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HARMONIC FINSLER MANIFOLDS WITH MINIMAL HOROSPHERES

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Abstract. In this paper we show that complete noncompact harmonic Finsler manifolds with minimal horospheres are flat.

Introduction

The Lichnerowicz conjecture asserts that all harmonic Riemannian manifolds are either flat or locally symmetric spaces of rank one. Szabò [12] proved the Lichnerowicz conjecture for *compact* simply connected Riemannian manifolds. Besson, Courtois, and Gallot [2] confirmed the conjecture for Riemannian manifolds of negative curvature admitting a compact quotient.

On the other hand, Damek and Ricci [5] provided examples showing that in the *noncompact* Riemannian case the conjecture is wrong. The classification of all noncompact harmonic Riemannian manifolds is still a very difficult open problem. Ranjan and Shah [10] showed that noncompact, simply connected harmonic manifolds with minimal horospheres are flat. In this paper, using the Borsuk-Ulam theorem and the existence of stable Jacobi tensors, we extend this result to Finsler manifolds.

Theorem. *Complete noncompact harmonic Finsler manifolds with minimal horospheres are flat.*

For compact Finsler manifolds M without conjugate points it is clear that the volume entropy $h_\mu \geq 0$, and it was conjectured by Burago and Ivanov that either $h_\mu > 0$ or the geodesic flow of M is conjugate to the flow of a flat Finsler metric ([4, 7, 13]). However, to the best of our knowledge, this conjecture is not yet solved. In this paper we provide convincing evidence for this conjecture.

Corollary. *If M is a compact harmonic Finsler manifold without conjugate points such that the volume entropy is zero, then M is flat.*

In particular, harmonic Finsler tori without conjugate points are flat (cf. [3, 9]).

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1. Preliminaries

In this section we recall some basics in Finsler geometry and prove some auxiliary facts. We follow the presentation in [8], where most concepts are developed from Riemannian point of view. We refer to [11] as more exhaustive references in Finsler geometry.

A function $F : TM \to [0, \infty)$ will be called a *Finsler metric* on the manifold M if it is smooth outside the zero section and its restriction to each tangent space T_xM is a quadratically convex norm $F(x, \cdot)$. By a *geodesic* we always mean an affine geodesic, i.e., a constant-speed one. For a nonzero $v \in TM$, we denote by γ_v the unique geodesic with initial velocity $\dot\gamma_v(0) = v$. For every $p \in M$ and $v \in T_pM \setminus \{0\}$ there is a unique positive definite quadratic form g_v on T_pM such that g_v and $F^2|_{T_pM}$ agree to second order at v. With the Chern connection, we can define the covariant derivative $D_{\dot\gamma_v(t)}J(t)$ of a vector field $J(t)$ along a geodesic $\gamma_v(t)$. A vector field $J(t)$ along $\gamma_v(t)$ is called a *Jacobi field* if it satisfies

$$D_{\dot\gamma_v(t)}D_{\dot\gamma_v(t)}J(t) + R_{\dot\gamma(t)}\big(J(t)\big) = 0,$$

where R is the curvature tensor.

The point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along geodesic γ, if there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(t_0)$. We have the stable Jacobi tensors $C_v(t)$ with $C_v(0) = Id$ of Finsler manifolds without conjugate points as in the case of Riemannian manifolds (see [6, 8]). Making the change of variables

$$U_v(t) := \big(\ln C_v(t)\big)' = C_v'(t) \cdot C_v^{-1}(t)$$

for t values for which $\det C_v(t) \neq 0$. In fact, if $\gamma_v(t)$ has no points conjugate to $\gamma(0)$ on $(0, \infty)$, then $U_v(t)$ is defined for all $t \in (0, \infty)$. Then the tensor U defined on the unit tangent bundle SM such that for every $v \in SM$, $U_v(t)$ is a self-adjoint linear operator on

$$\dot\gamma_v(t)^\perp := \{w \in T_{\gamma_v(t)}M \mid g_{\dot\gamma_v(t)}(\dot\gamma_v(t), w) = 0\}$$

and satisfies the *Riccati equation*

$$U_v'(t) + U_v^2(t) + R_v(t) = 0. \tag{1.1}$$

We will let $u(v) := \operatorname{tr}\big(U_v(0)\big)$ be the *mean curvature* (the trace of the second fundamental form) of the stable horosphere through v in the universal covering space. It is also known that $u(v)$ are measurable function, and the volume entropy

$$h_\mu = \int_{SM} u(v)\, d\mu(v). \tag{1.2}$$

2. Harmonic Finsler manifolds

A compact Finsler manifold is called a *Blaschke* manifold if every minimal geodesic of length less than the diameter is the unique shortest path between any of its points. Equivalently, for which all cut loci are round spheres of constant radius and dimension.

Remark 2.1. For a Blaschke manifold the exponential map restricted to the unit tangent sphere defines a great sphere foliation. Since every great sphere foliation of sphere is homeomorphic to a Hopf fibration, a simply connected Blaschke manifold is actually homeomorphic to compact rank one symmetric spaces.

A complete Finsler manifold is called *harmonic* if the mean curvature of all geodesic spheres is a function depending only on the radius. A historical break in the theory of harmonic Riemannian manifolds was made by Allamigeon when he proved the following: A simply connected harmonic Riemannian manifold is either diffeomorphic to Euclidean space or is a Blaschke manifold. The following theorem is to put them in a Finsler-geometric setting. For the sake of completeness we sketch the proof.

Theorem 2.2. *A simply connected harmonic Finsler manifold M is either diffeomorphic to Euclidean space or is a Blaschke manifold.*

Proof. Suppose there is no conjugate points. Then the exponential map is a covering map and since M is simply connected, a diffeomorphism. So take a $0 \neq v_0 \in T_xM$ and an $r_0 \in \mathbb{R}$ such that the first conjugate point along γ_{v_0} is $\gamma_{v_0}(r_0)$. Then the first conjugate point along γ_v is $\gamma_v(r_0)$ for all $v \in T_xM$, since the mean curvature is radial. Note that r_0 is the same for every point in M. This means that M is a Blaschke manifold by the Allamigeon-Warner theorem, cf. [1, Corollary 5.31]. □

Therefore, noncompact harmonic Finsler manifolds have nonnegative constant mean curvature of the stable horospheres. Hence, we try to classify harmonic Finsler manifolds according to mean curvature stable horospheres. A harmonic Finsler manifold having stable horospheres of zero mean curvature is called as *manifold with minimal horospheres.*

The following proposition follows from standard techniques.

Proposition 2.3. *Let M be a complete noncompact harmonic Finsler manifold with minimal horospheres. Then for all $v \in SM$ we have*

$$U_v(0) = -U_{-v}(0).$$

Proof. Since M has no conjugate points, it is clear that the tensor $U_v(0) + U_{-v}(0)$ is nonpositive. Therefore we have

$$\operatorname{tr}\Big(U_v(0) + U_{-v}(0)\Big) \leq 0.$$

Furthermore M have minimal horospheres, we have $\operatorname{tr} U_v(0) = 0$ for all $v \in SM$, and hence

$$\operatorname{tr}\Big(U_v(0) + U_{-v}(0)\Big) = 0.$$

Since $U_v(0) + U_{-v}(0) \leq 0$ is symmetric negative semi-definite tensor, we can conclude that the tensor itself must be zero. □

The above proposition shows that the stable and unstable horospheres of harmonic Finsler manifolds with minimal horospheres coincide like flat spaces.

The following lemma is the final ingredient needed for main theorem.

Lemma 2.4. *If M is as above, then for every point $x \in M$, there exists $v \in S_xM$ such that $U_v(0) = 0$.*

Proof. Since $U_v(0)$ is a symmetric matrix on $v^\perp$, there exists a basis of $v^\perp$ such that $U_v(0)$ is represented by a diagonal matrix. Let $\lambda_1(v) \leq \lambda_2(v) \leq \cdots \leq \lambda_{n-1}(v)$ be the eigenvalues of $U_v(0)$. Now consider the continuous function $f : S^{n-1} \to \mathbb{R}^{n-1}$ defined by

$$f(v) = \big(\lambda_1(v), \lambda_2(v), \ldots, \lambda_{n-1}(v)\big).$$

Then by the Borsuk-Ulam theorem, there exists $v \in S^{n-1}$ such that $f(v) = f(-v)$. Therefore $\lambda_i(v) = \lambda_i(-v)$ for all $i = 1, 2, \ldots, n-1$. Hence $U_v(0) = U_{-v}(0)$, and by Proposition 2.3, we have $U_v(0) = 0$. □

Now we are ready to prove main theorem using Lemma 2.4.

Theorem 2.5. *Complete noncompact harmonic Finsler manifolds with minimal horospheres are flat.*

Proof. By Lemma 2.4, we have $U_v(0) = 0$. Inserting this back into the Riccati equation (1.1), we end up with $R_v(0) = 0$, which shows that given underline manifolds are flat, finishing the proof. □

By using Theorem 2.5, we obtain the following Corollary.

Corollary. *If M is a compact harmonic Finsler manifold without conjugate points such that the volume entropy is zero, then M is flat.*

Proof. Since M have nonnegative constant mean curvature of the stable horospheres and the left-hand side of equation (1.2) has zero, we have that the integral term on the right-hand side of equation (1.2) has zero. This mean that the universal covering space $\widetilde{M}$ of M is a complete noncompact harmonic Finsler manifold with minimal horospheres. By Theorem 2.5, $\widetilde{M}$ is flat, we can conclude that M itself must be flat. □

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