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SOME NEW RESULTS ON HYPERSTABILITY OF THE GENERAL LINEAR EQUATION IN $(2,\beta)$-BANACH SPACES

Iz-iddine EL-Fassi

Abstract. In this paper, we first introduce the notions of $(2,\beta)$-Banach spaces and we will reformulate the fixed point theorem [10, Theorem 1] in this space. We also show that this theorem is a very efficient and convenient tool for proving the new hyperstability results of the general linear equation in $(2,\beta)$-Banach spaces. Our main results state that, under some weak natural assumptions, functions satisfying the equation approximately (in some sense) must be actually solutions to it. Our results are improvements and generalizations of the main results of Piszczek [34], Brzdęk [6, 7] and Bahyrycz et al. [2] in $(2,\beta)$-Banach spaces.

1. Introduction and preliminaries

In this paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the sets of all positive integers, real numbers, non-negative real numbers and complex numbers, respectively; and we put $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and let $\mathbb{F}$, $\mathbb{K}$ denote the fields of real or complex numbers.

The next definition describes the notion of hyperstability that we apply here (A^B denotes the family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$).

Definition 1.1. Let A be a nonempty set, (Z,d) be a metric space, $\gamma : A^n \to \mathbb{R}_+$, $B \subset A^n$ be nonempty, and $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{D} \to Z^{A^n}$ be two mappings with $\mathcal{D}$ is a nonempty set of Z^A. We say that the conditional equation

$$\mathcal{F}_1\varphi(x_1,\dots,x_n) = \mathcal{F}_2\varphi(x_1,\dots,x_n), \quad (x_1,\dots,x_n) \in B \tag{1.1}$$

is γ-hyperstable, if every $\varphi_0 \in \mathcal{D}$ satisfying

$$d\big(\mathcal{F}_1\varphi_0(x_1,\dots,x_n), \mathcal{F}_2\varphi_0(x_1,\dots,x_n)\big) \leqslant \gamma(x_1,\dots,x_n), (x_1,\dots,x_n) \in B \tag{1.2}$$

is a solution of Eq. (1.1).

That notion is one of the notions related to the issue of Ulam stability for various (e.g., difference, differential, functional, integral, operator) equations.

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Let us recall that the study of such problems was motivated by the following question of Ulam (cf. [23, 40]) asked in 1940.

Ulam's question. *Let $(G_1, *)$, $(G_2, \star)$ be two groups and $\rho : G_2 \times G_2 \to [0, \infty)$ be a metric. Given $\epsilon > 0$, does there exist $\delta > 0$ such that if a function $g : G_1 \to G_2$ satisfies the inequality*

$$\rho(g(x * y), g(x) \star g(y)) \leqslant \delta$$

for all $x, y \in G_1$, then there is a homomorphism $h : G_1 \to G_2$ with

$$\rho(g(x), h(x)) \leqslant \epsilon$$

for all $x \in G_1$?

In 1941, Hyers [23] published the first answer to it, in the case of Banach space. The following theorem is the most classical result concerning the Hyers-Ulam stability of the Cauchy equation

$$f(x + y) = f(x) + f(y), \qquad x, y \in E_1, \tag{1.3}$$

where E_1 is a normed space.

Theorem 1.1. *Let E_1, E_2 be normed spaces and $f : E_1 \to E_2$ satisfy the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p) \tag{1.4}$$

for all $x, y \in E_1 \backslash \{0\}$, where θ and p are real constants with $\theta > 0$ and $p \neq 1$. Then the following two statements are valid.

(a) *If $p \geqslant 0$ and E_2 is complete, then there exists a unique solution $T : E_1 \to E_2$ of (1.3) such that*

$$\|f(x) - T(x)\| \leqslant \frac{\theta}{|1 - 2^{p-1}|} \|x\|^p, \qquad x \in E_1 \backslash \{0\}. \tag{1.5}$$

(b) *If $p < 0$, then f is additive, i.e., (1.3) holds.*

Note that Theorem 1.1 reduces to the first result of stability due to Hyers [23] if $p = 0$, Aoki [1] for $0 < p < 1$ (see also [37]). Afterward, Gajda [20] obtained this result for $p > 1$ and gave an example to show that Theorem 1.1 fails whenever $p = 1$. Also, Rassias [38] proved Theorem 1.1 for $p < 0$ (see [39, page 326] and [4]). Now, it is well-known that the statement (b) is valid, i.e., f must be additive in that case, which has been proved for the first time in [30] and next in [6] on the restricted domain.

The hyperstability term was used for the first time probably in [32]; however, it seems that the first hyperstability result was published in [3] and concerned the ring homomorphisms. For further information concerning the notion of hyperstability we refer to the survey paper [9] (for recent related results see, e.g., [2, 5–7, 13–15, 22, 26–28, 31, 33, 34, 42]).

The theory of 2-normed spaces was first developed by Gähler [18] in the mid 1960's, while that of 2-Banach spaces was studied later by Gähler [19] and White [41]. For more details, the readers refer to the papers [12, 16, 17].

Now, we give some basic concepts concerning $(2,\beta)$-normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leqslant 1$ and let E be a linear space over $\mathbb{K}$ with $\dim E > 1$. A function $\|\cdot,\cdot\|_\beta : E \times E \to \mathbb{R}_+$ is called a $(2,\beta)$-*norm* on E if and only if it satisfies:

(D1) $\|x,y\|_\beta = 0$ if and only if x and y are linearly dependent;
(D2) $\|x,y\|_\beta = \|y,x\|_\beta$;
(D3) $\|\lambda x,y\|_\beta = |\lambda|^\beta \|x,y\|_\beta$;
(D4) $\|x,y+z\|_\beta \leqslant \|x,y\|_\beta + \|x,z\|_\beta$

for all $x,y,z \in E$ and $\lambda \in \mathbb{K}$. The pair $(E,\|\cdot,\cdot\|_\beta)$ is called a $(2,\beta)$-*normed space*.

If $x \in E$ and $\|x,y\|_\beta = 0$ for all $y \in E$, then $x = 0$. Moreover, the function $x \to \|x,y\|_\beta$ is a continuous function of E into $\mathbb{R}_+$ for each fixed $y \in E$.

The basic definitions of a $(2,\beta)$-Banach space are given as follows:

(a) A sequence $\{x_n\}$ in a $(2,\beta)$-normed space E is called a *Cauchy sequence* if there are $y,z \in E$ such that y and z are linearly independent, $\lim_{n,m\to\infty} \|x_n - x_m, y\|_\beta = 0$ and $\lim_{n,m\to\infty} \|x_n - x_m, z\|_\beta = 0$.
(b) A sequence $\{x_n\}$ in a linear $(2,\beta)$-normed space E is called a *convergent sequence* if there is an $x \in E$ such that $\lim_{n\to\infty} \|x_n - x, y\|_\beta = 0$ for all $y \in E$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(c) A $(2,\beta)$-normed space in which every Cauchy sequence is a convergent sequence is called a $(2,\beta)$-*Banach space*.

We remark that the concept of a linear $(2,\beta)$-normed space is a generalization of a linear 2-normed space ($\beta = 1$). Now, we present an example about $(2,\beta)$-normed space.

Example 1. For $x = (x_1,x_2), y = (y_1,y_2) \in E = \mathbb{R}^2$, the $(2,\beta)$-norm on E is defined by

$$\|x,y\|_\beta = |x_1y_2 - x_2y_1|^\beta,$$

where β is a fixed real number with $0 < \beta \leqslant 1$.

Let X be a β-normed spaces and Y a $(2,\beta)$-normed spaces. We say that a function $f : X \to Y$ satisfies the general linear equation if

$$f(ax+by) = rf(x) + sf(y), \quad x,y \in X, \tag{1.6}$$

where $a,b \in \mathbb{F}\backslash\{0\}$ and $r,s \in \mathbb{K}$. We see that for $a = b = r = s = 1$ in (1.6) we get the Cauchy equation while the Jensen equation corresponds to $a = b = r = s = \frac{1}{2}$. The general linear equation has been studied by many authors, in particular the results of the stability can be found in [8, 11, 21, 24, 25, 29, 35, 36].

Let U be a nonempty subset of X. We say that a function $f : U \to Y$ fulfils equation (1.6) on U (or is a solution to (1.6) on U) provided it satisfies the

conditional functional equation

$$f(ax+by)=rf(x)+sf(y),\quad x,y\in U,\quad ax+by\in U, \tag{1.7}$$

where $a,b\in\mathbb{F}\backslash\{0\}$ and $r,s\in\mathbb{K}$.

If $U=X$, then we simply say that f fulfils (or is a solution to) equation (1.6) on X.

We consider functions $f:U\to Y$ fulfilling (1.7) approximately, i.e., satisfying the inequality

$$\left\|f(ax+by)-rf(x)-sf(y),z\right\|_\beta\leqslant\gamma(x,y,z),\ z\in Y,\ x,y\in U,\ ax+by\in U, \tag{1.8}$$

with $\gamma:U\times U\times Y\to\mathbb{R}_+$ is a given mapping. In this paper, we show that, for some conditions on γ (and under some additional assumptions on U), the conditional functional equation (1.7) is γ-hyperstable in the class of functions $f:U\to Y$, i.e., each $f:U\to Y$ satisfying inequality (1.8) with such γ must fulfil equation (1.7).

2. A fixed point theorem

In this section, we rewrite the fixed point theorem [10, Theorem 1] in $(2,\beta)$-Banach space. For it we need to introduce the following hypotheses.

(H1) W is a nonempty set and Y is a $(2,\beta)$-Banach space.

(H2) $f_1,\dots,f_k:W\to W$ and $L_1,\dots,L_k:W\to\mathbb{R}_+$ are given maps.

(H3) $\mathcal{T}:Y^W\to Y^W$ is an operator satisfying the inequality

$$\|\mathcal{T}\xi(x)-\mathcal{T}\mu(x),z\|_\beta\leqslant\sum_{i=1}^{k}L_i(x)\,\|\xi(f_i(x))-\mu(f_i(x)),z\|_\beta$$

for all $\xi,\mu\in Y^W$ and $(x,z)\in W\times Y$.

(H4) $\Lambda:{\mathbb{R}_+}^W\to\mathbb{R}_+^W$ is a linear operator defined by

$$\Lambda\delta(x):=\sum_{i=1}^{k}L_i(x)\delta(f_i(x))$$

for all $\delta\in\mathbb{R}_+^W$ and $x\in W$.

The basic tool in this paper is the following fixed point theorem.

Theorem 2.1. *Assume that hypotheses* **(H1)-(H4)** *are satisfied. Suppose that there are functions $\varepsilon:W\to\mathbb{R}_+$ and $\varphi:W\to Y$ such that*

$$\|\mathcal{T}\varphi(x)-\varphi(x),z\|_\beta\leqslant\varepsilon(x),\quad(x,z)\in W\times Y$$

$$\varepsilon^*(x):=\sum_{n=0}^{\infty}\Lambda^n\varepsilon(x)<\infty,\quad x\in W.$$

Then, there exists a unique fixed point ψ of $\mathcal{T}$ with

$$\|\varphi(x)-\psi(x),z\|_\beta\leqslant\varepsilon^*(x),\quad(x,z)\in W\times Y.$$

Moreover

$$\psi(x) = \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \quad x \in W.$$

Proof. As in the proof of [10, Theorem 1], we can prove Theorem 2.1. □

3. Hyperstability results for Eq. (1.7)

In the remaining part of the paper, X is a β-normed spaces, Y is a $(2,\beta)$-Banach space, $X_0 := X \setminus \{0\}$, and $\mathbb{N}_{m_0}$ denotes the set of all positive integers greater than or equal to a given $m_0 \in \mathbb{N}$.

The following theorems are the main results in this paper and concern the γ-hyperstability of (1.7). Namely, for

$$\gamma(x,y,z) = h_1(x,z)h_2(y,z),$$

with $h_1, h_2 : U \times Y \to \mathbb{R}_+$ being two functions and

$$\gamma(x,y,z) = h(x,z) + h(y,z),$$

with $h : U \times Y \to \mathbb{R}_+$ being a function, under some additional assumptions on the functions h, h_1, h_2 and on nonempty $U \subset X$, we show that the conditional functional equation (1.7) is γ-hyperstable in the class of functions f mapping U to a $(2,\beta)$-normed space.

Theorem 3.1. *Assume that $U \subset X_0$ is nonempty and there is $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$ with*

$$\left(1 - \frac{1}{n}\right)\frac{x}{a}, \quad \frac{x}{bn} \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geqslant n_0, \tag{3.1}$$

where $a, b \in \mathbb{F}\setminus\{0\}$. Let $r, s \in \mathbb{K}$, $0 < \beta \leqslant 1$ and $h_1, h_2 : U \times Y \to \mathbb{R}_+$ be two functions such that

(3.2)

$$\mathcal{M}_0 := \left\{ n \in \mathbb{N}_{n_0} : \alpha_n := |r|^\beta \sigma_1\left(\tfrac{n-1}{an}\right)\sigma_2\left(\tfrac{n-1}{an}\right) + |s|^\beta \sigma_1\left(\tfrac{1}{bn}\right)\sigma_2\left(\tfrac{1}{bn}\right) < 1 \right\} \neq \emptyset,$$

where $\sigma_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx,z) \leqslant t h_i(x,z) \ \text{ for all } \ (x,z) \in U \times Y\}$ for $n \in \mathbb{F}\setminus\{0\}$ and $i = 1,2$, such that

$$\begin{aligned} \lim_{n\to\infty} \sigma_1\left(\frac{n-1}{an}\right)\sigma_2\left(\frac{n-1}{an}\right) &= \lim_{n\to\infty} \sigma_1\left(\frac{1}{bn}\right)\sigma_2\left(\frac{1}{bn}\right) \\ &= \lim_{n\to\infty} \sigma_1\left(\frac{n-1}{an}\right)\sigma_2\left(\frac{1}{bn}\right) = 0, \end{aligned} \tag{3.3}$$

where $n \to \infty$ in $\mathbb{F}$ if and only if $|n| \to \infty$. Suppose that $f : U \to Y$ satisfies the inequality

$$\begin{aligned} &\|f(ax+by) - rf(x) - sf(y), z\|_\beta \\ &\leqslant h_1(x,z)h_2(y,z), \ z \in Y,\ x,y \in U, \ \ ax+by \in U, \end{aligned} \tag{3.4}$$

with $r, s \in \mathbb{K}$, then (1.7) holds.

Proof. Replacing (x,y) by $\left(\left(1-\frac{1}{m}\right)\frac{x}{a},\frac{x}{bm}\right)$ in (3.4), we get

$$\begin{aligned}&\left\|f(x)-rf\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)-sf\left(\frac{x}{bm}\right),z\right\|_\beta\\&\leqslant h_1\left(\left(1-\frac{1}{m}\right)\frac{x}{a},z\right)h_2\left(\frac{x}{bm},z\right)\end{aligned}\tag{3.5}$$

for all $m\in\mathbb{N}_{n_0}$ and $(x,z)\in U\times Y$. Fix $m\in\mathbb{N}_{n_0}$ and we define

$$\mathcal{T}_m\xi(x):=r\xi\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)+s\xi\left(\frac{x}{bm}\right),\ \ \xi\in Y^U,$$

$$\Lambda_m\delta(x):=|r|^\beta\delta\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)+|s|^\beta\delta\left(\frac{x}{bm}\right),\quad \delta\in\mathbb{R}_+^U$$

for every $x\in U$. Further, observe that

$$\begin{aligned}\varepsilon_m(x):&=h_1\left(\left(1-\frac{1}{m}\right)\frac{x}{a},z\right)h_2\left(\frac{x}{bm},z\right)\\&\leqslant\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)h_1(x,z)h_2(x,z)\end{aligned}\tag{3.6}$$

for all $(x,z)\in U\times Y$. Then inequality (3.5) takes the form

$$\|\mathcal{T}_mf(x)-f(x),z\|_\beta\leqslant\varepsilon_m(x),\qquad (x,z)\in U\times Y$$

and the operator Λ_m has the form described in (**H4**) with $k=2$,

$$f_1(x)=\left(1-\frac{1}{m}\right)\frac{x}{a},\qquad f_2(x)=\frac{x}{bm},\qquad L_1(x)=|r|^\beta,\qquad L_2(x)=|s|^\beta$$

for all $x\in U$. Moreover, for every $\xi,\mu\in Y^U$ and $(x,z)\in U\times Y$, we obtain

$$\begin{aligned}\|\mathcal{T}_m\xi(x)-\mathcal{T}_m\mu(x),z\|_\beta&\leqslant|r|^\beta\|(\xi-\mu)(f_1(x)),z\|_\beta+|s|^\beta\|(\xi-\mu)(f_2(x)),z\|_\beta\\&=\sum_{i=1}^2L_i(x)\|(\xi-\mu)(f_i(x)),z\|_\beta,\end{aligned}$$

where $(\xi-\mu)(x)\equiv\xi(x)-\mu(x)$. So, **(H3)** is valid for $\mathcal{T}_m$.

By using mathematical induction, we will show that for each $(x,z)\in U\times Y$ we have

$$\Lambda_m^n\varepsilon_m(x)\leqslant\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)\alpha_m^nh_1(x,z)h_2(x,z),\tag{3.7}$$

for all $n\in\mathbb{N}_0$ and $m\in\mathcal{M}_0$. From (3.6), we obtain that the inequality (3.7) holds for $n=0$. Next, we will assume that (3.7) holds for $n=l$, where $l\in\mathbb{N}_0$. Then we have

$$\begin{aligned}\Lambda_m^{l+1}\varepsilon_m(x)&=\Lambda_m(\Lambda_m^l\varepsilon_m(x))\\&=|r|^\beta\Lambda_m^l\varepsilon_m\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)+|s|^\beta\Lambda_m^l\varepsilon_m\left(\frac{x}{bm}\right)\end{aligned}$$

$$\begin{aligned}
&\leqslant |r|^{\beta}\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)\alpha_m^l h_1\left(\left(1-\frac{1}{m}\right)\frac{x}{a},z\right)\\
&\quad h_2\left(\left(1-\frac{1}{m}\right)\frac{x}{a},z\right)\\
&\quad + |s|^{\beta}\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)\alpha_m^l h_1\left(\frac{x}{bm},z\right)h_2\left(\frac{x}{bm},z\right)\\
&\leqslant \sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)\alpha_m^{l+1}h_1(x,z)h_2(x,z).
\end{aligned}$$

This shows that (3.7) holds for $n = l+1$. Now we can conclude that the inequality (3.7) holds for all $n \in \mathbb{N}_0$. Therefore, we obtain that

$$\begin{aligned}
\varepsilon_m^*(x) &:= \sum_{n=0}^{\infty}\Lambda_m^n\varepsilon_m(x)\\
&\leqslant \frac{\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)h_1(x,z)h_2(x,z)}{1-\alpha_m}, \quad (x,z)\in U\times Y,\ \ m\in\mathcal{M}_0.
\end{aligned}$$

Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the function $L_m : U \to Y$, given by $L_m(x) = \lim_{n\to\infty}\mathcal{T}_m^n f(x)$ for $x \in U$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$L_m(x) = rL_m\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right) + sL_m\left(\frac{x}{bm}\right)$$

for all $x \in U$; moreover

$$\begin{aligned}
&\|f(x)-L_m(x),z\|_{\beta}\\
&\leqslant \frac{\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right)h_1(x,z)h_2(x,z)}{1-\alpha_m}, \quad (x,z)\in U\times Y,\ \ m\in\mathcal{M}_0.
\end{aligned}$$

We show that

$$\|\mathcal{T}_m^n f(ax+by) - r\mathcal{T}_m^n f(x) - s\mathcal{T}_m^n f(y), z\|_{\beta} \leqslant \alpha_m^n h_1(x,z)h_2(y,z) \tag{3.8}$$

for every $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$, $z \in Y$ and $x, y \in U$ with $ax + by \in U$.

Clearly, if $n = 0$, then (3.8) is simply (3.4). So, fix $n \in \mathbb{N}\cup\{0\}$ and suppose that (3.8) holds for n and every $z \in Y$ and $x, y \in U$ with $ax + by \in U$. Then, for every $m \in \mathcal{M}_0$, $z \in Y$ and $x, y \in U$ with $ax + by \in U$,

$$\begin{aligned}
&\left\|\mathcal{T}_m^{n+1}f(ax+by) - r\mathcal{T}_m^{n+1}f(x) - s\mathcal{T}_m^{n+1}f(y), z\right\|_{\beta}\\
&= \left\|r\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{ax+by}{a}\right) + s\mathcal{T}_m f\left(\frac{ax+by}{bm}\right) - r^2\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)\right.\\
&\quad \left. - rs\mathcal{T}_m f\left(\frac{x}{bm}\right) - rs\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{y}{a}\right) - s^2\mathcal{T}_m f\left(\frac{y}{bm}\right), z\right\|_{\beta}\\
&\leqslant |r|^{\beta}\left\|\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{ax+by}{a}\right) - r\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{x}{a}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - s\mathcal{T}_m f\left(\left(1-\frac{1}{m}\right)\frac{y}{a}\right), z\Big\|_\beta \\
&\quad + |s|^\beta \left\|\mathcal{T}_m f\left(\frac{ax+by}{bm}\right) - r\mathcal{T}_m f\left(\frac{x}{bm}\right) - s\mathcal{T}_m f\left(\frac{y}{bm}\right), z\right\|_\beta \\
&\leqslant |r|^\beta \alpha_m^n h_1\left(\left(1-\frac{1}{m}\right)\frac{x}{a}, z\right) h_2\left(\left(1-\frac{1}{m}\right)\frac{y}{a}, z\right) \\
&\quad + |s|^\beta \alpha_m^n h_1\left(\frac{x}{bm}, z\right) h_2\left(\frac{y}{bm}, z\right) \\
&\leqslant \alpha_m^{n+1} h_1(x,z) h_2(y,z).
\end{aligned}
$$

Thus, by induction, we have shown that (3.8) holds for all $z \in Y$ and $x, y \in U$ such that $ax + by \in U$ and for all $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (3.8), we obtain that

$$
L_m(ax+by) = rL_m(x) + sL_m(y) \tag{3.9}
$$

for every $m \in \mathcal{M}_0$ and $x, y \in U$ with $ax + by \in U$.

In this way, for each $m \in \mathcal{M}_0$, we obtain a function L_m such that (3.9) holds for $x, y \in U$ with $ax + by \in U$ and

$$
\begin{aligned}
&\|f(x) - L_m(x), z\|_\beta \\
&\leqslant \frac{\sigma_1\left(\frac{m-1}{am}\right)\sigma_2\left(\frac{1}{bm}\right) h_1(x,z) h_2(x,z)}{1-\alpha_m}, \quad (x,z) \in U \times Y, \ \ m \in \mathcal{M}_0.
\end{aligned}
$$

It follows, with $m \to \infty$, that f fulfils (1.7). □

In a similar way we can prove the following theorem.

Theorem 3.2. *Let U be a nonempty subset of X_0 and there is $n_0 \in \mathbb{N}$, with*

$$
-\frac{n}{a}x, \ \ \frac{n+1}{b}x \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geqslant n_0, \tag{3.10}
$$

where $a, b \in \mathbb{F}\backslash\{0\}$. Let $r, s \in \mathbb{K}$, $0 < \beta \leqslant 1$ and $h : U \times Y \to \mathbb{R}_+$ be a function such that

$$
\mathcal{M}_0 := \left\{ n \in \mathbb{N}_{n_0} : d_n := |r|^\beta \sigma\left(-\frac{n}{a}\right) + |s|^\beta \sigma\left(\frac{n+1}{b}\right) < 1 \right\} \neq \emptyset, \tag{3.11}
$$

where $\sigma(n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant th(x, z) \ \ \text{for all} \ \ (x, z) \in U \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$, such that

$$
\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0, \tag{3.12}
$$

where $n \to \infty$ in $\mathbb{F}$ if and only if $|n| \to \infty$. If $f : U \to Y$ satisfies the functional inequality

$$
\begin{aligned}
\|f(ax+by) - rf(x) - sf(y), z\|_\beta \leqslant\ & h(x,z) + h(y,z), \\
& z \in Y,\ x, y \in U,\ \ ax + by \in U,
\end{aligned} \tag{3.13}
$$

then (1.7) *holds.*

Proof. Replacing (x,y) by $\left(-\frac{m}{a}x, \frac{m+1}{b}x\right)$ in (3.13), we get

(3.14)
$$\left\|f(x) - rf\Big(-\frac{m}{a}x\Big) - sf\Big(\frac{m+1}{b}x\Big), z\right\|_\beta \leqslant h\left(-\frac{m}{a}x, z\right) + h\left(\frac{m+1}{b}x, z\right)$$

for all $(x,z) \in U \times Y$ and $m \in \mathbb{N}_{n_0}$. Let

$$\begin{aligned} \varepsilon_m(x) :&= h\left(-\frac{m}{a}x, z\right) + h\left(\frac{m+1}{b}x, z\right) \\ &\leqslant \left(\sigma\left(-\frac{m}{a}\right) + \sigma\left(\frac{m+1}{b}\right)\right) h(x,z) \end{aligned} \tag{3.15}$$

$$\mathcal{T}_m\xi(x) := r\xi\Big(-\frac{m}{a}x\Big) + s\xi\Big(\frac{m+1}{b}x\Big)$$

for $x \in U$, $m \in \mathbb{N}_{n_0}$ and $\xi \in Y^U$. Then inequality (3.14) takes the form

$$\|\mathcal{T}_m f(x) - f(x), z\|_\beta \leqslant \varepsilon_m(x), \quad (x,z) \in U \times Y, \ \ m \in \mathbb{N}_{n_0}.$$

Write

$$\Lambda_m\delta(x) = |r|^\beta\delta\Big(-\frac{m}{a}x\Big) + |s|^\beta\delta\Big(\frac{m+1}{b}x\Big)$$

for $x \in U$, $m \in \mathbb{N}_{n_0}$ and $\delta \in \mathbb{R}_+^U$. Then, for each $m \in \mathbb{N}_{n_0}$, operator Λ_m has the form described in **(H4)** with $k = 2$ and

$$f_1(x) \equiv -\frac{m}{a}x, \qquad f_2(x) \equiv \frac{m+1}{b}x, \qquad L_1(x) \equiv |r|^\beta, \qquad L_2(x,y) \equiv |s|^\beta.$$

Moreover, for every $\xi, \mu \in Y^U$, $m \in \mathbb{N}_{n_0}$ and $(x,z) \in U \times Y$, we have

$$\begin{aligned} \|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x), z\|_\beta &= \|r(\xi-\mu)(f_1(x)) + s(\xi-\mu)(f_2(x)), z\|_\beta \\ &\leqslant \sum_{i=1}^{2} L_i(x)\, \|(\xi-\mu)(f_i(x,y)), z\|_\beta . \end{aligned}$$

So, **(H3)** is valid for $\mathcal{T}_m$.

Next, it is clear that, by induction on n, from (3.15) we obtain

$$\Lambda_m^n\varepsilon_m(x) \leqslant \left(\sigma\left(-\frac{m}{a}\right) + \sigma\left(\frac{m+1}{b}\right)\right) d_m^n h(x,z) \tag{3.16}$$

for all $n \in \mathbb{N}_0$ and $m \in \mathcal{M}_0$. Therefore, we obtain that

$$\begin{aligned} \varepsilon_m^*(x) :&= \sum_{n=0}^{\infty} \Lambda_m^n\varepsilon_m(x) \\ &\leqslant \frac{\left(\sigma\left(-\frac{m}{a}\right) + \sigma\left(\frac{m+1}{b}\right)\right) h(x,z)}{1-d_m}, \quad (x,z) \in U \times Y, \ \ m \in \mathcal{M}_0. \end{aligned}$$

Thus, according to Theorem 2.1, for each $m \in \mathcal{M}_0$ the function $L_m : U \to Y$, given by $L_m(x) = \lim_{n\to\infty} \mathcal{T}_m^n f(x)$ for $x \in U$, is a unique fixed point of $\mathcal{T}_m$, i.e.,

$$L_m(x) = rL_m\Big(-\frac{m}{a}x\Big) + sL_m\Big(\frac{m+1}{b}x\Big)$$

for all $x \in U$; moreover

$$\|f(x) - L_m(x), z\|_\beta \leqslant \frac{\big(\sigma\left(-\frac{m}{a}\right) + \sigma\left(\frac{m+1}{b}\right)\big)\, h(x,z)}{1-d_m},\ (x,z) \in U \times Y,\ \ m \in \mathcal{M}_0.$$

Similarly as in the proof of Theorem 3.1 we show that

$$\|\mathcal{T}_m^n f(ax+by) - r\mathcal{T}_m^n f(x) - s\mathcal{T}_m^n f(y), z\|_\beta \leqslant d_m^n\big(h(x,z) + h(y,z)\big) \tag{3.17}$$

for every $n \in \mathbb{N}_0$, $m \in \mathcal{M}_0$, $z \in Y$ and $x, y \in U$ with $ax + by \in U$. Also the remaining reasonings are analogous as in the proof of that theorem. □

The next theorems shows the hyperstability of the general linear equation for $U = X$. In 2015, Brzdęk [8] proved the following result.

Lemma 3.1 ([8])**.** *Assume that E_1 is a linear space over a field $\mathbb{F}$, E_2 is a linear space over a field $\mathbb{K}$, $a, b \in \mathbb{F}\backslash\{0\}$, $r, s \in \mathbb{K}$ and $f : E_1 \to E_2$ satisfies*

$$f(ax+by) = rf(x) + sf(y), \quad x, y \in E_1\backslash\{0\}.$$

Then f satisfies the equation

$$f(ax+by) = rf(x) + sf(y), \quad x, y \in E_1.$$

Using the above theorems and Lemma 3.1, we obtain the following results.

Theorem 3.3. *Let $r, s \in \mathbb{K}$, $0 < \beta \leqslant 1$ and $h_1, h_2 : X \times Y \to \mathbb{R}_+$ be two functions such that* (3.2) *is an infinite set, where $\sigma_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leqslant th_i(x,z)$ for all $(x,z) \in X \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$ and $i = 1, 2$, such that* (3.3) *holds. If $f : X \to Y$ satisfies* (3.4) *for all $z \in Y$ and $x, y \in X\backslash\{0\}$, then f satisfies the equation*

$$f(ax+by) = rf(x) + sf(y), \quad x, y \in X.$$

Theorem 3.4. *Let $r, s \in \mathbb{K}$, $0 < \beta \leqslant 1$ and $h : X \times Y \to \mathbb{R}_+$ be a functions such that* (3.11) *is an infinite set, where $\sigma(n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant th(x,z)$ for all $(x,z) \in X \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$, such that*

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

If $f : X \to Y$ satisfies (3.13) *for all $z \in Y$ and $x, y \in X\backslash\{0\}$, then f satisfies the equation*

$$f(ax+by) = rf(x) + sf(y), \quad x, y \in X.$$

According to Theorems 3.1, 3.2 and the same technique as in the proof of [8, Corollary 4.8], we get the following corollaries.

Corollary 3.1. *Let U be a nonempty subset of X_0 fulfilling condition* (3.1) *with some $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$. Let $F : U^2 \to Y$ be a given mapping and $h_1, h_2 : U \times Y \to \mathbb{R}_+$ be two functions such that* (3.2) *is an infinite set, where $\sigma_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leqslant t h_i(x, z) \text{ for all } (x, z) \in U \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$ and $i = 1, 2$, such that* (3.3) *holds. Suppose that $f : U \to Y$ satisfies the condition*

$$\begin{aligned} &\|f(ax + by) - rf(x) - sf(y) - F(x, y), z\|_\beta \\ &\leqslant h_1(x, z)h_2(y, z), \quad z \in Y, \quad x, y \in U, \ ax + by \in U, \end{aligned} \tag{3.18}$$

and the functional equation

$$g(ax + by) = rg(x) + sg(y) + F(x, y), \quad x, y \in U, \ ax + by \in U \tag{3.19}$$

has a solution $f_0 : U \to Y$, where $a, b \in \mathbb{F}\backslash\{0\}$ and $r, s \in \mathbb{K}$. Then f is a solution of (3.19).

Corollary 3.2. *Let U be a nonempty subset of X_0 fulfilling condition* (3.10) *with some $n_0 \in \mathbb{N}$. Let $F : U^2 \to Y$ be a given mapping and $h : U \times Y \to \mathbb{R}_+$ be a function such that* (3.11) *is an infinite set, where $\sigma(n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant t h(x, z) \text{ for all } (x, z) \in U \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$, such that*

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

Suppose that $f : U \to Y$ satisfies the condition

$$\begin{aligned} &\|f(ax + by) - rf(x) - sf(y) - F(x, y), z\|_\beta \\ &\leqslant h(x, z) + h(y, z), \quad z \in Y, \quad x, y \in U, \ ax + by \in U, \end{aligned} \tag{3.20}$$

and the functional equation

$$g(ax + by) = rg(x) + sg(y) + F(x, y), \quad x, y \in U, \ ax + by \in U, \tag{3.21}$$

has a solution $f_0 : U \to Y$, where $a, b \in \mathbb{F}\backslash\{0\}$ and $r, s \in \mathbb{K}$. Then f is a solution of (3.21).

Remark 1. It is easily seen that (under the assumptions of Corollary 3.1 (or Corollary 3.2)) in the case $r + s \neq 1$ and K is a constant function, $F(x, y) \equiv K$, (3.19) (or (3.21)) admits a constant solution of the form

$$f_0(x) = \frac{K}{1 - r - s}, \quad x \in U.$$

4. Some particular cases

At the end of this paper, we derive some corollaries of our main results.

Corollary 4.1. *Assume that $U \subset X_0$ is nonempty and there is $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$ with*

$$2\left(1 - \frac{1}{n}\right)x, \ \frac{2x}{n} \in U, \qquad x \in U, \ n \in \mathbb{N}, \ n \geqslant n_0. \tag{4.1}$$

Let $h_1, h_2 : U \times Y \to \mathbb{R}_+$ be two functions such that

(4.2)
$$\mathcal{M}_0 := \left\{n \in \mathbb{N}_{n_0} : 2^{-\beta}\sigma_1\left(\tfrac{2n-2}{n}\right)\sigma_2\left(\tfrac{2n-2}{n}\right) + 2^{-\beta}\sigma_1\left(\tfrac{2}{n}\right)\sigma_2\left(\tfrac{2}{n}\right) < 1\right\} \neq \emptyset,$$

where $\sigma_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leqslant t h_i(x, z) \ \text{ for all } \ (x, z) \in U \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$ and $i = 1, 2$, such that

$$\lim_{n\to\infty} \sigma_1\left(\frac{2n-2}{n}\right)\sigma_2\left(\frac{2n-2}{n}\right) = \lim_{n\to\infty} \sigma_1\left(\frac{2}{n}\right)\sigma_2\left(\frac{2}{n}\right) = \lim_{n\to\infty} \sigma_1\left(\frac{2n-2}{n}\right)\sigma_2\left(\frac{2}{n}\right) = 0. \tag{4.3}$$

If $f : U \to Y$ satisfies

$$\begin{aligned}&\left\|f\Big(\frac{1}{2}(x+y)\Big) - \frac{1}{2}(f(x)+f(y)), z\right\|_\beta \\ &\leqslant h_1(x,z)h_2(y,z), \quad z \in Y,\ x, y \in U,\ \ \frac{1}{2}(x+y) \in U,\end{aligned}$$

then f is Jensen on U, i.e.,

$$f\Big(\frac{1}{2}(x+y)\Big) = \frac{1}{2}(f(x)+f(y)), \qquad x, y \in U,\ \ \frac{1}{2}(x+y) \in U. \tag{4.4}$$

Proof. Letting $a = b = r = s = \frac{1}{2}$ in Theorem 3.1, we get the desired result. □

Corollary 4.2. *Let U be a nonempty subset of X_0 and there is $n_0 \in \mathbb{N}$, with*

$$-x,\ nx \in U, \qquad x \in U,\ n \in \mathbb{N},\ n \geqslant n_0. \tag{4.5}$$

Let $h : U \times Y \to \mathbb{R}_+$ be a function such that

$$\mathcal{M}_0 := \left\{n \in \mathbb{N}_{n_0} : 2^{-\beta}\sigma(-2n) + 2^{-\beta}\sigma\left(2n+2\right) < 1\right\} \neq \emptyset, \tag{4.6}$$

where $\sigma(\pm n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant t h(x, z) \ \text{ for all } \ (x, z) \in U \times Y\}$ for $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

If $f : U \to Y$ satisfies

$$\begin{aligned}&\left\|f\Big(\frac{1}{2}(x+y)\Big) - \frac{1}{2}(f(x)+f(y)), z\right\|_\beta \\ &\leqslant h(x,z) + h(y,z), \quad z \in Y,\ x, y \in U,\ \ \frac{1}{2}(x+y) \in U,\end{aligned}$$

then (4.4) *holds.*

Proof. Letting $a = b = r = s = \frac{1}{2}$ in Theorem 3.2, we get the desired result. □

Corollary 4.3. *Let U be a nonempty subset of X_0 fulfilling condition* (4.1) *with some $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$. Let $J : U^2 \to Y$ be a given mapping and $h_1, h_2 : U \times Y \to \mathbb{R}_+$ be two functions such that* (4.2) *is an infinite set, where $\sigma_i(n) := \inf\{t \in \mathbb{R}_+ : h_i(nx, z) \leqslant t h_i(x, z) \text{ for all } (x, z) \in U \times Y\}$ for $n \in \mathbb{F}\backslash\{0\}$ and $i = 1, 2$, such that* (4.3) *holds. Suppose that $f : U \to Y$ satisfies the condition*

$$\begin{aligned}&\Big\|f\Big(\frac{1}{2}(x+y)\Big) - \frac{1}{2}(f(x)+f(y)) - J(x,y), z\Big\|_\beta \\ &\leqslant h_1(x,z)h_2(y,z), \quad z \in Y,\ x, y \in U,\ \ \frac{1}{2}(x+y) \in U,\end{aligned}$$

and the functional equation

$$g\Big(\frac{1}{2}(x+y)\Big) = \frac{1}{2}(g(x)+g(y)) + J(x,y), \quad x, y \in U,\ \frac{1}{2}(x+y) \in U, \tag{4.7}$$

has a solution $g_0 : U \to Y$. Then f is a solution of (4.7).

Corollary 4.4. *Let U be a nonempty subset of X_0 fulfilling condition* (4.5) *with some $n_0 \in \mathbb{N}$. Let $J : U^2 \to Y$ be a given mapping and $h : U \times Y \to \mathbb{R}_+$ be a function such that* (4.6) *is an infinite set, where $\sigma(\pm n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant t h(x, z) \text{ for all } (x, z) \in U \times Y\}$ for $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

Suppose that $f : U \to Y$ satisfies the condition

$$\begin{aligned}&\Big\|f\Big(\frac{1}{2}(x+y)\Big) - \frac{1}{2}(f(x)+f(y)) - J(x,y), z\Big\|_\beta \\ &\leqslant h(x,z) + h(y,z), \quad z \in Y,\ x, y \in U,\ \ \frac{1}{2}(x+y) \in U,\end{aligned}$$

and the equation (4.7) *has a solution $f_0 : U \to Y$. Then f is a solution of* (4.7).

Corollary 4.5. *Let U be a nonempty subset of X_0 fulfilling condition* (4.5) *with some $n_0 \in \mathbb{N}$. Let $h : U \times Y \to \mathbb{R}_+$ be a function such that*

$$\mathcal{M}_0 := \left\{ n \in \mathbb{N}_{n_0} : \sigma(-n) + \sigma(n+1) < 1 \right\} \neq \emptyset, \tag{4.8}$$

where $\sigma(\pm n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant t h(x, z) \text{ for all } (x, z) \in U \times Y\}$ for $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

If $f : U \to Y$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y), z\|_\beta \leqslant h(x,z) + h(y,z), \quad z \in Y,\ x, y \in U,\ \ x+y \in U,$$

then f is additive on U, i.e.,

$$f(x+y) = f(x) + f(y), \quad x, y \in U,\ \ x+y \in U.$$

Proof. Letting $a = b = r = s = 1$ in Theorem 3.2, we get the desired result. □

Corollary 4.6. *Let U be a nonempty subset of X_0 fulfilling condition* (4.5) *with some $n_0 \in \mathbb{N}$. Let $C : U^2 \to Y$ be a given mapping and $h : U \times Y \to \mathbb{R}_+$ be a function such that* (4.8) *is an infinite set, where $\sigma(\pm n) := \inf\{t \in \mathbb{R}_+ : h(nx, z) \leqslant th(x, z)$ for all $(x, z) \in U \times Y\}$ for $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \sigma(n) = \lim_{n\to\infty} \sigma(-n) = 0.$$

If $f : U \to Y$ satisfies the functional inequality

$$\begin{aligned} &\|f(x+y) - f(x) - f(y) - C(x,y), z\|_\beta \\ &\leqslant h(x,z) + h(y,z), \quad z \in Y,\ x, y \in U,\ \ x + y \in U, \end{aligned}$$

and the functional equation

$$h(x+y) = h(x) + h(y) + C(x,y), \quad x, y \in U,\ x + y \in U, \tag{4.9}$$

has a solution $h_0 : U \to Y$, then f is a solution of (4.9).

According to Theorem 3.1, Theorem 3.2 and Corollary 3.1 and Corollary 3.2 with $h(x, z) := c\|x\|_\beta^p \|z, w\|_\beta$ and $h_i(x, z) := c_i\|x\|_\beta^{p_i} \|z, w\|_\beta^{1/2}$ for all $(x, z) \in U \times Y$ and for some arbitrary element $w \in Y$ where $c, p, c_i, p_i \in \mathbb{R}$ for $i = 1, 2$, we get the improvement of the main result of Piszczek [34] in $(2, \beta)$-Banach space as follows:

Corollary 4.7. *Let U be a nonempty subset of X_0 fulfilling condition* (3.1) *with some $n_0 \in \mathbb{N}$, $n_0 \geqslant 2$. If $f : U \to Y$ satisfies the functional inequality*

$$\begin{aligned} &\|f(ax+by) - rf(x) - sf(y), z\|_\beta \\ &\leqslant c\|x\|_\beta^p \|y\|_\beta^q \|z, w\|_\beta, \ \ z \in Y,\ x, y \in U,\ ax + by \in U, \end{aligned}$$

for some fixed element $w \in Y$, $c \geqslant 0$, $r, s \in \mathbb{K}$ and $p, q \in \mathbb{R}$ such that $p + q > 0$, $q > 0$ and $|r| < |a|^{p+q}$, then (1.7) *holds.*

Corollary 4.8. *Let U be a nonempty subset of X_0 fulfilling condition* (3.10) *with some $n_0 \in \mathbb{N}$. If $f : U \to Y$ satisfies the functional inequality*

$$\begin{aligned} &\|f(ax+by) - rf(x) - sf(y), z\|_\beta \\ &\leqslant c(\|x\|_\beta^p + \|y\|_\beta^p)\|z, w\|_\beta, \ \ z \in Y,\ x, y \in U,\ ax + by \in U, \end{aligned}$$

for some fixed element $w \in Y$, $c \geqslant 0$, $r, s \in \mathbb{K}$ and $p \in \mathbb{R}$ such that $p < 0$, then (1.7) *holds.*

Corollary 4.9. *Let U be a nonempty subset of X_0, $F : U^2 \to Y$ be a given mapping and w be a fixed element of Y. Suppose that $f : U \to Y$ satisfies the condition*

$$\begin{aligned} &\|f(ax+by) - rf(x) - sf(y) - F(x,y), z\|_\beta \\ &\leqslant c\|x\|_\beta^p \|y\|_\beta^q \|z, w\|_\beta, \ \ z \in Y,\ x, y \in U,\ ax + by \in U, \end{aligned} \tag{4.10}$$

or

$$\|f(ax+by) - rf(x) - sf(y) - F(x,y), z\|_\beta \tag{4.11}$$

$$\leqslant c\big(\|x\|_\beta^p + \|y\|_\beta^p\big)\|z,w\|_\beta, \quad z \in Y,\ x,y \in U,\ ax+by \in U,$$

and the functional equation

$$g(ax+by) = rg(x) + sg(y) + F(x,y), \quad x,y \in U,\ ax+by \in U, \tag{4.12}$$

has a solution $f_0 : U \to Y$, *where* $c \geqslant 0$, $a,b \in \mathbb{F}\backslash\{0\}$, $r,s \in \mathbb{K}$ *and* $p,q \in \mathbb{R}$. *Assume that one of the following conditions is valid.*

(i) $p+q > 0$, $q > 0$ $|r| < |a|^{p+q}$ *and* (3.1) *holds with some* $n_0 \in \mathbb{N}$ *and* $n_0 \geqslant 2$, *in the case* f *satisfies* (4.10),

(ii) $p < 0$ *and* (3.10) *holds with some* $n_0 \in \mathbb{N}$, *in the case* f *satisfies* (4.11).

Then f *is a solution to* (4.12).

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Iz-iddine EL-Fassi
Department of Mathematics
Faculty of Sciences
Ibn Tofail University
BP 133, Kenitra, Morocco
Email address: izelfassi.math@gmail.com; izidd-math@hotmail.fr