approximations for a Cauchy problem for the heat equation*, Inverse Problems **3** (1987), no. 2, 263–273.

[3] L. Elden, F. Berntsson, and T. Reginska, *Wavelet and Fourier methods for solving the sideways heat equation*, SIAM J. Sci. Comput. **21** (2000), no. 6, 2187–2205.

[4] C.-L. Fu, X.-T. Xiong, and P. Fu, *Fourier regularization method for solving the surface heat flux from interior observations*, Math. Comput. Modelling **42** (2005), no. 5-6, 489–498.

[5] R. Gorenflo and S. Vessella, *Abel Integral Equations*, Lecture Notes in Mathematics, **1461**, Springer-Verlag, Berlin, 1991.

[6] C. W. Groetsch, *A numerical method for some inverse problems in nonlinear heat transfer*, Comput. Math. Appl. (Athens) **1** (1994), 49–59.

[7] C. W. Groetsch and M. Hanke, *A general framework for regularized evaluation of unstable operators*, J. Math. Anal. Appl. **203** (1996), no. 2, 451–463.

[8] D. N. Hào, H. J. Reinhardt, and A. Schneider, *Numerical solution to a sideways parabolic equation*, Internat. J. Numer. Methods Engrg. **50** (2001), no. 5, 1253–1267.

[9] D. A. Murio, *The Mollification Method and the Numerical Solution of Ill-posed Problems*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1993.

[10] G. Talenti and S. Vessella, *A note on an ill-posed problem for the heat equation*, J. Austral. Math. Soc. Ser. A **32** (1982), no. 3, 358–368.

[11] V. S. Vladimirov, *Equations of Mathematical Physics*, Translated from the Russian by Audrey Littlewood. Edited by Alan Jeffrey. Pure and Applied Mathematics, **3**, Marcel Dekker, Inc., New York, 1971.

Leïla Alem
Laboratory of Applied Mathematics
Badji Mokhtar University
Annaba 23000, Algeria
Email address: alemleila@yahoo.fr

Lahcène Chorfi
Laboratory of Applied Mathematics
Badji Mokhtar University
Annaba 23000, Algeria
Email address: l_chorfi@hotmail.com