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L^p SOLUTIONS FOR GENERAL TIME INTERVAL MULTIDIMENSIONAL BSDES WITH WEAK MONOTONICITY AND GENERAL GROWTH GENERATORS

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Abstract. This paper is devoted to the existence and uniqueness of L^p $(p > 1)$ solutions for general time interval multidimensional backward stochastic differential equations (BSDEs for short), where the generator g satisfies a $(p \wedge 2)$-order weak monotonicity condition in y and a Lipschitz continuity condition in z, both non-uniformly in t. The corresponding stability theorem and comparison theorem are also proved.

1. Preliminaries

In this paper, we assume that k and d are two given positive integers, and $0 \leq T \leq +\infty$ is an extended real number. Let $\mathbf{R}^+ := [0, +\infty)$ and let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$, and $(\mathcal{F}_t)_{t\geq 0}$ be the natural σ-algebra generated by $(B_t)_{t\geq 0}$. We assume that $\mathcal{F}_T = \mathcal{F}$ and $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and complete. The main purpose of this paper is to study the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s,\ t \in [0, T], \tag{1.1}$$

where the terminal condition ξ is an $\mathcal{F}_T$-measurable and k-dimensional random vector, T is called the time horizon, and the generator $g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbf{R}^k \times \mathbf{R}^{k\times d} \mapsto \mathbf{R}^k$ is $(\mathcal{F}_t)$-progressively measurable for each (y, z). The triple (ξ, T, g) is called the parameters of BSDE (1.1), and a pair of $(\mathcal{F}_t)$-progressively measurable processes $(y_t, z_t)_{t\in[0,T]}$ satisfying (1.1) is called a solution of BSDE (1.1).

Now, we introduce some basic notations and definitions, which will be used in the whole paper. Firstly, for each subset $A \subset \Omega \times [0, T]$, let $\mathbf{1}_A = 1$ in

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case of $(\omega,t)\in A$, otherwise $\mathbf{1}_A=0$. Let the Euclidean norm of space $\mathbf{R}^d$ be denoted by $|y|$ for a vector $y\in\mathbf{R}^k$ and let $\langle x,y\rangle$ represent the inner product of $x,y\in\mathbf{R}^k$. Then, for each real number $p>1$, let $L^p(\mathbf{R}^k)$ represent the set of all $\mathbf{R}^k$-valued and $\mathcal{F}_T$-measurable random variables ξ such that $\mathbf{E}[|\xi|^p]<+\infty$ and let $S^p(0,T;\mathbf{R}^k)$ (or S^p simply) denote the set of $\mathbf{R}^k$-valued, $(\mathcal{F}_t)$-adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{S^p}:=\left(\mathbf{E}\left[\sup_{t\in[0,T]}|Y_t|^p\right]\right)^{\frac{1}{p}}<+\infty.$$

Furthermore, let $M^p(0,T;\mathbf{R}^{k\times d})$ (or M^p simply) denote the set of $(\mathcal{F}_t)$-progressively measurable $\mathbf{R}^{k\times d}$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{M^p}:=\left(\mathbf{E}\left[\left(\int_0^T|Z_t|^2\mathrm{d}t\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}<+\infty.$$

Obviously, both S^p and M^p are Banach spaces for each $p>1$. At last, we let $\mathbb{S}$ denote the set of all nondecreasing and concave continuous functions $\kappa(\cdot):\mathbf{R}^+\mapsto\mathbf{R}^+$ satisfying $\kappa(0)=0$, $\kappa(x)>0$ for $x>0$ and $\int_{0^+}\kappa^{-1}(x)\mathrm{d}x=+\infty$.

In this paper, we use the following definition concerning the $L^p\,(p>1)$ solutions of BSDE (1.1).

Definition 1.1. If $(y_t,z_t)_{t\in[0,T]}\in S^p(0,T;\mathbf{R}^k)\times M^p(0,T;\mathbf{R}^{k\times d})$ for some $p>1$ and dP-*a.s.*, BSDE (1.1) holds for each $t\in[0,T]$, then $(y_t,z_t)_{t\in[0,T]}$ is called an L^p solution of BSDE (1.1).

The nonlinear version of finite time interval multidimensional BSDEs were initially introduced in [19], where the authors established the existence and uniqueness for L^2 solutions of BSDEs under the Lipschitz assumption of the generator g. Since then, more and more scholars have been starting to investigate them with great interest, and BSDEs have gradually become an important mathematical tool in many fields such as financial mathematics, stochastic games, optimal control and PDEs and so on, see [1, 14], etc.

On the other hand, many investigators devoted themselves to improving the existence and uniqueness result of [19]. They have extended them to more general case by weakening the assumptions on g, or the L^2 integrability assumptions on ξ and $g(t,0,0)$, or relaxing the finite time terminal to the infinite case, see [1–6, 7–17, 20], etc. We especially mention that [5] first investigated the general time interval BSDEs, and [6] further developed them. Recently, [8] established the existence and uniqueness of an $L^p\,(p>1)$ solution for finite time interval multidimensional BSDEs under a $(p\wedge 2)$-order weak monotonicity condition together with a general growth condition in y of the generator g.

In the light of aforementioned works, the present paper is devoted to the general time interval multidimensional BSDEs with weak monotonicity and general growth generators, which extends the results in [8] to the general time interval case, and the existence and uniqueness result in [20] to the L^p $(p>1)$ solution case. More precisely, in this paper the generator g of BSDE (1.1) satisfies a $(p\wedge 2)$-order weak monotonicity and general growth condition in y and a Lipschitz continuity condition in z, both non-uniformly in time t. The paper is built up as follows. In Section 2, we first introduce some useful assumptions and lemmas, and establish two nonstandard a priori estimates for L^p $(p>1)$ solutions of general time interval multidimensional BSDEs. Then, we prove a stability theorem and an existence and uniqueness theorem in Section 3, and introduce several examples and corollaries in Section 4. Finally, in Section 5 we put forward a new comparison theorem of L^p $(p>1)$ solutions for general time interval one-dimensional BSDEs.

2. Lemmas and a priori estimates

In this section, we introduce several useful lemmas and establish some crucial priori estimates with respect to L^p $(p>1)$ solutions of BSDE (1.1), which will play an important role in proving our main results. The following Lemma 2.1 is a general Gronwall's inequality, which comes from [11].

Lemma 2.1 (Gronwall's inequality)**.** *Let $0\le T\le +\infty$, $\alpha(\cdot):[0,T]\mapsto \mathbf{R}^+$ be a decreasing function, $\beta(\cdot):[0,T]\mapsto \mathbf{R}^+$ satisfy $\int_0^T\beta(t)\mathrm{d}t<+\infty$ and $h(\cdot):[0,T]\mapsto \mathbf{R}^+$ be a continuous function such that*

$$h(t)\le \alpha(t)+\int_t^T \beta(s)h(s)\mathrm{d}s,\ t\in[0,T].$$

Then we have

$$h(t)\le \alpha(t)e^{\int_t^T\beta(s)\mathrm{d}s},\ t\in[0,T].$$

Now, we introduce the following Lemma 2.2, which comes from [10]. It will be frequently used later.

Lemma 2.2. *Suppose that $\kappa(\cdot):\mathbf{R}^+\mapsto \mathbf{R}^+$ is a nondecreasing and concave function with $\kappa(0)=0$. Then, it increases at most linearly, i.e., there exists a constant $A>0$ such that*

$$\kappa(x)\le A(x+1),\ \forall\ x\ge 0.$$

Moreover, for each $m\ge 1$, we have

$$\kappa(x)\le (m+2A)x+\kappa(\frac{2A}{m+2A}),\ \forall\ x\in \mathbf{R}^+.$$

The following Lemma 2.3 is a direct corollary of Lemma 5 in [12], which is a general version of Bihari's inequality.

Lemma 2.3 (Bihari's inequality)**.** *Assume that $0\le T\le +\infty$, $\beta(\cdot):[0,T]\mapsto \mathbf{R}^+$ satisfies $\int_0^T\beta(s)\mathrm{d}s<+\infty$, $h(\cdot):[0,T]\mapsto \mathbf{R}^+$ satisfies $\sup_{t\in[0,T]}h(t)<+\infty$ and*

$$h(t)\le \int_t^T\beta(s)\kappa(h(s))\mathrm{d}s,\ t\in[0,T],$$

where $\kappa \in \mathbb{S}$. Then we have that $h(t) = 0$ for all $t \in [0, T]$.

Next, we establish two nonstandard a priori estimates. The following assumptions on the generator g will be used, where $p > 1$, and $0 \le T \le +\infty$.

(A1) There exist two nonnegative functions $\mu(\cdot), \lambda(\cdot) : [0, T] \mapsto \mathbf{R}^+$ with $\int_0^T (\mu(t)+\lambda^2(t))\mathrm{d}t < +\infty$ such that $\mathrm{d}P\times\mathrm{d}t-a.e.$, $\forall\, (y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\langle y, g(\omega, t, y, z)\rangle \le \mu(t)|y|^2 + \lambda(t)|y||z| + |y|f_t + \varphi_t,$$

where $(f_t)_{t\in[0,T]}$ and $(\varphi_t)_{t\in[0,T]}$ are two nonnegative and $(\mathcal{F}_t)$-progressively measurable processes with

$$\mathbf{E}\left[\left(\int_0^T f_t \mathrm{d}t\right)^p\right] < +\infty \text{ and } \mathbf{E}\left[\left(\int_0^T \varphi_t \mathrm{d}t\right)^{p/2}\right] < +\infty.$$

(A2) There exist two nonnegative functions $\mu(\cdot), \lambda(\cdot) : [0, T] \mapsto \mathbf{R}^+$ with $\int_0^T (\mu(t)+\lambda^2(t))\mathrm{d}t < +\infty$ such that $\mathrm{d}P\times\mathrm{d}t-a.e.$, $\forall\, (y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$|y|^{p-1}\langle \frac{y}{|y|}\mathbf{1}_{|y|\neq 0}, g(\omega, t, y, z)\rangle \le \mu(t)\psi(|y|^p) + \lambda(t)|y|^{p-1}|z| + |y|^{p-1}f_t,$$

where $\psi \in \mathbb{S}$, $(f_t)_{t\in[0,T]}$ is a nonnegative and $(\mathcal{F}_t)$-progressively measurable process with

$$\mathbf{E}\left[\left(\int_0^T f_t \mathrm{d}t\right)^p\right] < +\infty.$$

Proposition 2.1. *Assume that $p > 1$, $0 \le T \le +\infty$, and assumption (A1) holds for the generator g. Let $(y_t, z_t)_{t\in[0,T]}$ solve BSDE (1.1), and $(y_t)_{t\in[0,T]}$ belong to $S^p(0, T; \mathbf{R}^k)$. Then $(z_t)_{t\in[0,T]}$ belongs to $M^p(0, T; \mathbf{R}^{k\times d})$, and for each $0 \le u \le t \le T$, we have*

$$\begin{aligned}
&\mathbf{E}\left[\left(\int_t^T |z_s|^2\mathrm{d}s\right)^{p/2}\middle|\mathcal{F}_u\right]\\
&\le C_p\left[3 + 2\int_t^T \big(\mu(s) + \lambda^2(s)\big)\,\mathrm{d}s\right]^{p/2}\cdot \mathbf{E}\left[\sup_{s\in[t,T]}|y_s|^p\middle|\mathcal{F}_u\right]\\
&\quad + C_p\mathbf{E}\left[\left(\int_t^T f_s\mathrm{d}s\right)^p\middle|\mathcal{F}_u\right] + C_p\mathbf{E}\left[\left(\int_t^T \varphi_s\mathrm{d}s\right)^{p/2}\middle|\mathcal{F}_u\right],
\end{aligned}$$

where C_p is a nonnegative constant depending only on p.

Proof. For each integer $n \ge 1$, we introduce the following $(\mathcal{F}_t)$-stopping time

$$\tau_n = \inf\left\{t \in [0, T] : \int_0^t |z_s|^2\mathrm{d}s \ge n\right\} \wedge T.$$

Then we apply the Itô's formula to $|y_t|^2$, so that for each $n \geq 1$,

$$\begin{aligned} &|y_{t\wedge\tau_n}|^2 + \int_{t\wedge\tau_n}^{\tau_n} |z_s|^2 \mathrm{d}s \\ &= |y_{\tau_n}|^2 + 2\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, g(s, y_s, z_s)\rangle \mathrm{d}s - 2\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, z_s \mathrm{d}B_s\rangle,\ t \in [0, T]. \end{aligned}$$

According to (A1), we can get that $\mathrm{d}P \times \mathrm{d}s - a.e.$,

$$2\langle y_s, g(s, y_s, z_s)\rangle \leq 2(\mu(s) + \lambda^2(s))|y_s|^2 + \frac{|z_s|^2}{2} + 2|y_s| f_s + 2\varphi_s.$$

Then,

$$\begin{aligned} \frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n} |z_s|^2 \mathrm{d}s \leq \left[2 + 2\int_{t\wedge\tau_n}^{\tau_n} (\mu(s) + \lambda^2(s))\mathrm{d}s\right] \sup_{s\in[t\wedge\tau_n, T]} |y_s|^2 + \left(\int_{t\wedge\tau_n}^{T} f_s \mathrm{d}s\right)^2 \\ + 2\int_{t\wedge\tau_n}^{T} \varphi_s \mathrm{d}s + 2\left|\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, z_s \mathrm{d}B_s\rangle\right|,\ t \in [0, T]. \end{aligned}$$

Using the inequality $(a+b)^{p/2} \leq 2^p(a^{p/2} + b^{p/2})$, we can deduce that, for some constant $c_p > 0$ depending only on p,

$$\begin{aligned} &\left(\int_{t\wedge\tau_n}^{\tau_n} |z_s|^2 \mathrm{d}s\right)^{p/2} \\ &\leq c_p \left[2 + 2\int_{t\wedge\tau_n}^{\tau_n} (\mu(s) + \lambda^2(s))\mathrm{d}s\right]^{p/2} \sup_{s\in[t\wedge\tau_n, T]} |y_s|^p + c_p \left(\int_{t\wedge\tau_n}^{T} f_s \mathrm{d}s\right)^p \\ &+ c_p \left(\int_{t\wedge\tau_n}^{T} \varphi_s \mathrm{d}s\right)^{p/2} + c_p \left|\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, z_s \mathrm{d}B_s\rangle\right|^{p/2},\ t \in [0, T]. \end{aligned} \tag{2.1}$$

Furthermore, we can take the conditional mathematical expectation with respect to $\mathcal{F}_u$ in both sides of (2.1) and use the Burkholder-Davis-Gundy (BDG for short) inequality to the process $\{M_t := \int_0^t \langle y_s, z_s \mathrm{d}B_s\rangle\}_{t\in[0,T]}$. Finally, letting $n \to +\infty$ and applying Yong's inequality, Fatou's lemma and Lebesgue's dominated convergence theorem, we can deduce the desired result. □

Proposition 2.2. *Assume that $p > 1$, $0 \leq T \leq +\infty$, and assumption (A2) holds for the generator g. Let $(y_t, z_t)_{t\in[0,T]}$ be an L^p solution of BSDE (1.1). Then there exists a nonnegative constant K_p depending only on p such that for each $0 \leq u \leq t \leq T$,*

$$\begin{aligned} &\mathbf{E}\left[\sup_{s\in[t,T]} |y_s|^p \middle| \mathcal{F}_u\right] \\ &\leq K_p e^{K_p \int_t^T \lambda^2(s)\mathrm{d}s} \left\{\mathbf{E}\left[|\xi|^p \middle| \mathcal{F}_u\right] + \int_t^T \mu(s)\psi\left(\mathbf{E}\left[|y_s|^p \middle| \mathcal{F}_u\right]\right)\mathrm{d}s\right. \end{aligned}$$

$$+\mathbf{E}\left[\left(\int_t^T f_s\mathrm{d}s\right)^p \Big| \mathcal{F}_u\right]\Bigg\}.$$

Proof. By Corollary 2.3 in [2] and assumption (A2), we can obtain that, with probability one, for each $t\in[0,T]$,

$$\begin{aligned}&|y_t|^p + c(p)\int_t^T |y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}|z_s|^2\mathrm{d}s\\ \le\ & |\xi|^p + p\int_t^T [\mu(s)\psi(|y_s|^p)+\lambda(s)|y_s|^{p-1}|z_s| + |y_s|^{p-1}f_s]\mathrm{d}s\\ &- p\int_t^T |y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle,\end{aligned}$$

where $c(p)=p[(p-1)\wedge 1]/2$. It is straightforward to show that $\mathrm{d}P-a.s$,

$$\int_t^T |y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}|z_s|^2\mathrm{d}s \ < \ +\infty,$$

and $\mathrm{d}P\times\mathrm{d}s-a.e.$,

$$\begin{aligned}p\lambda(s)|y_s|^{p-1}|z_s| &= p\left(\frac{\sqrt{2}\lambda(s)}{\sqrt{(p-1)\wedge 1}}|y_s|^{p/2}\right)\left(\sqrt{\frac{(p-1)\wedge 1}{2}}|y_s|^{\frac{p-2}{2}}\mathbf{1}_{|y_s|\neq 0}|z_s|\right)\\ &\le \frac{p\lambda^2(s)}{(p-1)\wedge 1}|y_s|^p + \frac{c(p)}{2}|y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}|z_s|^2.\end{aligned}$$

Then for each $t\in[0,T]$, we have

(2.2)
$$|y_t|^p + \frac{c(p)}{2}\int_t^T |y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}|z_s|^2\mathrm{d}s \le X_t - p\int_t^T |y_s|^{p-2}\mathbf{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle,$$

where

$$X_t = |\xi|^p + \frac{p}{(p-1)\wedge 1}\int_t^T \lambda^2(s)|y_s|^p\mathrm{d}s + p\int_t^T \mu(s)\psi(|y_s|^p)\mathrm{d}s + p\int_t^T |y_s|^{p-1}f_s\mathrm{d}s.$$

Furthermore, by virtue of (2.2), the BDG inequality and Young's inequality, a similar argument as in the proof of Proposition 3 in [8] yields the existence of a constant $k_p>0$ depending only on p such that

$$\begin{aligned}&\mathbf{E}\left[\sup_{s\in[t,T]}|y_s|^p\Big|\mathcal{F}_u\right]\\ \le\ & 2k_p\mathbf{E}\left[|\xi|^p + \frac{p}{(p-1)\wedge 1}\int_t^T \lambda^2(s)|y_s|^p\mathrm{d}s + p\int_t^T \mu(s)\psi(|y_s|^p)\mathrm{d}s\Big|\mathcal{F}_u\right]\\ &+ k_p\mathbf{E}\left[\left(\int_t^T f_s\mathrm{d}s\right)^p\Big|\mathcal{F}_u\right],\ 0\le u\le t\le T.\end{aligned}$$

Now, let

$$h_t = \mathbf{E}\left[\sup_{s\in[t,T]} |y_s|^p|\mathcal{F}_u\right].$$

In the previous inequality, applying Fubini's theorem, Jensen's inequality, and in view of the concavity of $\psi(\cdot)$, we can easily get that for each $0 \le u \le t \le T$,

$$\begin{aligned} h_t \le 2k_p\mathbf{E}\left[|\xi|^p\middle|\mathcal{F}_u\right] + 2pk_p\int_t^T \mu(s)\psi\left(\mathbf{E}\left[|y_s|^p\middle|\mathcal{F}_u\right]\right)\mathrm{d}s \\ + k_p\mathbf{E}\left[\left(\int_t^T f_s\mathrm{d}s\right)^p\middle|\mathcal{F}_u\right] + \frac{2pk_p}{(p-1)\wedge 1}\int_t^T \lambda^2(s)h_s\mathrm{d}s. \end{aligned}$$

At last, by Lemma 2.1 the desired result follows immediately. □

3. Existence and uniqueness theorem and stability theorem

In this section, we will put forward and prove our main results. Firstly, we introduce the following assumptions on the generator g. In stating them, we always suppose that $0 \le T \le +\infty$, $p > 1$ and $u(\cdot), v(\cdot) : [0,T] \mapsto \mathbf{R}^+$ are two deterministic functions with $\int_0^T (u(t) + v^2(t))\mathrm{d}t < +\infty$.

$(\mathrm{H1})_p$ g satisfies a p-order weak monotonicity condition in y, non-uniformly in t, i.e., there exists a function $\rho(\cdot) \in \mathbb{S}$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k\times d}$,

$$|y_1 - y_2|^{p-1}\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbf{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \le u(t)\rho(|y_1 - y_2|^p);$$

(H2) $\mathrm{d}P \times \mathrm{d}t - a.e.$, $\forall\, z \in \mathbf{R}^{k\times d}$, $y \mapsto g(\omega, t, y, z)$ is continuous;

(H3) g has a general growth with respect to y, i.e.,

$$\forall\, \alpha > 0,\ \phi_\alpha(t) := \sup_{|y|\le\alpha} |g(\omega, t, y, 0) - g(\omega, t, 0, 0)| \in L^1([0,T] \times \Omega);$$

(H4) g satisfies a Lipschitz continuity condition in z, non-uniformly in t, i.e., $\mathrm{d}P \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbf{R}^k,\ z_1, z_2 \in \mathbf{R}^{k\times d}$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \le v(t)|z_1 - z_2|;$$

$(\mathrm{H5})_p$ $\mathbf{E}\left[|\xi|^p + \left(\int_0^T |g(t, 0, 0)|\mathrm{d}t\right)^p\right] < +\infty.$

Remark 3.1. By a similar argument to that in the proof of Proposition 1 in [8] we know that $(\mathrm{H1})_p$ implies $(\mathrm{H1})_q$ for each $1 \le p \le q < +\infty$.

The following Proposition 3.1 is a direct corollary of Theorem 6 in [20], which will be used later.

Proposition 3.1. *Assume that* $0 \le T \le +\infty$ *and the generator* g *satisfies* $(\mathrm{H1})_2$, (H2)-(H4) *and* $(\mathrm{H5})_2$. *Then, BSDE* (1.1) *admits a unique* L^2 *solution.*

The following Theorem 3.1 is one of the main results in this paper, which generalizes Proposition 3.1 to the L^p $(p > 1)$ solution case.

Theorem 3.1. *Assume that* $p > 1, 0 \le T \le +\infty$ *and the generator* g *satisfies* $(\mathrm{H1})_{p\wedge 2}$, (H2)-(H4) *and* $(\mathrm{H5})_p$. *Then, BSDE* (1.1) *admits a unique* L^p *solution.*

Before proving Theorem 3.1, we first introduce and prove the following stability theorem. Suppose that $p > 1, 0 \le T \le +\infty$ and for each $n \ge 1$, $(y_t, z_t)_{t\in[0,T]}$ and $(y_t^n, z_t^n)_{t\in[0,T]}$ are, respectively, an L^p solution of BSDE (1.1) and the following BSDE depending on parameter n:

$$y_t^n = \xi^n + \int_t^T g^n(s, y_s^n, z_s^n)\mathrm{d}s - \int_t^T z_s^n \mathrm{d}B_s,\ t \in [0,T].$$

We need the following assumptions.

(B1) $\xi^n \in L^p(\mathbf{R}^k)$ for each $n \ge 1$ and all of g^n satisfy assumptions $(\mathrm{H1})_{p\wedge 2}$ and (H4) with the same $\rho(\cdot)$, $u(\cdot)$ and $v(\cdot)$.

(B2) $\lim\limits_{n\to+\infty} \mathbf{E}\left[|\xi^n - \xi|^p + \left(\int_0^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)|\mathrm{d}s\right)^p\right] = 0.$

Theorem 3.2. *Assume that assumptions* (B1) *and* (B2) *hold true. Then we have*

$$\lim_{n\to+\infty} \mathbf{E}\left[\sup_{s\in[0,T]} |y_s^n - y_s|^p + \left(\int_0^T |z_s^n - z_s|^2 \mathrm{d}s\right)^{p/2}\right] = 0. \tag{3.1}$$

Proof. To begin with, by virtue of (B1) and Remark 3.1, we deduce that for each $n \ge 1$,

(a) $(\mathrm{H1})_p$ holds true for each g^n, together with $u(\cdot)$ and a new and same function $\hat{\rho}(x) \in \mathbb{S}$;
(in case of $1 < p \le 2$, $\hat{\rho}(x) \equiv \rho(x)$)

(b) $(\mathrm{H1})_2$ holds true for each g^n, together with $u(\cdot)$ and a new and same function $\bar{\rho} \in \mathbb{S}$.
(in case of $p \ge 2$, $\bar{\rho}(x) \equiv \rho(x)$)

In the sequel, for each $n \ge 1$, set $\hat{y}_\cdot^n := y_\cdot^n - y_\cdot$, $\hat{z}_\cdot^n := z_\cdot^n - z_\cdot$, and $\hat{\xi}^n := \xi^n - \xi$. Then

$$\hat{y}_t^n = \hat{\xi}^n + \int_t^T \hat{g}^n(s, \hat{y}_s^n, \hat{z}_s^n)\mathrm{d}s - \int_t^T \hat{z}_s^n \mathrm{d}B_s,\ t \in [0,T], \tag{3.2}$$

where for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\hat{g}^n(s, y, z) := g^n(s, y + y_s, z + z_s) - g(s, y_s, z_s).$$

By virtue of assumption (B1) together with (a), it is easy to verify that $\mathrm{d}P \times \mathrm{d}s - a.e.$, for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$,

$$\begin{aligned}&|y|^{p-1}\langle \frac{y}{|y|}\mathbf{1}_{|y|\neq 0}, \hat{g}^n(s, y, z)\rangle \\ &\le u(s)\hat{\rho}(|y|^p) + v(s)|y|^{p-1}|z| + |y|^{p-1}|g^n(s, y_s, z_s) - g(s, y_s, z_s)|.\end{aligned}$$

It follows that the generator $\hat{g}^n$ of BSDE (3.2) satisfies assumption (A2) with

$$\mu(\cdot) = u(\cdot),\ \psi(\cdot) = \hat{\rho}(\cdot),\ \lambda(\cdot) = v(\cdot) \text{ and } f_t = |g^n(t, y_t, z_t) - g(t, y_t, z_t)|.$$

Then, Proposition 2.2 with $u = 0$ yields the existence of a nonnegative constant K_p depending only on p such that for each $n \ge 1$ and each $t \in [0, T]$,

$$\begin{aligned} &\mathbf{E}\left[\sup_{r\in[t,T]} |\hat{y}_r^n|^p\right] \\ &\le K_p e^{K_p \int_t^T v^2(s)\mathrm{d}s} \left\{ \mathbf{E}\left[|\hat{\xi}^n|^p\right] + \int_t^T u(s)\hat{\rho}\left(\mathbf{E}\left[\sup_{r\in[s,T]} |\hat{y}_r^n|^p\right]\right)\mathrm{d}s \right. \\ &\left. + \mathbf{E}\left[\left(\int_t^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)|\mathrm{d}s\right)^p\right]\right\}. \end{aligned} \tag{3.3}$$

Moreover, using (B2), Lemma 2.2 and Lemma 2.1, we can deduce that

$$\sup_{n\ge1} \mathbf{E}\left[\sup_{r\in[0,T]} |\hat{y}_r^n|^p\right] < +\infty.$$

Then, taking account of (B2) and taking limsup in (3.3) with respect to n and using Fatou's lemma, the monotonicity and continuity of $\hat{\rho}(\cdot)$ and Lemma 2.3 we obtain that

$$\lim_{n\to+\infty} \mathbf{E}\left[\sup_{s\in[0,T]} |y_s^n - y_s|^p\right] = 0. \tag{3.4}$$

On the other hand, in view of (B1), (b) and Lemma 2.2 we can also get that $\mathrm{d}P \times \mathrm{d}s - a.e.$, for each $(y, z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$ and $m \ge 1$,

$$\begin{aligned} \langle y, \hat{g}^n(s, y, z)\rangle &\le u(s)(m + 2A)|y|^2 + v(s)|y||z| + u(s)\bar{\rho}(\frac{2A}{m + 2A}) \\ &\quad + |y||g^n(s, y_s, z_s) - g(s, y_s, z_s)|. \end{aligned}$$

It follows that the generator $\hat{g}^n$ of BSDE (3.2) satisfies assumption (A1) with

$$\mu(\cdot) = (m + 2A)u(\cdot),\ \lambda(\cdot) = v(\cdot),\ f_t = |g^n(t, y_t, z_t) - g(t, y_t, z_t)| \text{ and}$$
$$\varphi_t = \bar{\rho}(\frac{2A}{m + 2A})u(t)$$

for each $m \ge 1$. Thus, by Proposition 2.1 with $u = t = 0$ we know that there exists a constant $C_p > 0$ depending only on p such that for each $n, m \ge 1$,

$$\begin{aligned} &\mathbf{E}\left[\left(\int_0^T |\hat{z}_s^n|^2\mathrm{d}s\right)^{p/2}\right] \\ &\le C_p\left(3 + 2\int_0^T \left[(m + 2A)u(s) + v^2(s)\right]\mathrm{d}s\right)^{p/2} \mathbf{E}\left[\sup_{s\in[0,T]} |\hat{y}_s^n|^p\right] \\ &+ C_p\mathbf{E}\left[\left(\int_0^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)|\mathrm{d}s\right)^p\right] + C_p\left(\bar{\rho}(\frac{2A}{m + 2A})\int_0^T u(s)\mathrm{d}s\right)^{p/2}. \end{aligned}$$

Finally, by virtue of (3.4), (B2) and the fact that $\bar{\rho}$ is a continuous function with $\bar{\rho}(0)=0$, we first let $n\to\infty$, and then $m\to\infty$ in the previous inequality to obtain that

$$\lim_{n\to+\infty}\mathbf{E}\left[\left(\int_0^T |z_s^n - z_s|^2 \mathrm{d}s\right)^{p/2}\right]=0.$$

The proof of Theorem 3.2 is complete. □

Based on Theorem 3.2, we can prove Theorem 3.1.

Proof of Theorem 3.1. Assume that $p>1$, $0\le T\le+\infty$, and the generator g satisfies assumptions $(\mathrm{H1})_{p\wedge 2}$ with $u(\cdot)$ and $\rho(x)$, (H2)-(H4) with $v(\cdot)$ and $(\mathrm{H5})_p$. In view of Remark 3.1, we know that $(\mathrm{H1})_2$ also holds true for g with $u(\cdot)$ and a new function $\bar{\rho}(x)$ (in case of $p\ge 2,\ \bar{\rho}(x)\equiv\rho(x)$).

By Theorem 3.2, the uniqueness part of Theorem 3.1 follows immediately. Now, let us show the existence part. Firstly, for each $\gamma\in\mathbf{R}^+$and $x\in\mathbf{R}^k$, we define $q_\gamma(x):=\gamma x/(\gamma\vee|x|)$ and for each $n\ge 1$, let

$$\xi_n := q_n(\xi) \text{ and } g_n(t,y,z):=g(t,y,z)-g(t,0,0)+q_{ne^{-t}}(g(t,0,0)). \tag{3.5}$$

It is clear that, for each $n\ge 1$, the generator g_n satisfies assumptions $(\mathrm{H1})_2$ with $u(\cdot)$ and $\bar{\rho}(x)$, and (H2)-(H4) with $v(\cdot)$. Moreover, for each $n\ge 1$,

$$|\xi_n|\le n,\ \mathrm{d}P - a.s. \text{ and } |g_n(t,0,0)|\le ne^{-t},\ \mathrm{d}P\times\mathrm{d}t - a.e., \tag{3.6}$$

and by virtue of (H5), it follows that

$$\lim_{m,n\to+\infty}\mathbf{E}\left[|\xi_m-\xi_n|^p+\left(\int_0^T |q_{me^{-t}}(g(t,0,0))-q_{ne^{-t}}(g(t,0,0))|\mathrm{d}t\right)^p\right]=0. \tag{3.7}$$

By Proposition 3.1 we know that BSDE with parameters (ξ_n,T,g_n) admits a unique L^2 solution for each $n\ge 1$, denoted by $(y_t^n,z_t^n)_{t\in[0,T]}$.

Since the generator g satisfies $(\mathrm{H1})_2$ with $u(\cdot)$ and $\bar{\rho}(x)$, and (H4) with $v(\cdot)$, then $\mathrm{d}P\times\mathrm{d}t - a.e.$, for each $n\ge 1$ and $(y,z)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}$,

$$\langle y, g_n(t,y,z)\rangle \le u(t)\bar{\rho}(|y|^2)+v(t)|y||z|+|y||q_{ne^{-t}}(g(t,0,0))|. \tag{3.8}$$

Thus, assumption (A2) is satisfied by $g_n(t,y,z)$ with

$$p=2,\ \mu(\cdot)=u(\cdot),\ \psi(\cdot)=\bar{\rho}(\cdot),\ \lambda(\cdot)=v(\cdot),\ f_t=ne^{-t}.$$

From Proposition 2.2 together with (3.6) and Lemmas 2.1-2.2, it follows that for each $n\ge 1$, $(y_t^n)_{t\in[0,T]}$ is a bounded process and then belongs to $S^p(0,T;\mathbf{R}^k)$. Furthermore, combining (3.8) and Lemma 2.2 yields that $\mathrm{d}P\times\mathrm{d}t - a.e.$, for each $n\ge 1$ and $(y,z)\in\mathbf{R}^k\times\mathbf{R}^{k\times d}$,

$$\langle y, g_n(t,y,z)\rangle \le Au(t)|y|^2+v(t)|y||z|+ne^{-t}|y|+Au(t),$$

so that $g_n(t,y,z)$ satisfies assumption (A1) with

$$\mu(t)=Au(t),\ \lambda(t)=v(t),\ f_t=ne^{-t},\ \varphi_t=Au(t).$$

Then, Proposition 2.1 together with (3.6) yields that for each $n \geq 1$, $(z_t^n)_{t\in[0,T]}$ belongs to $M^p(0,T;\mathbf{R}^{k\times d})$.

On the other hand, set $\hat{\xi}^{m,n} = \xi_m - \xi_n, \hat{y}^{m,n}_{\cdot} = y^m_{\cdot} - y^n_{\cdot}, \hat{z}^{m,n}_{\cdot} = z^m_{\cdot} - z^n_{\cdot}$. Then, $(\hat{y}^{m,n}_{\cdot}, \hat{z}^{m,n}_{\cdot})$ is an L^p solution of the following BSDE depending on (m,n):

$$\hat{y}_t^{m,n} = \hat{\xi}^{m,n} + \int_t^T \hat{g}^{m,n}(s, \hat{y}_s^{m,n}, \hat{z}_s^{m,n})\mathrm{d}s - \int_t^T \hat{z}_s^{m,n}\mathrm{d}B_s,\ t\in[0,T], \tag{3.9}$$

where $\hat{g}^{m,n}(s,y,z) := g_m(s, y+y_s^n, z+z_s^n) - g_n(s, y_s^n, z_s^n)$ for each $(y,z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$. It follows from (3.5) that $\mathrm{d}P \times \mathrm{d}t - a.e.$, for each $m,n \geq 1$,

$$\hat{g}^{m,n}(t,y,z) = q_{me^{-t}}(g(t,0,0)) - q_{ne^{-t}}(g(t,0,0)) + g(t, y+y_t^n, z+z_t^n) - g(t, y_t^n, z_t^n).$$

Then, by virtue of the assumptions of the generator g together with (3.7), we can deduce that the assumptions $(\mathrm{H1})_{p\wedge 2}$ and (H4) with $\rho(\cdot)$, $u(\cdot)$ and $v(\cdot)$ are satisfied by the generator $\hat{g}^{m,n}$ of BSDE (3.9) for each $m,n \geq 1$, and

$$\lim_{m,n\to+\infty} \mathbf{E}\left[|\hat{\xi}^{m,n} - 0|^p + \left(\int_0^T |\hat{g}^{m,n}(s,0,0) - \tilde{g}(s,0,0)|\mathrm{d}s\right)^p\right] = 0,$$

where for each $(y,z) \in \mathbf{R}^k \times \mathbf{R}^{k\times d}$, $\tilde{g}(s,y,z) := 0$. Thus, applying Theorem 3.2 to BSDE (3.9) yields that

$$\lim_{m,n\to+\infty} \mathbf{E}\left[\sup_{s\in[0,T]} |\hat{y}_s^{m,n} - 0|^p + \left(\int_0^T |\hat{z}_s^{m,n} - 0|\mathrm{d}s\right)^{p/2}\right] = 0.$$

It means that $\{(y_t^n, z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $S^p(0,T;\mathbf{R}^k) \times M^p(0,T;\mathbf{R}^{k\times d})$. At last, we denote by $(y_t, z_t)_{t\in[0,T]}$ the limit of the sequence $\{(y_t^n, z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ in $S^p(0,T;\mathbf{R}^k) \times M^p(0,T;\mathbf{R}^{k\times d})$, and pass to limit under the uniform convergence in probability for the BSDE with parameters (ξ_n, T, g_n), in view of (H2), (H3) and (H4), to see that $(y_t, z_t)_{t\in[0,T]}$ solves BSDE (1.1). The proof is complete. □

4. Corollaries and examples

In this section, we will introduce several corollaries of Theorem 3.1 and two examples. We suppose that $0 \leq T \leq +\infty$ and $u(\cdot) : [0,T] \to \mathbf{R}^+$ is a deterministic function with $\int_0^T u(t)\mathrm{d}t < +\infty$. The following assumptions on the generator g will be used.

$(\mathrm{H1a})_p$ g satisfies a p-order one-sided Mao condition in y, non-uniformly in t, i.e., there exists a function $\rho(\cdot) \in \mathbb{S}$ such that $\mathrm{d}P \times \mathrm{d}t - a.e.$, $\forall\ y_1, y_2 \in \mathbf{R}^k, z \in \mathbf{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbf{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z) - g(\omega,t,y_2,z)\rangle \leq u(t)\rho^{\frac{1}{p}}(|y_1-y_2|^p);$$

(H1b)$_p$ g satisfies a p-order one-sided Constantin condition in y, non-uniformly in t, i.e., there exists a nondecreasing and concave function $\rho(\cdot):\mathbf{R}^+\mapsto\mathbf{R}^+$ satisfying $\rho(0)=0$, $\rho(u)>0$ for $u>0$ and $\int_{0^+}\frac{u^{p-1}}{\rho^p(u)}\mathrm{d}u=+\infty$, such that $\mathrm{d}P\times\mathrm{d}t-a.e.$, $\forall\ y_1,y_2\in\mathbf{R}^k, z\in\mathbf{R}^{k\times d}$,

$$\langle\frac{y_1-y_2}{|y_1-y_2|}\mathbf{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z)-g(\omega,t,y_2,z)\rangle\leq u(t)\rho(|y_1-y_2|);$$

(H1*) g satisfies a one-sided Osgood condition in y, non-uniformly in t, i.e., there exists a function $\rho(\cdot)\in\mathbb{S}$ such that $\mathrm{d}P\times\mathrm{d}t-a.e.$, $\forall\ y_1,y_2\in\mathbf{R}^k, z\in\mathbf{R}^{k\times d}$,

$$\langle\frac{y_1-y_2}{|y_1-y_2|}\mathbf{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z)-g(\omega,t,y_2,z)\rangle\leq u(t)\rho(|y_1-y_2|).$$

By a similar argument to that in the proof of Proposition 1 in [8], we know that for each $p>1$, (H1a)$_p$ $\Leftrightarrow$(H1b)$_p$ $\Rightarrow$(H1*) $\Rightarrow$(H1)$_p$. Thus, by Theorem 3.1 and Remark 3.1 the following corollaries follow immediately.

Corollary 4.1. *Assume that* $0\leq T\leq+\infty$ *and the generator* g *satisfies assumptions* (H1)$_2$ *and* (H2)-(H4). *Then, if assumption* (H5)$_p$ *holds true for some* $p>2$*, then BSDE* (1.1) *admits a unique* L^p *solution.*

Corollary 4.2. *Suppose that* $p>1, 0\leq T\leq+\infty$ *and assumptions* (H1*) *and* (H2)-(H4) *are satisfied by the generator* g*. Then, if assumption* (H5)$_p$ *also holds true for some* $p>1$*, then BSDE* (1.1) *admits a unique* L^p *solution.*

Corollary 4.3. *Suppose that* $p>1, 0\leq T\leq+\infty$ *and assumptions* (H1a)$_p$ (*or* (H1b)$_p$), (H2)-(H4) *and* (H5)$_p$ *are satisfied by the generator* g*, then BSDE* (1.1) *admits a unique* L^p *solution.*

Example 4.1. Let $k=1, p>1, 0\leq T\leq+\infty$ and

$$g(\omega,t,y,z)=t^2e^{-t}\left(h(|y|)-e^{|B_t(\omega)|y}\right)+\frac{|z|}{\sqrt{1+t^2}},$$

where $h(x):=x|\ln x|^{1/p}\mathbf{1}_{0<x\leq\delta}+(h'(\delta-)(x-\delta)+h(\delta))\mathbf{1}_{x>\delta}$ with $\delta>0$ small enough.

It is not difficult to check that assumptions (H1b)$_p$, (H2)-(H4), and (H5)$_p$ with $u(t)=t^2e^{-t}, v(t)=1/\sqrt{1+t^2}$ and $\rho(x)=h(x)$ are satisfied by this g. Then, it follows from Corollary 4.3 that for each $\xi\in L^p(\mathbf{R}^k)$, BSDE (1.1) admits a unique L^p solution. We remark that this conclusion cannot be obtained by Theorem 1 in [8] and Theorem 6 in [20].

Example 4.2. Let $0\leq T\leq+\infty, y=(y_1,\ldots,y_k)$ and $g(t,y,z)=(g_1(t,y,z),\ldots,g_k(t,y,z))$, where for each $i=1,\ldots,k$,

$$g_i(\omega,t,y,z):=\frac{1}{(1+t)^2}\left(\sigma(y_i)+e^{-y_i}\right)+\frac{|z|}{\sqrt{1+t^2}}+\frac{t^2}{t^4+t},$$

where $\sigma(x):=x|\ln x|\ln|\ln x|\mathbf{1}_{0<x\leq\delta}+\left(\sigma'(\delta-)(x-\delta)+\sigma(\delta)\right)\mathbf{1}_{x>\delta}$ with $\sigma>0$ small enough.

It is straightforward to verify that assumptions (H1*) and (H2)-(H4) with $u(t) = \frac{1}{(1+t)^2}, v(t) = \frac{1}{\sqrt{1+t^2}}$ and $\rho(x) = \sigma(x)$ are satisfied by this g. Then, it follows from Corollary 4.2 that for each $p > 1$, and $\xi \in L^p(\mathbf{R}^k)$, BSDE (1.1) admits a unique L^p solution. This conclusion cannot be obtained by Theorem 1 in [8] and Theorem 6 in [20].

5. A comparison theorem

In this section, we only study the one-dimensional BSDE, i.e., $k = 1$, and establish a general comparison theorem of the L^p solutions for BSDE (1.1) with generators satisfying $(\mathrm{H1})_p$ and (H4), which extends Theorem 3 in [8] to the general time interval BSDEs.

Theorem 5.1. *Assume that $p > 1$, $0 \le T \le +\infty$, $\xi, \xi' \in L^p(\mathbf{R}^k)$, g and g' are two generators of BSDEs, and $(y_\cdot, z_\cdot)$ and $(y'_\cdot, z'_\cdot)$ are, respectively, L^p solutions to the BSDE with parameters (ξ, T, g) and (ξ', T, g). If $\xi \le \xi'$, $\mathrm{d}P - a.s.$, and one of the following two conditions is satisfied:*

(i) *g satisfies $(\mathrm{H1})_p$ and (H4), and $g(t, y'_t, z'_t) \le g'(t, y'_t, z'_t)$, $\mathrm{d}P \times \mathrm{d}t - a.e.$;*

(ii) *g' satisfies $(\mathrm{H1})_p$ and (H4), and $g(t, y_t, z_t) \le g'(t, y_t, z_t)$, $\mathrm{d}P \times \mathrm{d}t - a.e.$;*

then for each $t \in [0, T]$, we have $y_t \le y'_t$, $\mathrm{d}P - a.s.$.

Proof. We only prove the case that (i) is satisfied. The other case can be proved in the same way. Now, we assume that $\xi \le \xi'$, $\mathrm{d}P - a.s.$, and assumptions $(\mathrm{H1})_p$ with $u(\cdot)$ and $\rho(x)$, (H4) with $v(\cdot)$ and $g(t, y'_t, z'_t) \le g'(t, y'_t, z'_t)$, $\mathrm{d}P \times \mathrm{d}t - a.e.$ are satisfied by the generator g. Setting $\hat{y}_t := y_t - y'_t$, $\hat{z}_t := z_t - z'_t$, $\hat{\xi} := \xi - \xi'$. By virtue of the Itô-Tanaka formula (see Exercise VI.**1**.25 in [15]) we know that for each $t \in [0, T]$,

$$
\begin{aligned}
&(\hat{y}_t^+)^p + \frac{p[(p-1)\wedge 1]}{2} \int_t^T |\hat{y}_s|^{p-2} \mathbf{1}_{\hat{y}_s > 0} |\hat{z}_s|^2 \mathrm{d}s \\
&\le (\hat{\xi}^+)^p + p \int_t^T |\hat{y}_s|^{p-1} \mathbf{1}_{\hat{y}_s > 0} [g(s, y_s, z_s) - g'(s, y'_s, z'_s)] \mathrm{d}s \\
&\quad - p \int_t^T |\hat{y}_s|^{p-1} \mathbf{1}_{\hat{y}_s > 0} \hat{z}_s \mathrm{d}B_s.
\end{aligned} \tag{5.1}
$$

By adding and subtracting the term $g(s, y'_s, z'_s)$, in view of the fact that $g(s, y'_s, z'_s) - g'(s, y'_s, z'_s)$ is non-positive, and applying assumptions $(\mathrm{H1})_p$ and (H4) for the generator g together with Young's inequality, we can deduce that $\mathrm{d}P \times \mathrm{d}s - a.e.$,

$$
\begin{aligned}
&p|\hat{y}_s|^{p-1} \mathbf{1}_{\hat{y}_s > 0} [g(s, y_s, z_s) - g'(s, y'_s, z'_s)] \\
&\le pu(s)\rho((\hat{y}_s^+)^p) + pv(s)|\hat{y}_s^+|^{p-1}|\hat{z}_s| \\
&\le pu(s)\rho((\hat{y}_s^+)^p) + \frac{pv^2(s)}{(p-1)\wedge 1} |\hat{y}_s^+|^p + \frac{p[(p-1)\wedge 1]}{4} |\hat{y}_s|^{p-2} \mathbf{1}_{|\hat{y}_s| > 0} |\hat{z}_s|^2
\end{aligned} \tag{5.2}
$$

$$\leq \bar{u}(s)\bar{\rho}((\hat{y}_s^+)^p) + \frac{p[(p-1)\wedge 1]}{4}|\hat{y}_s|^{p-2}\mathbf{1}_{|\hat{y}_s|>0}|\hat{z}_s|^2,$$

where

$$\bar{u}(s) := pu(s) + \frac{p}{(p-1)\wedge 1}v^2(s) \text{ and } \bar{\rho}(u) := \rho(u) + u.$$

It is clear that $\int_0^T \bar{u}(t)\mathrm{d}t < +\infty$, and $\bar{\rho}(\cdot)$ is a nondecreasing concave function with $\bar{\rho}(0) = 0$ and $\bar{\rho}(u) > 0$ for $u > 0$. Furthermore, since $\xi \leq \xi'$ $\mathrm{d}P - a.s.$, it follows from (5.1) and (5.2) that

$$(\hat{y}_t^+)^p \leq \int_t^T \bar{u}(s)\bar{\rho}((\hat{y}_s^+)^p)\mathrm{d}s - p\int_t^T |\hat{y}_s|^{p-1}\mathbf{1}_{\hat{y}_s>0}\hat{z}_s\mathrm{d}B_s,\ t \in [0,T].$$

Taking mathematical expectation in the previous inequality and in view of fact that $\{p\int_t^T |\hat{y}_s|^{p-1}\mathbf{1}_{\hat{y}_s>0}\hat{z}_s\mathrm{d}B_s\}_{t\in[0,T]}$ is a martingale, we can deduce that, by virtue of Fubini's theorem and Jensen's inequality,

$$\mathbf{E}[(\hat{y}_t^+)^p] \leq \int_t^T \bar{u}(s)\bar{\rho}(\mathbf{E}[(\hat{y}_s^+)^p])\mathrm{d}s,\ t \in [0,T].$$

Since $\rho \in \mathbb{S}$, it is not difficult to verify that $\int_{0^+} \frac{1}{\bar{\rho}(u)}\mathrm{d}u = +\infty$, and then $\bar{\rho}(\cdot) \in \mathbb{S}$. Thus, Lemma 2.3 yields that for each $t \in [0,T]$, $\mathbf{E}[(\hat{y}_t^+)^p] = 0$ and then $y_t \leq y'_t$, $\mathrm{d}P - a.s.$. The proof is complete. □

References

[1] C. Bender and M. Kohlmann, *BSDEs with stochastic Lipschitz condition*, http://cofe.uni-konstanz.de /Papers/dp0008.pdf, 2000.

[2] P. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica, *L^p solutions of backward stochastic differential equations*, Stochastic Process. Appl. **108** (2003), no. 1, 109–129.

[3] P. Briand and R. Elie, *A simple constructive approach to quadratic BSDEs with or without delay*, Stochastic Process. Appl. **123** (2013), no. 8, 2921–2939.

[4] P. Briand and Y. Hu, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, Probab. Theory Related Fields **141** (2008), no. 3-4, 543–567.

[5] Z. J. Chen, *Existence of solutions to backward stochastic differential equations with stopping times*, Kexue Tongbao (Chinese) **42** (1997), no. 22, 2379–2382.

[6] Z. Chen and B. Wang, *Infinite time interval BSDEs and the convergence of g-martingales*, J. Aust. Math. Soc. Ser. A **69** (2000), no. 2, 187–211.

[7] F. Delbaen, Y. Hu, and A. Richou, *On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions*, Ann. Inst. Henri Poincaré Probab. Stat. **47** (2011), no. 2, 559–574.

[8] S. Fan, *L^p solutions of multidimensional BSDEs with weak monotonicity and general growth generators*, J. Math. Anal. Appl. **432** (2015), no. 1, 156–178.

[9] ———, *Bounded solutions, L^p $(p > 1)$ solutions and L^1 solutions for one dimensional BSDEs under general assumptions*, Stochastic Process. Appl. **126** (2016), no. 5, 1511–1552.

[10] S. Fan and L. Jiang, *Uniqueness result for the BSDE whose generator is monotonic in y and uniformly continuous in z*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 1-2, 89–92.

[11] ———, *L^p solutions of finite and infinite time interval BSDEs with non-Lipschitz coefficients*, Stochastics **84** (2012), no. 4, 487–506.

[12] S. Fan, L. Jiang, and D. Tian, *One-dimensional BSDEs with finite and infinite time horizons*, Stochastic Process. Appl. **121** (2011), no. 3, 427–440.

[13] Y. Hu and S. Tang, *Multi-dimensional backward stochastic differential equations of diagonally quadratic generators*, Stochastic Process. Appl. **126** (2016), no. 4, 1066–1086.
[14] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab. **28** (2000), no. 2, 558–602.
[15] J.-F. Le Gall, *Mouvement brownien, martingales et calcul stochastique*, Mathématiques & Applications (Berlin), **71**, Springer, Heidelberg, 2013.
[16] J. P. Lepeltier and J. San Martin, *Backward stochastic differential equations with continuous coefficient*, Statist. Probab. Lett. **32** (1997), no. 4, 425–430.
[17] ———, *Existence for BSDE with superlinear-quadratic coefficient*, Stochastics Stochastics Rep. **63** (1998), no. 3-4, 227–240.
[18] Y. Liu, D. Li, and S. Fan, *L^p $(p > 1)$ solutions of BSDEs with generators satisfying some non-uniform conditions in t and ω*, arXiv: 1603.00259v1 [math. PR] 1 Mar 2016.
[19] Pardoux and S. G. Peng, *Adapted solution of a backward stochastic differential equation*, Systems Control Lett. **14** (1990), no. 1, 55–61.
[20] L. Xiao and S. Fan, *General time interval BSDEs under the weak monotonicity condition and nonlinear decomposition for general g-supermartingales*, Stochastics **89** (2017), no. 5, 786–816.

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