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SOME RESULTS ON STABLE f-HARMONIC MAPS

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Abstract. In this paper, we prove that any stable f-harmonic map from sphere $\mathbb{S}^n$ to Riemannian manifold (N,h) is constant, where f is a smooth positive function on $\mathbb{S}^n \times N$ satisfying one condition with $n > 2$. We also prove that any stable f-harmonic map φ from a compact Riemannian manifold (M,g) to $\mathbb{S}^n$ $(n > 2)$ is constant where, in this case, f is a smooth positive function on $M \times \mathbb{S}^n$ satisfying $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$.

1. Preliminaries and notations

We give some definitions. (1) Let (M,g) be a Riemannian manifold. By R^M and Ric^M we denote respectively the Riemannian curvature tensor and the Ricci tensor of (M,g). Then R^M and Ric^M are defined by

$$R^M(X,Y)Z = \nabla^M_X \nabla^M_Y Z - \nabla^M_Y \nabla^M_X Z - \nabla^M_{[X,Y]} Z, \tag{1}$$

$$\mathrm{Ric}^M(X,Y) = g(R^M(X,e_i)e_i, Y), \tag{2}$$

where ∇^M is the Levi-Civita connection with respect to g, $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. The divergence of $(0,p)$-tensor α on M is defined by

$$(\mathrm{div}^M \alpha)(X_1, \ldots, X_{p-1}) = (\nabla^M_{e_i} \alpha)(e_i, X_1, \ldots, X_{p-1}), \tag{3}$$

where $X_1, \ldots, X_{p-1} \in \Gamma(TM)$, and $\{e_i\}$ is an orthonormal frame. Given a smooth function λ on M, the gradient of λ is defined by

$$g(\mathrm{grad}^M \lambda, X) = X(\lambda), \tag{4}$$

the Hessian of λ is defined by

$$(\mathrm{Hess}^M \lambda)(X,Y) = g(\nabla^M_X \mathrm{grad}\, \lambda, Y), \tag{5}$$

where $X, Y \in \Gamma(TM)$, the Laplacian of λ is defined by

$$\Delta^M(\lambda) = \mathrm{trace}_g \mathrm{Hess}^M \lambda, \tag{6}$$

(for more details, see for example [9]).

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(2) Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, $\tau(\varphi)$ the tension field of φ (see [1, 2, 6]), and let f be a smooth positive function on $M \times N$, the f-tension field of φ is given by

$$\tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) \circ \varphi, \tag{7}$$

where f_φ is a smooth positive function on M defined by

$$f_\varphi(x) = f(x, \varphi(x)), \quad \forall x \in M, \tag{8}$$

$e(\varphi)$ is the energy density of φ (see [1]), and grad^M (resp. grad^N) denotes the gradient operator with respect to g (resp. h). Then φ is called f-harmonic if the f-tension field vanishes, i.e., $\tau_f(\varphi) = 0$ (for more details on the concept of f-harmonic maps see [4, 5, 10]). We define the index form for f-harmonic maps by

$$I_f^\varphi(v, w) = \int_M h(J_f^\varphi(v), w) v^M \tag{9}$$

for all $v, w \in \Gamma(\varphi^{-1}TN)$, where

$$\begin{aligned} J_f^\varphi(v) = & -f_\varphi \operatorname{trace}_g R^N(v, d\varphi)d\varphi - \operatorname{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi v \\ & + e(\varphi)(\nabla_v^N \operatorname{grad}^N f) \circ \varphi - d\varphi(\operatorname{grad}^M v(f)) \\ & - v(f)\tau(\varphi) + \langle \nabla^\varphi v, d\varphi \rangle (\operatorname{grad}^N f) \circ \varphi, \end{aligned} \tag{10}$$

R^N is the curvature tensor of (N, h), ∇^N is the Levi-Civita connection of (N, h), ∇^φ denote the pull-back connection on $\varphi^{-1}TN$, and v^M is the volume form of (M, g) (see [1], [9]). If φ is a f-harmonic map and for any vector field v along φ, the index form satisfies $I_f^\varphi(v, v) \geq 0$, then φ is called a stable f-harmonic map. Note that, the definition of stable f-harmonic maps is a generalization of stable harmonic maps if $f = 1$ on M (see [11]).

2. Main results

Theorem 2.1. *Any stable f-harmonic map φ from sphere $(\mathbb{S}^n, g)$ $(n > 2)$ to Riemannian manifold (N, h) is constant, where f is a smooth positive function on $\mathbb{S}^n \times N$ satisfying* $\operatorname{trace}_g h((\nabla d\varphi)(\cdot, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(\cdot)) \geq 0$.

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$. Set

$$\lambda(x) = \langle \alpha, x \rangle_{\mathbb{R}^{n+1}}$$

for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \operatorname{grad}^{\mathbb{S}^n} \lambda$. Note that

$$v = \langle \alpha, e_i \rangle e_i, \ \nabla_X^{\mathbb{S}^n} v = -\lambda X \text{ for all } X \in \Gamma(T\mathbb{S}^n),$$
$$\operatorname{trace}_g (\nabla^{\mathbb{S}^n})^2 v = \nabla_{e_i}^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} v - \nabla_{\nabla_{e_i}^{\mathbb{S}^n} e_i}^{\mathbb{S}^n} v = -v,$$

where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric g of the sphere (see [11]). At point x_0, we have

$$\nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi d\varphi(v) = \nabla_{\operatorname{grad}^{\mathbb{S}^n} f_\varphi}^\varphi d\varphi(v) + f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi d\varphi(v), \tag{11}$$

the first term of (11) is given by

$$
\begin{aligned}
\nabla^{\varphi}_{\operatorname{grad} f_{\varphi}} d\varphi(v) &= \nabla^{\varphi}_{v} d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) + d\varphi([\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}, v]) \\
&= \nabla^{\varphi}_{v} d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) + d\varphi(\nabla^{\mathbb{S}^n}_{\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}} v) \\
&\quad - d\varphi(\nabla^{\mathbb{S}^n}_{v} \operatorname{grad}^{\mathbb{S}^n} f_{\varphi}),
\end{aligned} \tag{12}
$$

the seconde term of (11) is given by

$$
\begin{aligned}
f_{\varphi}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} d\varphi(v) &= f_{\varphi}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{v} d\varphi(e_i) + f_{\varphi}\nabla^{\varphi}_{e_i} d\varphi([e_i, v]) \\
&= f_{\varphi} R^N(d\varphi(e_i), d\varphi(v)) d\varphi(e_i) + f_{\varphi}\nabla^{\varphi}_{v}\nabla^{\varphi}_{e_i} d\varphi(e_i) \\
&\quad + f_{\varphi} d\varphi([e_i,[e_i,v]]) + 2 f_{\varphi}\nabla^{\varphi}_{[e_i,v]} d\varphi(e_i),
\end{aligned} \tag{13}
$$

from the definition of tension field, we get

$$
\begin{aligned}
f_{\varphi}\nabla^{\varphi}_{e_i}\nabla^{\varphi}_{e_i} d\varphi(v) = &- f_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + f_{\varphi}\nabla^{\varphi}_{v}\tau(\varphi) \\
&+ f_{\varphi}\nabla^{\varphi}_{v} d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\
&- f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_{\varphi}\nabla^{\varphi}_{[e_i,v]} d\varphi(e_i) \\
= &- f_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + \nabla^{\varphi}_{v} f_{\varphi}\tau(\varphi) - v(f_{\varphi})\tau(\varphi) \\
&+ f_{\varphi}\nabla^{\varphi}_{v} d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\
&- f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_{\varphi}\nabla^{\varphi}_{[e_i,v]} d\varphi(e_i),
\end{aligned} \tag{14}
$$

by equations (11), (12), (14), and the f-harmonicity condition of φ, we have

$$
\begin{aligned}
\nabla^{\varphi}_{e_i} f_{\varphi}\nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}} v) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) \\
&\quad - f_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\
&\quad + \nabla^{\varphi}_{v} e(\varphi)(\operatorname{grad}^N f)\circ\varphi - v(f_{\varphi})\tau(\varphi) \\
&\quad + f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{v}\nabla^{\mathbb{S}^n}_{e_i} e_i) + f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{e_i} v) \\
&\quad - f_{\varphi} d\varphi(\nabla^{\mathbb{S}^n}_{e_i}\nabla^{\mathbb{S}^n}_{v} e_i) + 2 f_{\varphi}\nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i),
\end{aligned} \tag{15}
$$

by the definition of Ricci tensor, we get

$$
\begin{aligned}
\nabla^{\varphi}_{e_i} f_{\varphi}\nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}} v) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) \\
&\quad - f_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + \nabla^{\varphi}_{v} e(\varphi)(\operatorname{grad}^N f)\circ\varphi \\
&\quad - v(f_{\varphi})\tau(\varphi) + f_{\varphi} d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) + f_{\varphi} d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) \\
&\quad + 2 f_{\varphi}\nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i),
\end{aligned} \tag{16}
$$

from the property $\nabla^{\mathbb{S}^n}_X v = -\lambda X$, we obtain

$$
\begin{aligned}
\nabla^{\varphi}_{e_i} f_{\varphi}\nabla^{\varphi}_{e_i} d\varphi(v) = &-\lambda d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) - d\varphi(\nabla^{\mathbb{S}^n}_{v} \operatorname{grad}^{\mathbb{S}^n} f_{\varphi}) \\
&- f_{\varphi} R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\
&+ \langle \nabla^{\varphi}_{e_i} d\varphi(v), d\varphi(e_i)\rangle (\operatorname{grad}^N f)\circ\varphi
\end{aligned}
$$

$$\begin{aligned}
&- h(d\varphi(\nabla^{\mathbb{S}^n}_{e_i} v), d\varphi(e_i))(\operatorname{grad}^N f)\circ\varphi \\
&+ e(\varphi)\nabla^{\varphi}_v(\operatorname{grad}^N f)\circ\varphi \\
&- v(f)\tau(\varphi) - d\varphi(v)(f)\tau(\varphi) + f_\varphi d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) \\
&+ f_\varphi d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) - 2\lambda f_\varphi \tau(\varphi).
\end{aligned}\tag{17}$$

From the definition of Jacobi operator (10) and equation (17) we have

$$\begin{aligned}
J^{\varphi}_f(d\varphi(v)) = {}& \lambda d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f_\varphi) \\
&+ \lambda h(d\varphi(e_i), d\varphi(e_i))(\operatorname{grad}^N f)\circ\varphi + v(f)\tau(\varphi) \\
&- f_\varphi d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) - f_\varphi d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) \\
&- d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(v)(f)) + 2\lambda f_\varphi \tau(\varphi),
\end{aligned}\tag{18}$$

since $\operatorname{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$ and $\operatorname{Ricci}^{\mathbb{S}^n} v = (n-1)v$ (see [1,11]), we conclude

$$\begin{aligned}
h(J^{\varphi}_f(d\varphi(v)), d\varphi(v)) = {}& \lambda h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\
&+ h(d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\
&+ 2\lambda h(e(\varphi)(\operatorname{grad}^N f)\circ\varphi, d\varphi(v)) \\
&+ v(f)h(\tau(\varphi), d\varphi(v)) \\
&- (n-2)f_\varphi h(d\varphi(v), d\varphi(v)) \\
&- h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(v)(f)), d\varphi(v)) \\
&+ 2\lambda f_\varphi h(\tau(\varphi), d\varphi(v)),
\end{aligned}\tag{19}$$

by (19) and the f-harmonicity condition of φ, it follows that

$$\begin{aligned}
\operatorname{trace}_\alpha h(J^{\varphi}_f(d\varphi(v)), d\varphi(v)) = {}& h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j} \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) \\
&+ h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) - (n-2)f_\varphi |d\varphi|^2 \\
&- h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)),
\end{aligned}\tag{20}$$

by the following formulas

$$\begin{aligned}
h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j} \operatorname{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) = {}& h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j} \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\
&+ h(\nabla^{\varphi}_{e_j} d\varphi(e_i), \operatorname{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)) \\
&+ h(d\varphi(e_i), \nabla^{\varphi}_{e_j} \operatorname{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)), \\
d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)) = {}& e_i[h(d\varphi(e_j), \operatorname{grad}^N f)]d\varphi(e_i) \\
= {}& h(\nabla^{\varphi}_{e_i} d\varphi(e_j), \operatorname{grad}^N f)d\varphi(e_i) \\
&+ h(d\varphi(e_j), \nabla^{\varphi}_{e_i} \operatorname{grad}^N f)d\varphi(e_i),
\end{aligned}$$

$$\begin{aligned}
-h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)) = {}& -h(\nabla^{\varphi}_{e_i} d\varphi(e_j), \operatorname{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)) \\
&- h(d\varphi(e_j), \nabla^{\varphi}_{e_i} \operatorname{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)),
\end{aligned}$$

and equation (20), it follows that

$$\begin{aligned}\operatorname{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j} \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) - (n-2) f_\varphi |d\varphi|^2,\end{aligned}$$

note that

$$\begin{aligned}h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) &= h(\nabla^\varphi_{e_i} d\varphi(e_i), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) \\ &= \operatorname{div}^{\mathbb{S}^n} \eta - h(d\varphi(e_i), \nabla^\varphi_{e_i} d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)),\end{aligned}$$

with $\eta(X) = h(d\varphi(X), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f))$, $\forall X \in \Gamma(T\mathbb{S}^n)$. We obtain

$$\begin{aligned}\operatorname{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) = &- h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &+ \operatorname{div}^{\mathbb{S}^n} \eta - (n-2) f_\varphi |d\varphi|^2,\end{aligned} \tag{21}$$

since $h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \geq 0$, from the stable f-harmonic condition, and equation (21), we get

$$\begin{aligned}0 &\leq \operatorname{trace}_\alpha I_f^\varphi(d\varphi(v), d\varphi(v)) + \int_{\mathbb{S}^n} h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) v^{\mathbb{S}^n} \\ &= -(n-2) \int_{\mathbb{S}^n} f_\varphi |d\varphi|^2 v^{\mathbb{S}^n} \leq 0.\end{aligned}$$

Consequently, $|d\varphi| = 0$, that is φ is constant, because $n > 2$. □

If $f = 1$ on $\mathbb{S}^n \times N$, we get the following result:

Corollary 2.2 ([12])**.** *Any stable harmonic map φ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

Using the similar technique we have:

Theorem 2.3. *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on $M \times \mathbb{S}^n$, with $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M. When the data is the same as those in the previous proof, we have

$$\nabla^\varphi_{e_i} f_\varphi \nabla^\varphi_{e_i}(v \circ \varphi) = \nabla^\varphi_{\operatorname{grad}^M f_\varphi}(v \circ \varphi) + f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi), \tag{22}$$

the first term of (22) is given by

$$\nabla^\varphi_{\operatorname{grad}^M f_\varphi}(v \circ \varphi) = -(\lambda \circ \varphi) d\varphi(\operatorname{grad}^M f_\varphi), \tag{23}$$

the seconde term of (22) is given by

$$\begin{aligned}f_\varphi \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi) &= -f_\varphi \nabla^\varphi_{e_i}(\lambda \circ \varphi) d\varphi(e_i) \\ &= -f_\varphi d\varphi(\operatorname{grad}^M(\lambda \circ \varphi)) - (\lambda \circ \varphi) f_\varphi \tau(\varphi),\end{aligned} \tag{24}$$

by the definition of gradient operator, we get

$$-f_\varphi d\varphi(\operatorname{grad}^M(\lambda \circ \varphi)) = -f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle d\varphi(e_i), \tag{25}$$

substituting the formulas (23), (24), (25) into (22) gives

$$\begin{aligned}\nabla^{\varphi}_{e_i} f_\varphi \nabla^{\varphi}_{e_i}(v\circ\varphi) = &-(\lambda\circ\varphi)d\varphi(\operatorname{grad}^M f_\varphi) - f_\varphi\langle d\varphi(e_i), v\circ\varphi\rangle d\varphi(e_i)\\ &-(\lambda\circ\varphi)f_\varphi\tau(\varphi),\end{aligned} \tag{26}$$

from the f-harmonicity condition of φ, and equation (26), we have

$$\begin{aligned}\langle\nabla^{\varphi}_{e_i} f_\varphi \nabla^{\varphi}_{e_i}(v\circ\varphi), v\circ\varphi\rangle = &-f_\varphi\langle d\varphi(e_i), v\circ\varphi\rangle\langle d\varphi(e_i), v\circ\varphi\rangle\\ &-(\lambda\circ\varphi)e(\varphi)\langle(\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi\rangle,\end{aligned} \tag{27}$$

since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned}&\langle f_\varphi R^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi\rangle\\ &= f_\varphi|d\varphi|^2\langle v\circ\varphi, v\circ\varphi\rangle - f_\varphi\langle d\varphi(e_i), v\circ\varphi\rangle\langle d\varphi(e_i), v\circ\varphi\rangle,\end{aligned} \tag{28}$$

by the definition of Jacobi operator and equations (27), (28), we get

$$\begin{aligned}\langle J^{\varphi}_f(v\circ\varphi), v\circ\varphi\rangle = &\, 2f_\varphi\langle d\varphi(e_i), v\circ\varphi\rangle\langle d\varphi(e_i), v\circ\varphi\rangle\\ &- f_\varphi|d\varphi|^2\langle v\circ\varphi, v\circ\varphi\rangle\\ &+(\lambda\circ\varphi)e(\varphi)\langle(\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi\rangle\\ &+e(\varphi)\langle(\nabla^{\mathbb{S}^n}_{v\circ\varphi}\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi\rangle\\ &-\langle d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi\rangle\\ &-\langle(v\circ\varphi)(f)\tau(\varphi), v\circ\varphi\rangle\\ &+\langle\nabla^{\varphi}v\circ\varphi, d\varphi\rangle\langle(\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi\rangle,\end{aligned} \tag{29}$$

so that

$$\begin{aligned}\operatorname{trace}_\alpha\langle J^{\varphi}_f(v\circ\varphi), v\circ\varphi\rangle = &\,(2-n)f_\varphi|d\varphi|^2\\ &+e(\varphi)\operatorname{trace}_\alpha(\operatorname{Hess}^{\mathbb{S}^n} f)(v\circ\varphi, v\circ\varphi)\\ &-\operatorname{trace}_\alpha\langle d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi\rangle\\ &-\operatorname{trace}_\alpha\langle\tau(\varphi), v\circ\varphi\rangle(v\circ\varphi)(f)\\ &+\operatorname{trace}_\alpha\langle\nabla^{\varphi}v\circ\varphi, d\varphi\rangle\langle(\operatorname{grad}^{\mathbb{S}^n} f)\circ\varphi, v\circ\varphi\rangle,\end{aligned} \tag{30}$$

where $\operatorname{Hess}^{\mathbb{S}^n} f$ is the hessian of the function f on $\mathbb{S}^n$, by the following formulas

$$\begin{aligned}d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)) &= e_i\langle v\circ\varphi, \operatorname{grad}^{\mathbb{S}^n} f\rangle d\varphi(e_i)\\ &=\langle\nabla^{\varphi}_{e_i}(v\circ\varphi), \operatorname{grad}^{\mathbb{S}^n} f\rangle d\varphi(e_i)\\ &\quad+\langle v\circ\varphi, \nabla^{\varphi}_{e_i}\operatorname{grad}^{\mathbb{S}^n} f\rangle d\varphi(e_i)\\ &= -(\lambda\circ\varphi)\langle d\varphi(e_i), \operatorname{grad}^{\mathbb{S}^n} f\rangle d\varphi(e_i)\\ &\quad+\langle v\circ\varphi, \nabla^{\mathbb{S}^n}_{d\varphi(e_i)}\operatorname{grad}^{\mathbb{S}^n} f\rangle d\varphi(e_i),\end{aligned}$$

$$-\operatorname{trace}_\alpha\langle d\varphi(\operatorname{grad}^M(v\circ\varphi)(f)), v\circ\varphi\rangle = -(\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)),$$

$$\begin{aligned}\langle \nabla^{\varphi} v \circ \varphi, d\varphi \rangle &= \langle \nabla^{\varphi}_{e_i} v \circ \varphi, d\varphi(e_i) \rangle \\ &= -(\lambda \circ \varphi) \langle d\varphi(e_i), d\varphi(e_i) \rangle \\ &= -(\lambda \circ \varphi) |d\varphi|^2,\end{aligned}$$

$$\operatorname{trace}_\alpha \langle \nabla^{\varphi} v \circ \varphi, d\varphi \rangle \langle (\operatorname{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle = 0,$$

$$\begin{aligned}-\operatorname{trace}_\alpha \langle \tau(\varphi), v \circ \varphi \rangle (v \circ \varphi)(f) &= -\langle \tau(\varphi), \operatorname{grad}^{\mathbb{S}^n} f \rangle \\ &= -\langle \nabla^{\varphi}_{e_i} d\varphi(e_i), \operatorname{grad}^{\mathbb{S}^n} f \rangle \\ &= -\operatorname{div} w + \langle d\varphi(e_i), \nabla^{\mathbb{S}^n}_{d\varphi(e_i)} \operatorname{grad}^{\mathbb{S}^n} f \rangle,\end{aligned}$$

where $w(X) = \langle d\varphi(X), \operatorname{grad}^{\mathbb{S}^n} f \rangle$, $\forall X \in \Gamma(TM)$, and (30), we have

$$\begin{aligned}\operatorname{trace}_\alpha \langle J^{\varphi}_f (v \circ \varphi), v \circ \varphi \rangle = (2-n) f_\varphi |d\varphi|^2 & \\ + e(\varphi) \Delta^{\mathbb{S}^n}(f) \circ \varphi & \\ - (\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)) & \\ - \operatorname{div} w + (\operatorname{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)), &\end{aligned}$$

where $\Delta^{\mathbb{S}^n}(f) \circ \varphi = \operatorname{trace}_\alpha (\operatorname{Hess}^{\mathbb{S}^n} f)(v \circ \varphi, v \circ \varphi)$, so that

$$\begin{aligned}\operatorname{trace}_\alpha I^{\varphi}_f (v \circ \varphi, v \circ \varphi) = (2-n) \int_M f_\varphi |d\varphi|^2 v^M & \\ + \int_M e(\varphi) [\Delta^{\mathbb{S}^n}(f) \circ \varphi] v^M. & \qquad (31)\end{aligned}$$

Hence Theorem 2 follows from (31) and the stable f-harmonicity condition of φ with $n > 2$ and $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$. □

From Theorem 2, we deduce:

Corollary 2.4 ([7,8])**.** *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant.*

Corollary 2.5 ([3])**.** *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth positive function on M.*

References

[1] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, **29**, The Clarendon Press, Oxford University Press, Oxford, 2003.
[2] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of S^3*, Internat. J. Math. **12** (2001), no. 8, 867–876.
[3] A. M. Cherif, M. Djaa, and K. Zegga, *Stable f-harmonic maps on sphere*, Commun. Korean Math. Soc. **30** (2015), no. 4, 471–479.
[4] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. **13** (2007), 423–435.

[5] M. Djaa, A. M. Cherif, K. Zagga, and S. Ouakkas, *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom. **5** (2012), no. 1, 90–100.

[6] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[7] R. Howard and S. W. Wei, *Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space*, Trans. Amer. Math. Soc. **294** (1986), no. 1, 319–331.

[8] Y. Ohnita, *Stability of harmonic maps and standard minimal immersions*, Tohoku Math. J. (2) **38** (1986), no. 2, 259–267.

[9] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.

[10] S. Ouakkas, R. Nasri, and M. Djaa, *On the f-harmonic and f-biharmonic maps*, J. P. J. Geom. Topol. **10** (2010), no. 1, 11–27.

[11] Y. Xin, *Geometry of Harmonic Maps*, Progress in Nonlinear Differential Equations and their Applications, **23**, Birkhäuser Boston, Inc., Boston, MA, 1996.

[12] Y. L. Xin, *Some results on stable harmonic maps*, Duke Math. J. **47** (1980), no. 3, 609–613.

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