REFINEMENT OF HOMOGENEITY AND RAMSEY NUMBERS

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Abstract. We introduce some variants of the finite Ramsey theorem. The variants are based on a refinement of homogeneity. In particular, they cover homogeneity, minimal homogeneity, end-homogeneity as special cases. We also show how to obtain upper bounds for the corresponding Ramsey numbers.

1. Introduction

The *finite Ramsey theorem* [11] states that there are always monochromatic cliques of required cardinalities in any edge labelling of a sufficiently large complete graph. It is a foundational result in combinatorics which guarantees regularity in disorder and initiated the so-called *Ramsey theory* in combinatorics.

In Ramsey theory, one studies inevitable substructures in large objects and is mainly interested in finding lower and upper bounds for how large a complete graph should be to allow regularity. For example, people have been interested in the case where two colors C_1, C_2 are used for edge labelling. In this case, the least number for how large a complete graph should be to allow a monochromatic clique is usually denoted by $R(m_1, m_2)$, where $m_i \in \mathbb{N}$, i.e., positive integers, are the cardinality of the required monochromatic clique of the color C_i. Numbers such as $R(m_1, m_2)$ are called *Ramsey numbers*.

One can easily extract an upper bound for $R(m_1, m_2)$ from the proof of the theorem in [11], and for lower bounds one has provided many other arguments. One of the early results is obtained by Erdös [3]: If $m \geq 3$, then

$$2^{m/2} < R(m, m) \leq 4^{m-1}.$$

There is usually a vast gap between the tightest lower and upper bounds, and only for very few numbers m_i the exact value of $R(m_1, m_2)$ is known. we refer to [2] for more about recent results about Ramsey numbers.

In this paper we introduce some variants of the finite Ramsey theorem and show how to obtain upper bounds for the corresponding Ramsey numbers. The variants are related to a general form of homogeneity. The proof of the main theorem of this paper is a revised and corrected version of that provided in [9].

This paper is organized as follows. Section 2 introduces two principles which have some historical meaning in mathematical logic. These two principles are based on homogeneity and min-homogeneity, respectively. In Section 3, we generalize the homogeneity property in order to get a variant of the finite Ramsey theory. Section 4 shows how to find upper bounds for the corresponding Ramsey numbers.

2. Paris-Harrington and Kanamori-McAloon principles

Before we show how to generalize the finite Ramsey theory in the next section, we would like to explain the importance of our interest by introducing two principles: the *Paris-Harrington principle* and the *Kanamori-McAloon principle*. In 1970s and 1980s, they played very important roles in the history of mathematical logic.

The Paris-Harrington principle is introduced in [10] as the first true sentence that mathematicians could encounter in their customary enterprise which cannot be proved in the first-order logical system called Peano arithmetic [8]. Until the paper was published in 1977, people believed that such a true sentence could be found only in relation with some meta-theoretic properties such as Gödel's incompleteness theorems in the 1930s. A similar result was introduced, 10 years later, by Kanamori and McAloon in [7]. They showed that one could define another principle using the so-called minimal homogeneity which is equivalent to the Paris-Harrington principle. That is, the Kanamori-McAloon principle is a true sentence which cannot be proved in Peano arithmetic.

The reason why the two principles are interesting can be found in the fact that they are simple variants of the finite Ramsey theorem. As mentioned before, the validity of the finite Ramsey theory can be shown in a very elementary way. On the other hand, the results in [7, 10] imply that the validity of the two variants requires much more complicated tools, so that they cannot be proved true in Peano arithmetic. One can even measure how complex the proofs could be. We refer the reader to [6] for the detail. In the rest of this section, we describe both principles. For that we first recall the finite Ramsey theorem represented in a number-theoretic way.

The finite Ramsey theorem

In the rest of this paper, small Latin letters $c, k, \ell, n, \ldots$ range over positive integers while capital Latin letters $X, Y, \ldots$ range over sets of positive integers.

Given a set X, a subset of X with n elements is called *n-subset* of X. Let $[X]^n$ denote the set of all *n-subset* of X, i.e.,

$$[X]^n := \{Y \subseteq X \mid card(Y) = n\}.$$

If C is a function defined on $[X]^n$ and if $x_1 < x_2 < \cdots < x_n$, then we write $C(x_1, \ldots, x_n)$ instead of $C(\{x_1, \ldots, x_n\})$. We now introduce a notation useful in a compact representation of the finite Ramsey theorem. Let the notation

$$\ell \to (k)_c^n \tag{1}$$

denote the following sentence:

> for any coloring $C \colon [X]^n \to \{1, \ldots, c\}$, where $card(X) = \ell$, there is a subset $H \subseteq X$ such that $card(H) \geq k$ and H is *homogeneous* for C, i.e., C is constant on $[H]^n$.

The finite Ramsey theorem [11] says that

> for all positive integers n, c, $k \in \mathbb{N}$, there exists some number $\ell \in \mathbb{N}$ large enough such that $\ell \to (k)_c^n$ holds.

Erdös and Rado [5] gives an upper bound for such a number ℓ depending super-exponentially on n, c, and k. We will later present a generalized result in Theorem 4.2.

The Paris-Harrinton theorem

The Paris-Harrington theorem is a variant of the finite Ramsey theorem which makes use of large homogeneous sets.

Definition (Largeness)**.** A set H of positive integers is called *large* if $card(H) \geq \min(H)$.

Definition (Homogeneity)**.** Given a function $C \colon [X]^n \to \mathbb{N}$, a subset $H \subseteq X$ is called *homogeneous* for C when $C(s) = C(t)$ for all s, $t \in [H]^n$.

Given positive integers ℓ, n, c, k, we use the following notation

$$\ell \to^* (k)_c^n \tag{2}$$

for the fact that

> for any coloring $C \colon [X]^n \to \{1, \ldots, c\}$, where $card(X) = \ell$, there exists a large subset $H \subseteq X$ such that it is homogeneous for C and $card(H) \geq k$.

Theorem 2.1 (Paris and Harrington [10])**.** *The following principle cannot be proved in the first-order Peano arithmetic although it is true.*

(PH) $:\equiv$ *for all n, c, $k \in \mathbb{N}$ there exists some $\ell \in \mathbb{N}$ such that $\ell \to^* (k)_c^n$*

We call (PH) *the Paris-Harrington principle.*

We will not discuss further the difference between trueness and provability. It would go far beyond the scope of this paper. We refer instead to [10] for readers interested in the difference.

The Kanamori-McAloon theorem

Kanamori and McAloon introduced, 10 years later, another variant of the finite Ramsey theorem which is also unprovable in the first-order Peano arithmetic although it is true. The so-called the *Kanamori-McAloon principle* uses the concept of *regressiveness*.

Definition (Regressiveness)**.** We call a function $C\colon [X]^n \to \mathbb{N}$ *regressive* when $C(s) < \min(s)$ holds for all $s \in [X]^n$ such that $\min(s) > 0$.

Definition (Minimal homogeneity)**.** Given a function $C\colon [X]^n \to \mathbb{N}$, a subset $H \subseteq X$ is called *min-homogeneous* for C when $C(s) = C(t)$ for all $s, t \in [H]^n$ such that $\min(s) = \min(t)$, that is, when f restricted to $[H]^n$ depends only on the minimum elements of the ordered input tuples.

Given positive integers ℓ, n, k the following notation

$$(3) \qquad\qquad\qquad\qquad \ell \to (k)^n_{reg}$$

denotes that

> for any regressive function $C\colon [X]^n \to \mathbb{N}$, where $card(X) = \ell$, there exists a subset $H \subseteq X$ such that it is min-homogeneous for C and $card(H) \geq k$.

Theorem 2.2 (Kanamori and McAloon [7])**.** *The validity of the following principle is equivalent to* (PH).

(KM) $:\equiv$ *for all n, $k \in \mathbb{N}$ there exists some $\ell \in \mathbb{N}$ such that $\ell \to (k)^n_{reg}$.*

We call (KM) *the Kanamori-McAloon principle.*

3. End-homogeneity and its generalized form

Having introduced two historically important principles (PH) and (KM), we now just focus on their purely combinatorial aspects, namely homogeneity and min-homogeneity. First we introduce another closely related homogeneity property, well known from set theory. It is the *end-homogeneity*. We refer the reader to Section 15 in [4] for more about the role of end-homogeneity in set theory. Here we focus on its finite version in combinatorics.

Definition (End-homogeneity)**.** Let $C\colon [X]^n \to \{1, \ldots, c\}$ be a coloring function. We call a subset $H \subseteq X$ *end-homogeneous for C* if for all $(n-1)$-element subset U of H and for all elements $v, w \in H$ such that $\max U < \min\{v, w\}$, we have

$$C(U \cup \{v\}) = C(U \cup \{w\}).$$

In fact, end-homogeneous sets are already used in the proof of the finite Ramsey theorem. We finally introduce here a general concept of homogeneity which covers all the three homogeneity properties. Let n, c, k, ℓ, be positive integers and s be a non-negative integer such that $0 \leq s \leq n \leq k$ holds.

Definition (*s*-homogeneity)**.** Let $C\colon [X]^n \to \{1,\dots,c\}$ be a coloring function. We call a subset $H \subseteq X$ *s-homogeneous* for C if for all *s*-element subset U of H and for all $(n-s)$-element subsets V, W of H such that $\max U < \min(V \cup W)$, we have

$$C(U \cup V) = C(U \cup W).$$

Note that *s*-homogeneity generalizes all the three homogeneity properties mentioned earlier.

Lemma 3.1. *Let* $C\colon [X]^n \to \{1,\dots,c\}$ *be a coloring function. Then the following hold by definition.*

(1) *0-homogeneous sets are homogeneous sets.*
(2) *1-homogeneous sets are min-homogeneous sets.*
(3) $(n-1)$*-homogeneous sets are end-homogeneous sets.*

Let

(4) $$\ell \to_s \langle k \rangle_c^n$$

denote that

> for any coloring $C\colon [X]^n \to \{1,\dots,c\}$, where $card(X) = \ell$, there exists a subset $H \subseteq X$ such that it is *s*-homogeneous for C and $card(H) \geq k$.

The following lemma shows a connection between *s*-homogeneity and homogeneity.

Lemma 3.2. *Let* $s \leq n$ *and assume*

(1) $\ell \to_s \langle k \rangle_c^n$,
(2) $k - n + s \to (m - n + s)_c^s$.

Then we have

$$\ell \to (m)_c^n.$$

Proof. Let $C\colon [X]^n \to \{1,\dots,c\}$ be given. Then by (1), there exists some $H \subseteq X$ such that $|H| = k$ and H is *s*-homogeneous for C. Let $z_1 < \cdots < z_{n-s}$ be the last $(n-s)$ elements of H and $H_0 := H \setminus \{z_1,\dots,z_{n-s}\}$. Then $card(H_0) = k - n + s$.

Define $D\colon [H_0]^s \to \{1,\dots,c\}$ by

$$D(x_1,\dots,x_s) := C(x_1,\dots,x_s,z_1,\dots,z_{n-s}).$$

By (2), there is Y_0 such that it is homogeneous for D and $Y_0 \subseteq H_0$, $card(Y_0) = m-n+s$. Therefore, $D \restriction [Y_0]^s = e$ for some $e \leq c$. Set $Y := Y_0 \cup \{z_1,\dots,z_{n-s}\}$. Then $card(Y) = m$ and Y is homogeneous for C. Indeed, we have for any sequence $x_1 < \cdots < x_n$ from Y

$$C(x_1,\dots,x_n) = C(x_1,\dots,x_s,z_1,\dots,z_{n-s}) = D(x_1,\dots,x_s) = e. \qquad \square$$

4. Ramsey numbers related to s-homogeneity

In this section we show how to obtain upper bounds for Ramsey numbers related to s-homogeneity which are defined by

$$(5) \qquad R_s^n(c,k) := \min\{\ell \mid \ell \to_s \langle k \rangle_c^n\},$$

where n, s are given such that $s \leq n$. R_s^n are called *Ramsey functions*.

Lemma 4.1. *Suppose n is a positive integer and s is a non-negative integer such that $s \leq n$. Then the following hold.*

    (1) $R_0^1(c,k) = c \cdot (k-1) + 1$.
    (2) $R_n^n(c,k) = R_s^n(1,k) = k$.
    (3) $R_s^n(c,n) = n$.
    (4) *If $0 < s$, then $R_s^n(c,k) \leq R_{s-1}^n(c,k)$.*

Note that (1) *corresponds to the pigeonhole principle.*[1]

We are going to give upper bounds for $R_s^n(c,k)$ which depend only on n, s, c, k. For typographical reasons, we will use a binary operation $*$ on positive integers defined by

$$x * y := x^y.$$

We assume that the operation $*$ is right-associative, that is,

$$x_1 * x_2 * \cdots * x_p := x_1 * (x_2 * (\cdots * (x_{p-1} * x_p) \cdots)).$$

We first remind Erdös and Rado's upper bound for $R_0^n(c,k)$:

$$(6) \qquad R_0^n(c,k) \leq c * (c^{n-1}) * (c^{n-2}) * \cdots * (c^2) * (c \cdot (k-n) + 1)$$

when $c \geq 2$ and $k \geq n \geq 2$. Note that the expression $(c^{n-1}) * (c^{n-2}) * \cdots * (c^2)$ does not exist when $n - 1 < 2$. We have for instance

$$(7) \qquad R_0^2(c,k) \leq c^{c \cdot (k-2) + 1}.$$

This upper bound turned out to be very useful in Weiermann [12]. As the main result of this paper, we generalize (6) to cover the s-homogeneity with $s > 0$.

Theorem 4.2. *Suppose $2 \leq n \leq k$, $0 < s \leq n$, and $2 \leq c$. Then the following holds.*

$$R_s^n(c,k) \leq c * (c^{n-1}) * (c^{n-2}) * \cdots * (c^{s+1}) * (k - n + s) * s.$$

Note that the expression $(c^{n-1}) * (c^{n-2}) * \cdots * (c^{s+1})$ does not exist when $n < s+2$. In particular, we have $R_1^2(c,k) \leq c^{k-1}$.

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[1] The pigeonhole principle states that if m containers contain in total n items and if $n > m > 0$, then at least one container must contain at least $\frac{n}{m}$ items.

Proof. The proof below is a refined reconstruction of that of Erdös and Rado [5]. We work with s-homogeneity instead of homogeneity. The main goal of the proof is to show the following recursive relation:

$$(8) \qquad R_s^n(c, k) * n \leq (c^n) * R_s^{n-1}(c, k-1) * (n-1)$$

from which the theorem follows by iterating it $(n - s)$ times.

Let $C \colon [X]^n \to \{1, \dots, c\}$ be a coloring function. In the following we will show that there is an s-homogeneous subset $H \subseteq X$ such that $card(H) \geq k$ under the assumption that $card(X)$ is large enough. How large it should be will be checked later, and the checking process will result in the relation (8).

Let

$$(9) \qquad \ell := 1 + R_s^{n-1}(c, k-1).$$

Not that $\ell \geq k > n$. Assuming X is very large, we are now going to find $x_1, \dots, x_\ell \in X$ and $X_n, \dots, X_{\ell+1} \subseteq X$, and construct functions $C_{n-1}, \dots, C_\ell$ such that for all $p \geq n$, we have

- X_p is not empty,
- C_p is a function defined on $X_p \setminus \{x_p\}$,
- X_{p+1} is homogeneous for C_p,
- $x_{p+1} = \min(X_{p+1})$, and
- $card(X_{p+1}) \geq \frac{card(X_p) - 1}{c * \binom{p-1}{n-2}}$.

First, let $x_1 < \dots < x_{n-1}$ denote the least $(n-1)$ elements of X and define a function $C_{n-1} \colon X \setminus \{x_1, \dots, x_{n-1}\} \to \{1, \dots, c\}$ by

$$C_{n-1}(x) := C(x_1, \dots, x_{n-1}, x).$$

Then by the pigeonhole principle, there exists some $X_n \subseteq X \setminus \{x_1, \dots, x_{n-1}\}$ such that it is homogeneous for C_{n-1} and

$$card(X_n) \geq \frac{card(X) - (n-1)}{c}.$$

Let $x_n := \min(X_n)$ and define a function C_n on $X_n \setminus \{x_n\}$ by

$$C_n(x) := \{(Y, C(Y \cup \{x_n, x\})) \mid Y \subseteq \{x_1, \dots, x_{n-1}\}\}.$$

Note that for each $x \in X_n \setminus \{x_n\}$, $C_n(x) \colon [\{x_1, \dots, x_{n-1}\}]^{n-2} \to \{1, \dots, c\}$. That is, the image of C_n contains at most $c * \binom{n-1}{n-2}$ many elements. Again by the pigeonhole principle, there exists some $X_{n+1} \subseteq X_n \setminus \{x_n\}$ such that C_n is constant on X_{n+1} and

$$card(X_{n+1}) \geq \frac{card(X_n) - 1}{c * \binom{n-1}{n-2}}.$$

The process above can be iterated for an arbitrary $p \geq n$ as follows. Let $p \geq n$ be given and suppose that $x_1, \dots, x_{p-1}$ and $X_n, X_{n+1}, \dots, X_p$ have been

defined. Suppose also X_p is not empty. Let $x_p := \backslash(X_p)$ and define a function C_p on $X_p \setminus \{x_p\}$ by

$$C_p(x) := \{(Y, C(Y \cup \{x_p, x\})) \mid Y \subseteq \{x_1, \ldots, x_{p-1}\}\}.$$

As explained above, the image of C_p containts at most $c * \binom{p-1}{n-2}$ many elements. Then by the pigeonhole principle, there exists some $X_{p+1} \subseteq X_p \setminus \{x_p\}$ such that C_p is constant on X_{p+1} and

$$card(X_{p+1}) \geq \frac{card(X_p) - 1}{c * \binom{p-1}{n-2}}.$$

Note again that, for any $p \geq n$, the above process can be performed if $card(X)$ is sufficiently large. For this proof, $p = \ell$ is enough. Then, in particular, we have $x_1 < \cdots < x_\ell$.

We now define a function $D \colon [\{1, \ldots, \ell-1\}]^{n-1} \to \{1, \ldots, c\}$ by

$$D(\rho_1, \ldots, \rho_{n-1}) := C(x_{\rho_1}, \ldots, x_{\rho_{n-1}}, x_\ell).$$

By (9), there exists some $Z \subseteq \{1, \ldots, \ell-1\}$ such that Z is s-homogeneous for D and $card(Z) = k - 1$. Finally, we put

$$X' := \{x_\rho \mid \rho \in Z\} \cup \{x_\ell\}.$$

Finally, we claim that X' is s-homogeneous for C.

Proof of the claim. Let

$$H := \{x_{\rho_1}, \ldots, x_{\rho_n}\} \quad \text{and} \quad H' = \{x_{\eta_1}, \ldots, x_{\eta_n}\}$$

be two subsets of X' such that $\rho_1 = \eta_1, \ldots, \rho_s = \eta_s$ and

$$1 \leq \rho_1 < \cdots < \rho_n \leq \ell, \quad 1 \leq \eta_1 < \cdots < \eta_n \leq \ell.$$

Since $x_{\rho_n}, x_\ell \in X_{\rho_n}$, we have $C_{\rho_{n-1}}(x_{\rho_n}) = C_{\rho_{n-1}}(x_\ell)$, hence

$$C(x_{\rho_1}, \ldots, x_{\rho_{n-1}}, x_{\rho_n}) = C(x_{\rho_1}, \ldots, x_{\rho_{n-1}}, x_\ell).$$

Similarly, we can show that

$$C(x_{\eta_1}, \ldots, x_{\eta_{n-1}}, x_{\eta_n}) = C(x_{\eta_1}, \ldots, x_{\eta_{n-1}}, x_\ell).$$

In addition, since $\{\rho_1, \ldots, \rho_{n-1}\} \cup \{\eta_1, \ldots, \eta_{n-1}\} \subseteq Z$, we have

$$D(\rho_1, \ldots, \rho_{n-1}) = D(\eta_1, \ldots, \eta_{n-1}),$$

i.e.,

$$C(x_{\rho_1}, \ldots, x_{\rho_{n-1}}, x_\ell) = C(x_{\eta_1}, \ldots, x_{\eta_{n-1}}, x_\ell).$$

We therefore have $C(H) = C(H')$. So X' is s-homogeneous for C. This proves the claim.

We now turn our attention to the question how large $card(X)$ should be such that the construction above can be carried through find $x_1, \ldots, x_\ell$. The answer can be given by carefully reviewing the above process.

Put first

$$t_n := \frac{card(X) - (n-1)}{c} \quad \text{and} \quad t_{p+1} := \frac{t_p - 1}{c * \binom{p-1}{n-2}},$$

where $n \le p < \ell$. Then

$$t_\ell = c^{-\binom{\ell-2}{n-2}} \cdot \left(c^{-\binom{\ell-3}{n-2}} \cdot \left(\cdots \left(c^{-\binom{n-1}{n-2}} \cdot (t_n - 1)\right) \cdots\right) - 1\right)$$

$$= c^{-\binom{\ell-2}{n-2} - \cdots - \binom{n-1}{n-2}} \cdot t_n - c^{-\binom{\ell-2}{n-2} - \cdots - \binom{n-1}{n-2}} - \cdots - c^{-\binom{\ell-2}{n-2} - \binom{\ell-3}{n-2}} - c^{-\binom{\ell-2}{n-2}}.$$

In order to show that $t_\ell > 0$, we need to show that

$$c^{-\binom{\ell-2}{n-2} - \cdots - \binom{n-1}{n-2}} \cdot t_n > c^{-\binom{\ell-2}{n-2} + \cdots - \binom{n-1}{n-2}} + \cdots + c^{-\binom{\ell-2}{n-2} - \binom{\ell-3}{n-2}} + c^{-\binom{\ell-2}{n-2}}.$$

Since $c = c^{\binom{n-2}{n-2}}$, a sufficient condition on $card(X)$ is then

(10) $\quad card(X) - n + 1 > c^{\binom{\ell-3}{n-2} + \cdots + \binom{n-2}{n-2}} + c^{\binom{\ell-4}{n-2} + \cdots + \binom{n-2}{n-2}} + \cdots + c^{\binom{n-2}{n-2}}.$

A possible value is

(11) $$card(X) = n + \sum_{p=n-1}^{\ell-2} c^{\binom{p}{n-1}}.$$

In fact, (10) can be proved using the so-called Pascal's rule:

$$\binom{q+1}{r+1} = \binom{q}{r} + \binom{q}{r+1}$$

for arbitrary positive integers q, r.

A simple reformulation of (11) results in

$$R_\mu^s(n, c, k) \le n + \sum_{p=n-1}^{\ell-2} c^{\binom{p}{n-1}}$$

$$\le n + \sum_{p=n-1}^{\ell-2} c^{p^{n-1}}$$

$$\le n + \sum_{p=n-1}^{\ell-2} \left(c^{(p+1)^{n-1}} - c^{p^{n-1}}\right) = n + c^{(\ell-1)^{n-1}} - c^{(n-1)^{n-1}}$$

$$\le c^{(\ell-1)^{n-1}} = c^{R_s^{n-1}(c, k-1)^{n-1}}.$$

Finally, we arrive at the recursive relation (8):

$$R_s^n(c, k) * n \le (c^n) * R_s^{n-1}(c, k-1) * (n-1).$$

Moreover, if we apply the above relation $(n-s)$ times, we get

$$R_s^n(c, k) * n \le (c^n) * (c^{n-1}) * \cdots * (c^{s+1}) * R_\mu^s(s, c, k-n+s) * s$$

$$= (c^n) * (c^{n-1}) * \cdots * (c^{s+1}) * (k-n+s) * s.$$

This completes the proof of the theorem. $\qquad\square$

5. Concluding remarks

We introduced and investigated some variants of the finite Ramsey theorem related to s-homogeneity, a general form of homogeneity. As the main result, we showed how to obtain upper bounds for the corresponding Ramsey numbers. This upper bounds could be very useful in getting independence results related to Kanamori-McAloon principle. Related results can be found in [1].

References

[1] L. Carlucci, G. Lee, and A. Weiermann, *Sharp thresholds for hypergraph regressive Ramsey numbers*, J. Combin. Theory Ser. A **118** (2011), no. 2, 558–585.

[2] D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory*, in Surveys in combinatorics 2015, 49–118, London Math. Soc. Lecture Note Ser., **424**, Cambridge Univ. Press, Cambridge, 2015.

[3] P. Erdös, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.

[4] P. Erdös, A. Hajnal, A. Máté, and R. Rado, *Combinatorial set theory*: *partition relations for cardinals*, Studies in Logic and the Foundations of Mathematics, **106**, North-Holland Publishing Co., Amsterdam, 1984.

[5] P. Erdös and R. Rado, *Combinatorial theorems on classifications of subsets of a given set*, Proc. London Math. Soc. (3) **2** (1952), 417–439.

[6] A. Freund, *Proof lengths for instances of the Paris-Harrington principle*, Ann. Pure Appl. Logic **168** (2017), no. 7, 1361–1382.

[7] A. Kanamori and K. McAloon, *On Gödel incompleteness and finite combinatorics*, Ann. Pure Appl. Logic **33** (1987), no. 1, 23–41.

[8] R. Kaye, *Models of Peano Arithmetic*, Oxford Logic Guides, **15**, The Clarendon Press, Oxford University Press, New York, 1991.

[9] G. Lee, *Phase Transitions in Axiomatic Thought*, PhD thesis, University of Münster, May 2005.

[10] J. Paris and L. Harrington, *A mathematical incompleteness in Peano arithmetic. I*, J. Barwise, ed., Handbook of Mathematical Logic, volume 90 of Studies in Logic and the Foundations of Mathematics, pages 1133–1142. North-Holland, 1977.

[11] F. P. Ramsey, *On a Problem of Formal Logic*, Proc. London Math. Soc. (2) **30** (1929), no. 4, 264–286.

[12] A. Weiermann, *A classification of rapidly growing Ramsey functions*, Proc. Amer. Math. Soc. **132** (2004), no. 2, 553–561.

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