

HOMOTOPY MINIMAL PERIODS OF MAPS ON THE KLEIN BOTTLE

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ABSTRACT. We determine the sets of homotopy minimal periods of all self-maps on the Klein bottle by using a single formula for homotopy minimal periods of maps on the infra-solvmanifolds of type (R). This provides an alternate but an easy proof for the main results of [12].

1. Introduction

Let $f : X \rightarrow X$ be a self-map on a topological space X . We define the following: The set of *periodic points* of f with *minimal period* n

$$P_n(f) = \text{Fix}(f^n) - \bigcup_{k < n} \text{Fix}(f^k)$$

and the set of *homotopy minimal periods* of f

$$\text{HPer}(f) = \bigcap_{g \simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}.$$

The famous Šarkovs'kii theorem characterizes the dynamics (minimal periods) of a map of interval [21]. The set of minimal periods of maps on the circle has been completely described in [1]. This led to a problem of study the set of *homotopy minimal periods* of self-maps. Such an invariant gives an information about rigid dynamics of self-maps.

This problem was successfully studied in [9] when the space is a torus of any dimension, and this was extended in [7] (see also [8, 15]) to any nilmanifold, and in [5, 16] and [10] to the special solvmanifolds modeled on Sol^3 and Sol_1^4 respectively. When X is the Klein bottle, the same problem was studied in [12], and when X is an infra-nilmanifold and f is an expanding map, it was shown in [2, 13, 14] that $\text{HPer}(f)$ is co-finite. More information was sought when $\text{HPer}(f)$ becomes infinite. When X is a flat manifold, some sufficient

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conditions on X and f for $\text{HPer}(f)$ to be infinite were found in [3, 19]. This was generalized from flat manifolds to infra-solvmanifolds of type (R) in [17].

The purpose of this paper is to give an alternate but an easy proof of the main results in [12] by using a single formula (Theorem 2.3) for the homotopy minimal periods of maps on the infra-solvmanifolds of type (R).

2. The Klein bottle maps

Let $\alpha = (\mathbf{a}, A)$ and $\tau_i = (\mathbf{e}_i, I_2)$ be elements of $\text{Aff}(\mathbb{R}^2) = \mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$, where

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then A has period 2, $(\mathbf{a}, A)^2 = (\mathbf{a} + A\mathbf{a}, I_2) = (\mathbf{e}_1, I_2)$, and $\tau_2\alpha = \alpha\tau_2^{-1}$. Let Γ be the subgroup generated by τ_1 and τ_2 . Then it forms a lattice of \mathbb{R}^2 and $\Gamma \backslash \mathbb{R}^2$ is the 2-torus. It is easy to check that the subgroup

$$\Pi = \langle \Gamma, (\mathbf{a}, A) \rangle \subset \text{Aff}(\mathbb{R}^2)$$

generated by the lattice Γ and the element (\mathbf{a}, A) is discrete and torsion free. Furthermore, Γ is a normal subgroup of Π of index 2. Thus Π is a Bieberbach group, which is the Klein bottle group, and the quotient space $\Pi \backslash \mathbb{R}^2$ is the Klein bottle. Thus $\Gamma \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ is a double covering projection.

Lemma 2.1 ([11, Lemma 2.1], [12, Lemma 3.1]). *Any homomorphism $\varphi : \Pi \rightarrow \Pi$ on the Klein bottle group Π is given as follows:*

$$\varphi(\alpha) = \alpha^r \tau_2^\ell, \quad \varphi(\tau_2) = \tau_2^q,$$

where either r is odd, or r is even and $q = 0$.

Let $f : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi \backslash \mathbb{R}^2$. Fix a lifting $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of f . Then the lifting \tilde{f} induces a homomorphism $\varphi : \Pi \rightarrow \Pi$ which is defined by the following rule:

$$\varphi(\beta) \circ \tilde{f} = \tilde{f} \circ \beta \quad \text{for all } \beta \in \Pi.$$

The homomorphism φ is called a *homomorphism of type (r, ℓ, q)* induced by f . In this case, f is said to be of *type (r, ℓ, q)* .

Recall from [12, Proposition 3.3 and Theorem 3.4] the following homotopy classification of all maps on the Klein bottle.

Lemma 2.2. *Every continuous map on the Klein bottle $\Pi \backslash \mathbb{R}^2$ is homotopic to a map of type (r, ℓ, q) where if r is odd, then $\ell = 0, 1$ and $q \geq 0$; and if r is even and $q = 0$, then $\ell \geq 0$. Furthermore, two such maps of type (r, ℓ, q) and (r', ℓ', q') are homotopic if and only if $r = r'$, $q = q'$ and $\ell = \ell'$.*

Let $f : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ induce a homomorphism φ on Π of type (r, ℓ, q) . By Lemma 2.1, φ must map Γ into Γ itself. Thus f always has a lifting $\tilde{f} : \Gamma \backslash \mathbb{R}^2 \rightarrow \Gamma \backslash \mathbb{R}^2$ so that the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{R}^2 & \xrightarrow{\tilde{f}} & \mathbb{R}^2 \\
 \downarrow p' & & \downarrow p' \\
 \Gamma \backslash \mathbb{R}^2 & \xrightarrow{\bar{f}} & \Gamma \backslash \mathbb{R}^2 \\
 \downarrow \bar{p} & & \downarrow \bar{p} \\
 \Pi \backslash \mathbb{R}^2 & \xrightarrow{f} & \Pi \backslash \mathbb{R}^2
 \end{array}$$

On the other hand, for such a homomorphism φ there exists an affine map $(\mathbf{c}, F) \in \mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$ such that

$$\varphi(\beta)(\mathbf{c}, F) = (\mathbf{c}, F)\beta, \quad \varphi(\tau_2)(\mathbf{c}, F) = (\mathbf{c}, F)\tau_2.$$

This is due to Theorem 1.1 of [18]. These equalities yield that

$$(\mathbf{c}, F) = \begin{cases} \left(\begin{pmatrix} -\frac{1}{2} & * \\ & \ell \end{pmatrix}, \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix} \right) & \text{if } r \text{ is odd;} \\ \left(\begin{pmatrix} * \\ * \\ * \end{pmatrix}, \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix} \right) & \text{if } r \text{ is even and } q = 0. \end{cases}$$

Furthermore, from the above equalities we see that the affine map (\mathbf{c}, F) on \mathbb{R}^2 induces a map $\Phi_{(\mathbf{c}, F)} : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$. Since $\varphi|_\Gamma = F|_\Gamma$, the endomorphism F on \mathbb{R}^2 induces a map $\phi_F : \Gamma \backslash \mathbb{R}^2 \rightarrow \Gamma \backslash \mathbb{R}^2$. Clearly the maps $\Phi_{(\mathbf{c}, F)}$ and ϕ_F induce homomorphisms φ and $\varphi|_\Gamma$, respectively. Since $\Pi \backslash \mathbb{R}^2$ and $\Gamma \backslash \mathbb{R}^2$ are $K(\pi, 1)$ manifolds, it follows that $\Phi_{(\mathbf{c}, F)} \simeq f$ and $\phi_F \simeq \bar{f}$.

Since the set of homotopy minimal periods is a homotopy invariant, we may assume in what follows that $\Phi_{(\mathbf{c}, F)} = f$ and $\phi_F = \bar{f}$ so that the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{R}^2 & \xrightarrow{(\mathbf{c}, F)} & \mathbb{R}^2 \\
 \downarrow p' & & \downarrow p' \\
 \Gamma \backslash \mathbb{R}^2 & \xrightarrow{\phi_F} & \Gamma \backslash \mathbb{R}^2 \\
 \downarrow \bar{p} & & \downarrow \bar{p} \\
 \Pi \backslash \mathbb{R}^2 & \xrightarrow{\Phi_{(\mathbf{c}, F)}} & \Pi \backslash \mathbb{R}^2
 \end{array}$$

Recall that the Klein bottle is a two-dimensional infra-solvmanifold of type (R). In what follows, we will use the following result to compute the homotopy minimal periods of maps on the Klein bottle.

Theorem 2.3 ([17, Theorem 2.4], [19, Theorem 3.1]). *Let $f : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ be a self-map on the Klein bottle $\Pi \backslash \mathbb{R}^2$ with an affine homotopy lift $(\mathbf{c}, F) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Let $\varphi : \Pi \rightarrow \Pi$ be the homomorphism induced by (\mathbf{c}, F) , i.e.,*

$$\varphi(\beta)(\mathbf{c}, F) = (\mathbf{c}, F)\beta \quad \forall \beta \in \Pi.$$

Then

$$\text{HPer}(f) = \left\{ k \mid \begin{array}{l} \exists \delta = (d, D) \in \Pi \text{ such that } \det(I - DF^k) \neq 0 \text{ and} \\ \forall d < k \text{ with } d \mid k, \forall \beta \in \Pi, \\ \delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta) \cdots \varphi^{k-d}(\beta) \end{array} \right\}.$$

Corollary 2.4. *Let $f : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ be a self-map on the Klein bottle $\Pi \backslash \mathbb{R}^2$ with an affine homotopy lift $(\mathbf{c}, F) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then*

- (1) $1 \in \text{HPer}(f)$ if and only if $\det(I - DF) \neq 0$ for some $\delta = (d, D) \in \Pi$.
- (2) For $k > 1$, $k \notin \text{HPer}(f)$ if and only if for any $\delta = (d, D) \in \Pi$,
 - (a) $\det(I - DF^k) = 0$, or
 - (b) there exist $d \mid k$ ($d < k$) and $\beta \in \Pi$ such that

$$\delta = \beta\varphi^d(\beta)\varphi^{2d}(\beta) \cdots \varphi^{k-d}(\beta).$$

Remark 2.5. The identity

$$\text{HPer}(f) = \{k \mid \text{NP}_k(f) \neq 0\}$$

on the Klein bottle was proved in [12] and [6, Theorem 3.1]. The set on the right hand side of the identity can be rephrased as the set in Theorem 2.3 when the space is the Klein bottle or in general an infra-solvmanifold of type (R), see for example [17]. Using this set in Theorem 2.3, we will determine the set of homotopy minimal periods.

3. Computation

Let $f : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$ be a self-map on the Klein bottle $\Pi \backslash \mathbb{R}^2$ with an affine homotopy lift $(\mathbf{c}, F) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Then we may assume that $f = \Phi_{(\mathbf{c}, F)}$, that is, f is the map induced by the affine map (\mathbf{c}, F) of \mathbb{R}^2 , where

$$F = \begin{cases} \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix} & \text{when } r \text{ is odd;} \\ \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix} & \text{when } r \text{ is even and } q = 0. \end{cases}$$

Note also that every element of the Klein bottle group $\Pi = \langle \tau_1, \tau_2, \alpha = (\mathbf{a}, A) \rangle$ is of the form $\tau_1^y \tau_2^z$ or $\alpha \tau_1^y \tau_2^z$. For any $\delta = (d, D)$ in Π , we thus have $D = I$ or $D = A$. I.e.,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix}.$$

3.1. Case: r is even

In this case, $F = \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix}$ and $q = 0$. Then

$$\varphi(\alpha) = \alpha^r \tau_2^\ell, \quad \varphi(\tau_1) = \tau_1^r \tau_2^{2\ell}, \quad \varphi(\tau_2) = 1$$

and

$$I - DF^k = \begin{bmatrix} 1 - r^k & 0 \\ \mp 2r^{k-1}\ell & 1 \end{bmatrix}.$$

Hence $\det(I - DF^k) = 1 - r^k \neq 0$ because r is even. By Corollary 2.4, $1 \in \text{HPer}(f)$. For $k > 1$, by Theorem 2.3, we have that $k \in \text{HPer}(f)$ if and only if there exists $\delta \in \Pi$ such that for all $d < k$ with $d \mid k$ and for all $\beta \in \Pi$,

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\dots\varphi^{k-d}(\beta)$$

if and only if

$$\bigcup_{\substack{d < k \\ d \mid k}} \{ \beta\varphi^d(\beta)\varphi^{2d}(\beta)\dots\varphi^{k-d}(\beta) \mid \beta \in \Pi \} \neq \Pi.$$

Proposition 3.1. *If $r = 0$, then $\text{HPer}(f) = \{1\}$.*

Proof. It remains to show that if $k > 1$, then $k \notin \text{HPer}(f)$. By Corollary 2.4, it suffices to show that

$$\{ \beta\varphi(\beta)\varphi^2(\beta)\dots\varphi^{k-1}(\beta) \mid \beta \in \Pi \} = \Pi.$$

It is easy to see that

$$\varphi(\beta) = \begin{cases} \tau_2^{2\ell y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \tau_2^{\ell+2\ell y} & \text{when } \beta = \alpha\tau_1^y \tau_2^z, \end{cases}$$

and so $\varphi^2(\beta) = \dots = \varphi^{k-1}(\beta) = 1$. Consequently, we have

$$\beta\varphi(\beta)\varphi^2(\beta)\dots\varphi^{k-1}(\beta) = \begin{cases} \tau_1^y \tau_2^{z+2\ell y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \alpha\tau_1^y \tau_2^{z+\ell+2\ell y} & \text{when } \beta = \alpha\tau_1^y \tau_2^z. \end{cases}$$

This proves that

$$\{ \beta\varphi(\beta)\varphi^2(\beta)\dots\varphi^{k-1}(\beta) \mid \beta \in \Pi \} = \Pi.$$

Consequently, if $k > 1$, then $k \notin \text{HPer}(f)$. This proves our proposition. □

Now, we shall consider the case where the even number r is $\neq 0$. For any integer d with $0 < d < k$ and $d \mid k$, we can observe easily that

$$\varphi^d(\beta) = \begin{cases} \tau_1^{r^d y} \tau_2^{2\ell r^{d-1} y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \tau_1^{r^d(\frac{1}{2}+y)} \tau_2^{2\ell r^{d-1}(\frac{1}{2}+y)} & \text{when } \beta = \alpha\tau_1^y \tau_2^z, \end{cases}$$

and hence

$$(3.1) \quad \begin{aligned} & \beta\varphi^d(\beta)\varphi^{2d}(\beta)\dots\varphi^{k-d}(\beta) \\ &= \begin{cases} \tau_1^{\frac{1-r^k}{1-r^d} y} \tau_2^{z+2\ell \frac{r^d-r^k}{r(1-r^d)} y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \alpha\tau_1^{\frac{1-r^k}{1-r^d} y + \frac{r^d-r^k}{2(1-r^d)}} \tau_2^{z+2\ell \frac{r^d-r^k}{r(1-r^d)} (\frac{1}{2}+y)} & \text{when } \beta = \alpha\tau_1^y \tau_2^z. \end{cases} \end{aligned}$$

The following lemma is crucial in using Theorem 2.3 and its corollary.

Lemma 3.2. *Let k and r be integers. If $|r| \geq 2$, then*

$$\gcd \left\{ \frac{1-r^k}{1-r^d} : d \mid k, d < k \right\} \neq 1$$

for all $k \geq 3$.

Proof. Let

$$\Phi_k(x) = \gcd \left\{ \frac{x^k - 1}{x^d - 1} : 0 < d < k, d \mid k \right\}.$$

Then it is obvious that

$$\Phi_k(x) = \frac{x^k - 1}{\text{lcm} \{x^d - 1 : 0 < d < k, d \mid k\}}.$$

This means that $\Phi_k(x)$ is the k th *cyclotomic polynomial* of x . We refer to [4] for a background on cyclotomic polynomials.

Assume $k \geq 3$. It is known that

$$(x - 1)^{\varphi(k)} < \Phi_k(x) < (x + 1)^{\varphi(k)}$$

for all $x > 0$. It follows that $\Phi_k(x) > (x - 1)^{\varphi(k)} \geq (2 - 1)^{\varphi(k)} = 1$ for all values of $x \geq 2$.

For $k \geq 3$ odd, it is also known that $\Phi_{2k}(x) = \Phi_k(-x)$. Hence $\Phi_k(-x) = \Phi_{2k}(x) > 1$ for all values of $x \geq 2$.

It remains to show that $\Phi_k(-x) \neq \pm 1$ when $k \geq 3$ is even and $x \geq 2$. Write $k = 2^m n$ where $m \geq 1$ and n is odd. In this case, it is known that

$$\Phi_k(-x) = \Phi_{2^m n}(-x) = \Phi_{2n}((-x)^{2^{m-1}}).$$

If $m \geq 2$ and $n \geq 3$ is odd, then $\Phi_k(-x) = \Phi_{2n}((-x)^{2^{m-1}}) = \Phi_{2n}(x^{2^{m-1}}) = \Phi_n(-x^{2^{m-1}}) > 1$ for all values of $x \geq 2$. If $m = 1$, then $n \geq 3$ (since $k \geq 3$) and so $\Phi_k(-x) = \Phi_{2n}(-x) = \Phi_n(x) > 1$ for all values of $x \geq 2$. If $m \geq 2$ and $n = 1$, then $\Phi_k(x) = \Phi_{2^m}(x) = \Phi_2(x^{2^{m-1}}) = x^{2^{m-1}} + 1$, and $\Phi_k(-x) = (-x)^{2^{m-1}} + 1 = x^{2^{m-1}} + 1 > 1$ for all values of $x \geq 2$.

In all, we have shown that when $k \geq 3$, $\Phi_k(x) \neq \pm 1$ for all values of $|x| \geq 2$. This proves our lemma. \square

Proposition 3.3. *If $r = -2$, then $\text{HPer}(f) = \mathbb{N} - \{2\}$.*

Proof. Since $1 \in \text{HPer}(f)$, we will show that $2 \notin \text{HPer}(f)$ and $k \in \text{HPer}(f)$ for all $k \geq 3$. By (3.1) with $k = 2$, we have

$$\{\beta\varphi(\beta) \mid \beta \in \Pi\} = \left\{ \tau_1^{-y} \tau_2^{z+2\ell y}, \alpha \tau_1^{-y-1} \tau_2^{z+2\ell(\frac{1}{2}+y)} \mid y, z \in \mathbb{Z} \right\} = \Pi.$$

This shows from Corollary 2.4 that $2 \notin \text{HPer}(f)$.

Next, we will show that $k \in \text{HPer}(f)$ for all $k > 2$. Observe from (3.1) with $k \geq 3$ that

$$\beta\varphi^d(\beta)\varphi^{2d}(\beta) \cdots \varphi^{k-d}(\beta)$$

$$= \begin{cases} \tau_1^{\frac{1-(-2)^k}{1-(-2)^d}y} \tau_2^{z+2\ell \frac{(-2)^d-(-2)^k}{(-2)(1-(-2)^d)}y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \alpha \tau_1^{\frac{1-(-2)^k}{1-(-2)^d}y + \frac{(-2)^d-(-2)^k}{2(1-(-2)^d)}} \tau_2^{z+2\ell \frac{(-2)^d-(-2)^k}{(-2)(1-(-2)^d)}(\frac{1}{2}+y)} & \text{when } \beta = \alpha \tau_1^y \tau_2^z. \end{cases}$$

As an example if $k = 3$, then $d = 1$ and so

$$\beta \varphi(\beta) \varphi^2(\beta) = \begin{cases} \tau_1^{3y} \tau_2^{z-2\ell y} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \alpha \tau_1^{3y+1} \tau_2^{z-2\ell(\frac{1}{2}+y)} & \text{when } \beta = \alpha \tau_1^y \tau_2^z. \end{cases}$$

We have a look at the exponent of τ_1 in $\beta \varphi(\beta) \varphi^2(\beta)$ when $\beta = \tau_1^y \tau_2^z$. The exponent of τ_1 is $3y$. This shows that the element $\delta = \tau_1$ cannot be expressed as $\beta \varphi(\beta) \varphi^2(\beta)$. Consequently, by Theorem 2.3, we have $3 \in \text{HPer}(f)$.

In general, we consider the exponents of τ_1 in $\beta \varphi^d(\beta) \varphi^{2d}(\beta) \dots \varphi^{k-d}(\beta)$ when $\beta = \tau_1^y \tau_2^z$, which are $\frac{1-(-2)^k}{1-(-2)^d}y$ where $d \mid k$ with $d < k$. From Lemma 3.2 it follows that all $k \geq 3$ belong to $\text{HPer}(f)$. \square

Proposition 3.4. *If r is even with $r \geq 2$ or $r \leq -4$, then $\text{HPer}(f) = \mathbb{N}$.*

Proof. We have shown before that $1 \in \text{HPer}(f)$. Consider $k = 2$ in (3.1). Then $\frac{1-r^k}{1-r^d} = \frac{1-r^2}{1-r} = 1+r$, which is not ± 1 since $r \neq 0, -2$. Hence by Theorem 2.3, $2 \in \text{HPer}(f)$.

Now consider $k \geq 3$. By Lemma 3.2, we have

$$\bigcup_{d \mid k, d < k} \{ \beta \varphi^d(\beta) \varphi^{2d}(\beta) \dots \varphi^{k-d}(\beta) \mid \beta \in \Pi \} \neq \Pi.$$

Therefore, by Theorem 2.3, $k \in \text{HPer}(f)$. \square

3.2. Case: r is odd

In this the case, $F = \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix}$. Then

$$\varphi(\alpha) = \alpha^r \tau_2^\ell, \quad \varphi(\tau_1) = \tau_1^r, \quad \varphi(\tau_2) = \tau_2^q$$

with $\ell = 0$ or 1 and $q \geq 0$ (see Lemma 2.2), and

$$I - DF^k = \begin{bmatrix} 1 - r^k & 0 \\ 0 & 1 \mp q^k \end{bmatrix}.$$

Hence $\det(I - DF^k) = (1 - r^k)(1 \mp q^k)$, which is 0 if and only if $r^k = 1$ or $q = 1$.

Proposition 3.5. *If $r = 1$, then $\text{HPer}(f) = \emptyset$.*

Proof. By the above observation, $\det(I - DF^k) = 0$ for all $k \geq 1$. This shows by Corollary 2.4 that $\text{HPer}(f) = \emptyset$. \square

Let r be an odd integer with $r \neq 1$. Then

$$(3.2) \quad \varphi^d(\beta) = \begin{cases} \tau_1^{r^d y} \tau_2^{q^d z} & \text{when } \beta = \tau_1^y \tau_2^z, \\ \alpha^{r^d} \tau_1^{r^d y} \tau_2^{q^d z + (q^{d-1} + \dots + q + 1)\ell} & \text{when } \beta = \alpha \tau_1^y \tau_2^z. \end{cases}$$

Proposition 3.6. *If $r = -1$, then $\text{HPer}(f) = \begin{cases} \{1\} & \text{when } q = 0, 1, \\ \mathbb{N} - 2\mathbb{N} & \text{when } q \geq 2. \end{cases}$*

Proof. Note that $\det(I - AF) = (1 - r)(1 + q) = 2(1 + q) \geq 2$ for in particular $(d, D) = (\mathbf{a}, A) \in \Pi$. By Corollary 2.4, $1 \in \text{HPer}(f)$.

Observe also that $\det(I - DF^{2k}) = (1 - (-1)^{2k})(1 \mp q^{2k}) = 0$ for all $(d, D) \in \Pi$. By Corollary 2.4, $2k \notin \text{HPer}(f)$ for all $k \geq 1$.

On the other hand, remark that $\det(I - DF^{2k+1}) = (1 - (-1)^{2k+1})(1 \mp q^{2k+1}) = 2(1 \mp q^{2k+1}) = 0$ if and only if $q = 1$ and $D = I$. Next we need to determine when $2k + 1 \in \text{HPer}(f)$.

STEP 1: $q = 1 \Rightarrow \text{HPer}(f) = \{1\}$.

In this case, it remains to show that all $2k + 1 \notin \text{HPer}(f)$ with $k > 0$. As observed just before, $\det(I - DF^{2k+1}) \neq 0$ for all $\delta = (d, D)$ of the form $\alpha \tau_1^y \tau_2^z$. By Corollary 2.4, it suffices to show that the condition (b) holds for all $\delta = (d, D)$ of the form $\alpha \tau_1^y \tau_2^z$.

Let $\beta = \alpha \tau_1^y \tau_2^z$. If $d \mid 2k + 1$, then we have d is odd and $(2k + 1) - d$ is an even multiple of d . Hence by (3.2)

$$\begin{aligned} & \beta \varphi^d(\beta) \varphi^{2d}(\beta) \dots \varphi^{(2k+1)-d}(\beta) \\ &= (\alpha \tau_1^y \tau_2^z)(\alpha^{-1} \tau_1^{-y} \tau_2^{z+d\ell})(\alpha \tau_1^y \tau_2^{z+2d\ell}) \dots (\alpha \tau_1^y \tau_2^{z+((2k+1)-d)\ell}) \\ &= \alpha \tau_1^y \tau_2^{z + \frac{(2k+1)-d}{2} \ell}. \end{aligned}$$

This implies that the condition (b) of Corollary 2.4 is true for all $\delta = (d, D)$ of the form $\alpha \tau_1^y \tau_2^z$.

STEP 2: $q = 0 \Rightarrow \text{HPer}(f) = \{1\}$.

If $q = 0$, then as observed before, $\det(I - DF^{2k+1}) \neq 0$ for all $(d, D) \in \Pi$. By (3.2), we have

$$\beta \varphi^d(\beta) \varphi^{2d}(\beta) \dots \varphi^{(2k+1)-d}(\beta) = \beta$$

for all $d \mid 2k + 1$ with $d < 2k + 1$ and all $\beta \in \Pi$. By Corollary 2.4, $2k + 1 \notin \text{HPer}(f)$ where $k > 0$. Together with the observation that $2k \notin \text{HPer}(f)$, we have $\text{HPer}(f) = \{1\}$.

STEP 3: $q \geq 2 \Rightarrow \text{HPer}(f) = \mathbb{N} - 2\mathbb{N}$.

If $q \geq 2$, then as observed before, $\det(I - DF^{2k+1}) \neq 0$ for all $(d, D) \in \Pi$. We will show that all $2k + 1 \in \text{HPer}(f)$. For this purpose, we shall show that

$$(3.3) \quad \Pi \neq \bigcup_{\substack{d < 2k+1 \\ d \mid 2k+1}} \left\{ \beta \varphi^d(\beta) \varphi^{2d}(\beta) \dots \varphi^{(2k+1)-d}(\beta) \mid \beta \in \Pi \right\}.$$

We remark from (3.2) that

$$\begin{aligned} & \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{(2k+1)-d}(\beta) \\ &= \begin{cases} \tau_1^y\tau_2^{\frac{1-q^{2k+1}}{1-q^d}z} & \text{when } \beta = \tau_1^y\tau_2^z, \\ \alpha\tau_1^y\tau_2^{\frac{1+q^{2k+1}}{1+q^d}z + \frac{1-q^d}{1-q}\frac{q^d-q^{2k+1}}{1-q^{2d}}\ell} & \text{when } \beta = \alpha\tau_1^y\tau_2^z. \end{cases} \end{aligned}$$

By Lemmas 3.2, $\gcd\left\{\frac{1-q^{2k+1}}{1-q^d} : d < 2k + 1, d \mid 2k + 1\right\} \neq 1$. This implies that (3.3) holds. By Theorem 2.3, $2k + 1 \in \text{HPer}(f)$. □

Proposition 3.7. *If r is odd with $|r| \geq 3$, then $\text{HPer}(f) = \mathbb{N}$.*

Since $\det(I - DF^k) = (1 - r^k)(1 \mp q^k) = 0$ implies that $q = 1$ and $D = I$, we have $\det(I - DF^k) \neq 0$ for all $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$. In particular, by Corollary 2.4, $1 \in \text{HPer}(f)$.

Next, for $k > 0$ we will show that there exists $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ such that

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{k-d}(\beta)$$

for all $d < k$ with $d \mid k$ and for all $\beta \in \Pi$. Then by Theorem 2.3, $k \in \text{HPer}(f)$.

For $\beta = \tau_1^y\tau_2^z$, we observe from (3.2) that

$$\beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{k-d}(\beta) = \tau_1^{\frac{1-r^{2k+1}}{1-r^d}y}\tau_2^{\frac{1-q^{2k+1}}{1-q^d}z}.$$

This implies that for all $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$,

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{k-d}(\beta)$$

for all $d < k$ with $d \mid k$ and for all $\beta \in \Pi$ of the form $\tau_1^y\tau_2^z$. Consequently, we need to show that for some $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$,

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{k-d}(\beta)$$

for all $\beta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ and all $d \mid k$ with $d < k$.

STEP 1: $2k + 1 \in \text{HPer}(f)$.

Let $d \mid 2k + 1$ with $k > 0$ and $d < 2k + 1$. Let $\beta = \alpha\tau_1^y\tau_2^z$. Then by (3.2)

$$\begin{aligned} \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{(2k+1)-d}(\beta) &= \alpha^{\frac{1-r^{2k+1}}{1-r^d}}\tau_1^{\frac{1-r^{2k+1}}{1-r^d}y}\tau_2^{\frac{1+q^{2k+1}}{1+q^d}z + \frac{q^d-q^{(2k+1)-d}}{(1-q)(1+q^d)}} \\ &= \alpha\tau_1^{\frac{1-r^{2k+1}}{1-r^d}y + \frac{r^d-r^{2k+1}}{2(1-r^d)}}\tau_2^{\frac{1+q^{2k+1}}{1+q^d}z + \frac{q^d-q^{(2k+1)-d}}{(1-q)(1+q^d)}} \end{aligned}$$

since $\frac{1-r^{2k+1}}{1-r^d}$ is odd. By Lemma 3.2, $\gcd\left\{\frac{1-r^{2k+1}}{1-r^d} : d < 2k + 1, d \mid 2k + 1\right\} \neq 1$. This implies that we can choose $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ such that

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{(2k+1)-d}(\beta)$$

for all $\beta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ and all $d \mid 2k + 1$ with $d < 2k + 1$. In conclusion, by Theorem 2.3 we have shown that all $2k + 1 \in \text{HPer}(f)$.

STEP 2: $\overline{2k} \in \text{HPer}(f)$.

Let $d \mid 2k$ with $d < 2k$. Let $\beta = \alpha\tau_1^y\tau_2^z$. Then we have

$$\begin{aligned} & \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{2k-d}(\beta) \\ = & \begin{cases} \alpha\tau_1^{\frac{1-r2k}{1-r^d}y+\frac{r^d-r2k}{2(1-r^d)}\tau_2^{\frac{1+q2k}{1+q^d}z+\frac{q^d-q2k}{(1-q)(1+q^d)}}} & \text{when } \frac{2k}{d} \text{ is odd,} \\ \tau_1^{\frac{1-r2k}{1-r^d}y+\frac{1-r2k}{2(1-r^d)}\tau_2^{-\frac{1-q2k}{1+q^d}z+\frac{1-q2k}{(1-q)(1+q^d)}}} & \text{when } \frac{2k}{d} \text{ is even.} \end{cases} \end{aligned}$$

This computation shows that when $\frac{2k}{d}$ is even, $\beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{k-d}(\beta)$ is of the form $\tau_1^*\tau_2^*$, and hence it cannot be $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$. When $\frac{2k}{d}$ is odd, it follows from Lemma 3.2 again that we can choose $\delta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ such that

$$\delta \neq \beta\varphi^d(\beta)\varphi^{2d}(\beta)\cdots\varphi^{(2k+1)-d}(\beta)$$

for all $\beta \in \Pi$ of the form $\alpha\tau_1^y\tau_2^z$ and all $d \mid 2k$ with $d < 2k$. In conclusion, by Theorem 2.3 we have shown that all $2k \in \text{HPer}(f)$.

4. Summary

Let $f : \Pi \setminus \mathbb{R}^2 \rightarrow \Pi \setminus \mathbb{R}^2$ be any continuous map on the Klein bottle $\Pi \setminus \mathbb{R}^2$ of type (r, ℓ, q) . By Lemma 2.2, the coordinates r, ℓ and q satisfy the following: if r is odd, then $\ell = 0, 1$ and $q \geq 0$; and if r is even and $q = 0$, then $\ell \geq 0$. Furthermore, the type (r, ℓ, q) is unique up to homotopy.

Then we can summarize what we have obtained as follows:

$$\text{HPer}(f) = \begin{cases} \emptyset & \text{when } r = 1, \\ \{1\} & \text{when } r = 0, \text{ or when } r = -1 \text{ and } q = 0, 1, \\ \mathbb{N} - 2\mathbb{N} & \text{when } r = -1 \text{ and } q \geq 2, \\ \mathbb{N} - \{2\} & \text{when } r = -2, \\ \mathbb{N} & \text{when } r \leq -3 \text{ or } r \geq 2. \end{cases}$$

This result reconfirms the main results in [20, Theorem], [12, Theorem 5.5] and [6, Theorem 4.2]. In [20], Llibre introduced the notion ‘‘minimal periods’’ but could not determine them for all cases of maps on the Klein bottle. The computation was completed independently in [6] and [12].

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