Commun. Korean Math. Soc. **33** (2018), No. 3, pp. 833–851
https://doi.org/10.4134/CKMS.c170282
pISSN: 1225-1763 / eISSN: 2234-3024

NORMALITY CRITERIA FOR A FAMILY OF MEROMORPHIC FUNCTIONS WITH MULTIPLE ZEROS

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Abstract. In this article, we prove some normality criteria for a family of meromorphic functions having zeros with some multiplicity. Our main result involves sharing of a holomorphic function by certain differential polynomials. Our results generalize some of the results of Fang and Zalcman [4] and Chen *et al.* [2] to a great extent.

1. Introduction and main results

One important aspect of the theory of complex analytic functions is to find normality criteria for families of meromorphic functions. The notion of normal families was introduced by Paul Montel in 1907. Let us begin by recalling the definition. *A family of meromorphic* (*holomorphic*) *functions defined on a domain $D \subset \mathbb{C}$ is said to be* normal *in the domain, if every sequence in the family has a subsequence which converges spherically uniformly on compact subsets of D to a meromorphic* (*holomorphic*) *function or to ∞* [1,5,10,14,15].

In [9], Mues and Steinmetz proved a uniqueness theorem which says that: *If a non-constant meromorphic function f in the plane, shares three distinct complex numbers a_1, a_2, a_3 with its first order derivative f', then $f \equiv f'$.* Wilhelm Schwick [11] was the first who gave a connection between normality and shared values and proved a theorem related to above result of [9] which says that: *A family $\mathcal{F}$ of meromorphic functions on a domain D is normal, if f and f' share a_1, a_2, a_3 for every $f \in \mathcal{F}$, where a_1, a_2, a_3 are distinct complex numbers.*

Let us recall the definition of shared value. Let f be a meromorphic function on a domain $D \subset \mathbb{C}$. For $p \in \mathbb{C}$, let

$$E_f(p) = \{z \in D : f(z) = p\}$$

and let

$$E_f(\infty) = \text{poles of } f \text{ in } D.$$

Received July 10, 2017; Accepted August 29, 2017.
2010 *Mathematics Subject Classification.* 30D45.
Key words and phrases. meromorphic functions, holomorphic functions, shared values, normal families.

For $p \in \mathbb{C} \cup \{\infty\}$, two meromorphic functions f and g of D share the value p if $E_f(p) = E_g(p)$.

In 2008, Fang and Zalcman [4] proved the following normality criteria:

Theorem A ([4])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, let $n \geq 2$ be a positive integer, and $a(\neq 0), b \in \mathbb{C}$. If for each $f \in \mathcal{F}$, all zeros of f are multiple and $f + a(f')^n \neq b$ on D, then $\mathcal{F}$ is normal on D.*

Related to the above result of [4], Wang [12] proved the following result on normality and sharing value:

Theorem B ([12])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain D, let $n \geq 3$ be a positive integer. Let a, b be two finite complex numbers such that $a \neq 0$. If all zeros of f are multiple for each $f \in \mathcal{F}$, and $f + a(f')^n$ and $g + a(g')^n$ share b in D for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

Extending the result of [4], Xu, Wu and Liao [13] proved the following normality criteria:

Theorem C ([13])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on a plane domain D, let $a(\neq 0), b \in \mathbb{C}$, and n, k be two positive integers such that $n \geq k + 1$. If for each $f \in \mathcal{F}$, f has only zeros of multiplicity at least $k + 1$, and $f + a\left(f^{(k)}\right)^n \neq b$ on D, then $\mathcal{F}$ is normal on D.*

Related to the result of [13], Chen et al. [2] proved the following normality criteria concerning shared values:

Theorem D ([2])**.** *Let $\mathcal{F}$ be a family of meromorphic functions on the plane domain D, let n, k be positive integers such that $n \geq k + 2$, and a, b be two finite complex numbers such that $a \neq 0$. If all zeros of f have multiplicity at least $k + 1$ for each $f \in \mathcal{F}$, and $f + a\left(f^{(k)}\right)^n$ and $g + a\left(g^{(k)}\right)^n$ share b in D for every pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

It is evident from the following questions that the above theorem has not been stated in its full generality:

Q.1. Can we weaken the condition on n?

Q.2. Can we replace value b by any holomorphic function?

In this paper, we try to give the answers to these questions and see that after weakening the condition on $n \geq k + 2$ to $n > 2$, the theorem is valid for the case where multiplicities of zeros of $f \in \mathcal{F}$ are at least $2k + 1$, and b is a non-vanishing holomorphic function. Now we state our main result.

Theorem 1.1. *Let $\alpha \not\equiv 0$ be a holomorphic function with zeros of multiplicity at most m in D. Let $a \in \mathbb{C}$ be a non-zero constant, and n, k be positive integers such that $n > k + 1$, and $m < k$. Let $\mathcal{F}$ be a family of meromorphic functions in the domain D. Suppose that for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $2k + 2$ and all poles* (*if exist*) *have multiplicity at least $2k + 3$. If for* *each pair f, g in $\mathcal{F}$, $f + a\left(f^{(k)}\right)^n$ and $g + a\left(g^{(k)}\right)^n$ share α in D, then $\mathcal{F}$ is normal in D.*

When α is non-vanishing holomorphic function in D, we get the following strengthened result:

Theorem 1.2. *Let $\alpha \neq 0$ be a holomorphic function with zeros of multiplicity at most m in D. Let a be a non-vanishing holomorphic function in D and n, k be positive integers such that $n > 2$. Let $\mathcal{F}$ be a family of meromorphic functions in the domain D. Suppose that for each $f \in \mathcal{F}$, all zeros of f have multiplicity at least $2k+1$. If for each pair f, g in $\mathcal{F}$, $f + a\left(f^{(k)}\right)^n$ and $g + a\left(g^{(k)}\right)^n$ share α in D, then $\mathcal{F}$ is normal in D.*

We give the following example in support of Theorem 1.2.

Example. Let $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$, $n = 3$ and $k = 1$. Consider the family $\mathcal{F} = \{jz^3 : j \in \mathbb{N}\}$ and $a(z) = 1/z^3$, $\alpha(z) = z^3$. Clearly $\mathcal{F}$ satisfies all the conditions of $\mathcal{F}$ and $\mathcal{F}$ is normal in D.

We also improved Theorem C in the following manner:

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, let n, k be positive integers such that $n > 2$. Let b be a non-zero finite complex number and a be a non-vanishing holomorphic function. If for each function $f \in \mathcal{F}$, all zeros of f have multiplicity at least $2k+1$, and $f + a\left(f^{(k)}\right)^n - b$ has at most one zero in D, then $\mathcal{F}$ is normal in D.*

Also we prove a theorem on the value distribution of a transcendental meromorphic function. The following theorem on value distribution of a zero-free transcendental meromorphic function is due to Li [8] (also see [7]).

Theorem E. *Let f be a transcendental meromorphic function with $f \neq 0$, let a be non-zero finite complex number, and let $n \geq 2$ and k be two positive integers. Then $f + a\left(f^{(k)}\right)^n$ assumes each value $b \in \mathbb{C}$ infinitely often.*

In the following theorem, we prove above result for the case where $f \not\equiv 0$.

Theorem 1.4. *Let $f \not\equiv 0$ be a transcendental meromorphic function, let a be non-zero finite complex number, and let $n \geq 3$ be a positive integer. Then $f + a\left(f^{(k)}\right)^n$ assumes each value $b \in \mathbb{C}$ infinitely often.*

2. Some notations and results of Nevanlinna theory

Let $\Delta = \{z : |z| < 1\}$ be the unit disk and $\Delta(z_0, r) := \{z : |z - z_0| < r\}$. We use the following standard functions of value distribution theory, namely

$$T(r, f), m(r, f), N(r, f) \text{ and } \overline{N}(r, f).$$

We denote by $S(r, f)$ any function satisfying

$$S(r, f) = o\big(T(r, f)\big) \text{ as } r \to +\infty,$$

possibly outside of a set with finite measure.

Second Fundamental Theorem. *Suppose $f(z)$ is meromorphic in the finite plane and non-degenerate into a constant. If $a_\nu (\nu = 1, 2, \ldots, q)$ are $q (\geq 3)$ distinct complex numbers (one of them may be infinity), then*

$$(q-2)T(r,f) \leq \sum_{\nu=1}^{q} \overline{N}\left(r, \frac{1}{f-a_\nu}\right) + S(r,f). \tag{2.1}$$

3. Some lemmas

In order to prove our results we need the following lemmas. The well known Zalcman Lemma is a very important tool in the study of normal families. The following is a new version due to Zalcman [17] (also see [16], p. 814).

Lemma 3.1. *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disk Δ, with the property that for every function $f \in \mathcal{F}$, the zeros of f are of multiplicity at least l and the poles of f are of multiplicity at least k. If $\mathcal{F}$ is not normal at z_0 in Δ, then for $-l < \alpha < k$, there exist*

(1) *a sequence of complex numbers $z_n \to z_0$, $|z_n| < r < 1$,*
(2) *a sequence of functions $f_n \in \mathcal{F}$,*
(3) *a sequence of positive numbers $\rho_n \to 0$,*

such that $g_n(\zeta) = \rho_n^\alpha f_n(z_n + \rho_n \zeta)$ converges to a non-constant meromorphic function g on $\mathbb{C}$ with $g^\#(\zeta) \leq g^\#(0) = 1$. Moreover, g is of order at most two. Here, $g^\#(z) = \frac{|g'(z)|}{1+|g(z)|^2}$ is the spherical derivative of g.

Let f be a non-constant meromorphic function in $\mathbb{C}$. A differential polynomial P of f is defined by $P(z) := \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} \left(f^{(j)}(z)\right)^{S_{ij}}$, where S_{ij}'s are non-negative integers and $\alpha_i(z) \not\equiv 0$ are small functions of f, that is $T(r, \alpha_i) = o\big(T(r,f)\big)$. The lower degree of the differential polynomial P is defined by

$$d(P) := \min_{1 \leq i \leq n} \sum_{j=0}^{p} S_{ij}.$$

The following result was proved by Dethloff et al. in [3].

Lemma 3.2. *Let $a_1, \ldots, a_q$ be distinct non-zero complex numbers. Let f be a non-constant meromorphic function and let P be a non-constant differential polynomial of f with $d(P) \geq 2$. Then*

$$T(r,f) \leq \left(\frac{q\theta(P)+1}{qd(P)-1}\right) \overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{qd(P)-1} \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{P-a_j}\right) + S(r,f)$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgue measure, where $\theta(P) := \max_{1 \leq i \leq n} \sum_{j=0}^{p} jS_{ij}$.

Moreover, in the case of an entire function, we have

$$T(r,f) \leq \left(\frac{q\theta(P)+1}{qd(P)}\right) \overline{N}\left(r, \frac{1}{f}\right) + \frac{1}{qd(P)} \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{P-a_j}\right) + S(r,f)$$

for all $r \in [1, +\infty)$ *excluding a set of finite Lebesgue measure.*

This result was proved by Hinchliffe in [6] for $q = 1$.

We now prove some lemmas to establish our results in the next section.

Lemma 3.3. *Let* f *be a transcendental meromorphic function in* $\mathbb{C}$. *If all zeros of* $f(z)$ *has multiplicity at least* $2k+1$, *then for a positive integer* $n > 2$, $\left(f^{(k)}\right)^n$ *assumes every non-zero finite value* b *infinitely often.*

Proof. Suppose on the contrary that $\left(f^{(k)}\right)^n$ assumes the value b only finitely many times. Then

$$N\left(r, \frac{1}{\left(f^{(k)}\right)^n - b}\right) = O(\log r) = S(r, f). \tag{3.1}$$

Without loss of generality we may assume $b = 1$. Let $P = \left(f^{(k)}\right)^n$. It is easy to see that

$$d(P) = n \text{ and } \theta(P) = nk.$$

Clearly $d(P) > 2$. So by Lemma 3.2, we get

$$T(r, f) \leq \left(\frac{nk+1}{n-1}\right)\overline{N}\left(r, \frac{1}{f}\right) + \left(\frac{1}{n-1}\right)\overline{N}\left(r, \frac{1}{P-1}\right) + S(r, f),$$

and this gives

$$T(r, f) \leq \left(\frac{nk+1}{(n-1)(2k+1)}\right)N(r, f) + S(r, f),$$

and so

$$\left(\frac{(n-2)(k+1)}{(n-1)(2k+1)}\right)T(r, f) \leq S(r, f).$$

But this is a contradiction and hence establishes the lemma. □

Lemma 3.4. *Let* f *be a non-constant rational function in* $\mathbb{C}$ *and* $n > 2$ *be a positive integer. If all zeros of* $f(z)$ *has multiplicity at least* $2k+1$ *then* $\left(f^{(k)}\right)^n$ *has at least two distinct* b*-points, where* b *is a non-zero complex number.*

Proof. On the contrary, assume that $\left(f^{(k)}\right)^n$ has at most one b-point. Now there are two cases to consider.

<u>Case 1.</u> Let $\left(f^{(k)}\right)^n - b = 0$ has exactly one zero and let this zero be at z_0.

First we assume that f is a non-constant polynomial. Set $\left(f^{(k)}(z)\right)^n - b = A(z - z_0)^l$, where A is a non-zero constant and l is a positive integer such that $l \geq 2(k+1)$. Then $\left(\left(f^{(k)}(z)\right)^n\right)' = Al(z - z_0)^{l-1}$. This shows that z_0 is the only zero of $\left(\left(f^{(k)}(z)\right)^n\right)'$. Since zeros of $\left(f^{(k)}(z)\right)^n$ are multiple, we deduce that z_0 is a zero of $\left(f^{(k)}(z)\right)^n$, which is a contradiction, since $b \neq 0$.

Now suppose that f is a non-polynomial rational function with zeros of multiplicity at least $2k+1$. Clearly, $f^{(k)}(z)$ is non-constant. Let $b_1, b_2, \ldots, b_n$ be n distinct zeros of $w^n = b$. Let $w = f^{(k)}(z)$ then we obtain

$$\left(f^{(k)} - b_1\right)\left(f^{(k)} - b_2\right)\cdots\left(f^{(k)} - b_n\right) = 0.$$

Since z_0 is a zero of $\left(f^{(k)}\right)^n - b = 0$, so for one $j \in \{1, 2, \ldots, n\}$, $f^{(k)}(z_0) = b_j$ and $f^{(k)}(z_0) \neq b_i$ for $i(\neq j) \in \{1, 2, \ldots, n\}$. Thus we have

$$f^{(k)}(z) = b_j + \frac{A(z-z_0)^l}{Q(z)} \equiv b_i + \frac{B}{Q(z)}, \tag{3.2}$$

where A, B are non-zero constants. By (3.2) we obtain

$$(b_i - b_j)Q(z) + B = A(z-z_0)^l. \tag{3.3}$$

From (3.3) we get $l \geq k+1$, since zeros of $Q(z)$ are of multiplicity at least $k+1$. Again from (3.3), we obtain $Q(z_0) \neq 0$. After differentiating (3.2), we have

$$f^{(k+1)}(z) = \frac{A(z-z_0)^{l-1}Q(z) - A(z-z_0)^l Q'(z)}{Q^2(z)} \equiv \frac{-BQ'(z)}{Q^2(z)}, \tag{3.4}$$

which gives, $A(z-z_0)^{l-1}\left(Q(z) - (z-z_0)Q'(z)\right) = -BQ'(z)$. Since $Q(z)$ is a polynomial of degree $l \geq k+1$, whose zeros are other than z_0. This shows that $Q(z) - (z-z_0)Q'(z)$ is a non-constant polynomial. Thus we observe that the degree of $Q'(z)$ is at least l. This is a contradiction to the fact that the degree of $Q(z)$ is l.

Case 2. Let $\left(f^{(k)}\right)^n \neq b$. Let $b_1, b_2, \ldots, b_n$ be $n(\geq 3)$ distinct solutions of $w^n = b$. By Nevanlinna's second fundamental theorem,

$$T\left(r, f^{(k)}\right) \leq \sum_{\nu=1}^{n} \overline{N}\left(r, \frac{1}{f^{(k)} - b_\nu}\right) + S\left(r, f^{(k)}\right).$$

It follows that $T(r, f^{(k)}) = S(r, f^{(k)})$, which is a contradiction. This completes the proof of lemma. $\square$

Lemma 3.5 ([13]). *Let $n \geq 2$, k be positive integers, let p be a non-zero constant and let $P(z)$ be a polynomial. Then the solution of the differential equation $p\left(W^{(k)}(z)\right)^n + W(z) = P(z)$ must be a polynomial.*

Lemma 3.6. *Let f be a transcedental meromorphic function on the complex plane $\mathbb{C}$, let $a(\neq 0)$ be a complex number and let n, m, k be three positive integers such that $n \geq k+1$ and $m < k$.*

(1) *If $n \geq k+2$, then*

$$(n-1)T(r, f^{(k)}) \leq (k^2+k+1)\overline{N}(r, f) + (k+1)^2 N\left(r, \frac{1}{f + a(f^{(k)})^n - z^m}\right) + S(r, f^{(k)}).$$

(2) *If $n = k+1$, then*

$$kT(r, f^{(k)}) \leq (k^2+1)\overline{N}(r, f) + (k^2+k)N\left(r, \frac{1}{f + a(f^{(k)})^n - z^m}\right) + S(r, f^{(k)}).$$

Proof. Let

$$g = f + a(f^{(k)}) - z^m, \tag{3.5}$$

and

$$h = \frac{g^{(k)}}{g}. \tag{3.6}$$

Then $h \not\equiv 0$. Otherwise, if $h \equiv 0$, then $g^{(k)} \equiv 0$, so we conclude that g is a polynomial with degree at most $k-1$. Noting that $n = k+1 \geq 2$, we conclude from (3.5) that f must be a polynomial, which is a contradiction. By simple calculation we have

$$g^{(k)} = f^{(k)} + a(f^{(k)})^{(k)} = f^{(k)}(1 + P(f^{(k)})),$$

where

$$P(f^{(k)}) = a\left(f^{(k)}\right)^{n-k-1} \times \left(\frac{n!}{(n-k)!}\left(f^{(k+1)}\right)^k + \cdots + n\left(f^{(k)}\right)^{k-1} f^{(2k)}\right),$$

and $P(f^{(k)})$ is a homogeneous differential equation in $f^{(k)}$ of degree $n-1$. Then

$$gh = f^{(k)}(1 + P(f^{(k)}). \tag{3.7}$$

It follows from (3.5) that $T(r, g) \leq O(T(r, f))$, and so $S(r, g) = S(r, f)$. This and (3.6) gives

$$m(r, h) = S(r, f). \tag{3.8}$$

Using(3.6)-(3.8) and Nevanlinna's first fundamental theorem, we obtain

$$\begin{aligned} N\left(r, \frac{1}{f^{(k)}}\right) + N\left(r, \frac{1}{P(f^{(k)})+1}\right) &\leq N\left(r, \frac{1}{h}\right) + N\left(r, \frac{1}{g}\right) \\ &\leq N(r, h) + N\left(r, \frac{1}{g}\right) + S(r, f) \\ &\leq kN(r, f) + (k+1)N\left(r, \frac{1}{g}\right) + S(r, f). \end{aligned} \tag{3.9}$$

On the other hand, by Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} & m\left(r, \frac{1}{(f^{(k)})^{n-1}}\right) + m\left(r, \frac{1}{P(f^{(k)})+1}\right) \\ \leq\ & m\left(r, \frac{P(f^{(k)})}{(f^{(k)})^{n-1}}\right) + m\left(r, \frac{1}{P(f^{(k)})}\right) + m\left(r, \frac{1}{P(f^{(k)})+1}\right) \\ \leq\ & m\left(r, \frac{1}{P(f^{(k)})}\right) + m\left(r, \frac{1}{P(f^{(k)})+1}\right) + S(r, f) \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&\leq m\left(r, \frac{1}{P(f^{(k)})} + \frac{1}{P(f^{(k)})+1}\right) + S(r,f)\\
&\leq m\left(r, \frac{1}{[P(f^{(k)})]'}\right) + m\left(r, \frac{[P(f(k))]'}{P(f^{(k)})} + \frac{[P(f(k))]'+1}{P(f^{(k)})+1}\right) + S(r,f)\\
&\leq m\left(r, \frac{1}{[P(f^{(k)})]'}\right) + S(r,f)\\
&= T(r,[P(f^{(k)})]') - N\left(r, \frac{1}{[P(f^{(k)})]'}\right) + S(r,f).
\end{aligned}
$$

We deduce from (3.10) and Nevanlinna's first fundamental theorem that

$$
\begin{aligned}
(n-1)T(r,f^{(k)}) \leq{}& N(r,f) + (n-1)N\left(r, \frac{1}{f^{(k)}}\right)\\
&+ N\left(r, \frac{1}{P(f^{(k)})+1}\right) - N\left(r, \frac{1}{[P(f^{(k)})]'}\right) + S(r,f). \qquad (3.11)
\end{aligned}
$$

If $n = k+1$, from (3.10) and (3.11), we have

$$
kT(r,f^{(k)}) \leq (k^2+1)N(r,f) + (k^2+k)N\left(r, \frac{1}{g}\right) + S(r,f).
$$

This prove second part of the lemma. If $n \geq k+2$ and supposing that z_0 is a zero of $f^{(k)}$ of multiplicity l, we see that z_0 is a zero of $[P(f^{(k)})]'$ of multiplicity at least $(n-1)l - k - 1$. This gives

$$
\begin{aligned}
&(n-1)N\left(r, \frac{1}{f^{(k)}}\right) + N\left(r, \frac{1}{P(f^{(k)})+1}\right) - N\left(r, \frac{1}{[P(f^{(k)})]'}\right)\\
&\leq (k+1)\overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{P(f^{(k)})+1}\right). \qquad (3.12)
\end{aligned}
$$

Sustituting (3.12) in (3.11), we have

$$
(n-1)T(r,f^{(k)}) \leq \overline{N}(r,f) + (k+1)\overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{P(f^{(k)})+1}\right),
$$

which, together with (3.9) leads to

$$
(n-1)T(r,f^{(k)}) \leq (k^2+k+1)\overline{N}(r,f) + (k+1)^2 N\left(r, \frac{1}{g}\right) + S(r,f).
$$

This completes the proof of the lemma. □

Lemma 3.7. *Let f be a transcendental meromorphic function on the complex palne $\mathbb{C}$, let a be a non-zero finite complex number and let n, k and m be three positive integers such that $n \geq k+1$ and $m < k$ then $f + af^{(k)} - z^m$ assumes infinitely many zeros.*

Proof. Suppose that $f + af^{(k)} - z^m$ has finitely many zeros. Since f is transcendental, we have

$$N\left(r, \frac{1}{f + a(f^{(k)})^n - z^m}\right) = O(\log r) = S(r,f). \tag{3.13}$$

If $n = k+1$. It follows from (3.13) and first part of Lemma 3.6 that

$$\begin{aligned} kT(r,f^{(k)}) &\leq (k^2+1)\overline{N}(r,f) + S(r,f) \\ &\leq \frac{k^2+1}{k+1} N(r,f^{(k)}) + S(r,f) \\ &\leq \frac{k^2+1}{k+1} T(r,f^{(k)}) + S(r,f), \end{aligned}$$

that is,

$$\frac{k-1}{k+1} T(r,f^{(k)}) \leq S(r,f^{(k)}).$$

This contradicts the fact that f is transcendental. If $n \geq k+2$, then using (3.13) and second part of Lemma 3.6, we obtain

$$\begin{aligned} (k+1)T(r,f^{(k)}) &\leq (n-1)T(r,f^{(k)}) \\ &\leq (k^2+k+1)\overline{N}(r,f) + S(r,f) \\ &\leq \frac{k^2+k+1}{k+1} N(r,f^{(k)}) + S(r,f) \\ &\leq \frac{k^2+k+1}{k+1} T(r,f^{(k)}) + S(r,f). \end{aligned}$$

Then $T(r,f^{(k)}) \leq S(r,f^{(k)})$. But this is impossible since f is transcendental. Hence lemma is proved. □

Lemma 3.8. *Let f be a non-constant rational function and let n, m, k be three positive integers such that $n > 2$ and $m < k$. Suppose that every zero of f has multiplicity at least $2k+2$ and every pole (if exists) of f has multiplicity at least $2k+3$. Then $f + \left(f^{(k)}\right)^n - z^m$ has at least two distinct zeros.*

Proof. Let us assume that $D(f)(z) - z^m := f + \left(f^{(k)}\right)^n - z^m$ has atmost one zero. Now we consider the following cases:

Case 1. $D(f)(z) - z^m$ has exactly one zero z_0 with multiplicity l.

Case 1.1. Suppose that f is a non constant polynomial, then we set

$$f(z) = A(z-\alpha_1)^{m_1} \cdots (z-\alpha_s)^{m_s}, \tag{3.14}$$

where A is a non-zero constant, $m_i \geq 2k+2$ are integers. Now differentiating (3.14) k-times, we get

$$f^{(k)}(z) = (z-\alpha_1)^{m_1-k} \cdots (z-\alpha_s)^{m_s-k} h_1(z), \tag{3.15}$$

where h_1 is a non zero polynomial with $\deg(h_1) \leq k(s-1)$. From (3.14) and (3.15), we see that

$$\begin{aligned} D(f)(z) - z^m &= f(z) + (f^{(k)})^n(z) \\ &= A(z-\alpha_1)^{m_1} \cdots (z-\alpha_s)^{m_s} \\ &\quad + (z-\alpha_1)^{n(m_1-k)} \cdots (z-\alpha_s)^{n(m_s-k)} h_1^n(z) - z^m. \end{aligned} \tag{3.16}$$

Now, differentiating (3.16) $m+1$-times we get

$$\begin{aligned} (D(f)(z))^{(m+1)} &= (z-\alpha_1)^{m_1-m-1} \cdots (z-\alpha_s)^{m_s-m-1} \\ &\quad \cdot [1 + (z-\alpha_1)^{(n-1)m_1-nk} \cdots (z-\alpha_s)^{(n-1)m_s-nk} g_1(z)] \\ &= (z-\alpha_1)^{m_1-m-1} \cdots (z-\alpha_s)^{m_s-m-1} h_2(z), \end{aligned} \tag{3.17}$$

where h_2 is a non zero polynomial.

As $D(f)(z) - z^m$ has only one zero then from (3.17), we get a contradiction.

Case 1.2. Suppose that $f(z)$ is a non-polynomial rational function defined as

$$f(z) = A \frac{(z-\alpha_1)^{m_1} \cdots (z-\alpha_s)^{m_s}}{(z-\beta_1)^{n_1} \cdots (z-\beta_t)^{n_t}}, \tag{3.18}$$

where A is a non-zero constant, $m_i \geq 2k+2$ $(i = 1, 2, \ldots, s)$ and $n_j \geq 2k+3$ $(j = 1, 2, \ldots, t)$.

Let us define

$$\sum_{i=1}^{s} m_i = M \geq (2k+2)s \text{ and } \sum_{j=1}^{t} n_j = N \geq (2k+3)t. \tag{3.19}$$

From (3.18), it follows that

$$f^{(k)}(z) = A \frac{(z-\alpha_1)^{m_1-k} \cdots (z-\alpha_s)^{m_s-k}}{(z-\beta_1)^{n_1+k} \cdots (z-\beta_t)^{n_t+k}} g_1(z), \tag{3.20}$$

where g_1 is a non zero polynomial with $\deg(g_1) \leq k(s+t-1)$ from (3.18) and (3.20), then

$$D(f) = \frac{(z-\alpha_1)^{m_1} \cdots (z-\alpha_s)^{m_s}}{(z-\beta_1)^{n(n_1+k)} \cdots (z-\beta_t)^{n(n_t+k)}} g(z), \tag{3.21}$$

where g is a non zero polynomial and

$$\deg(g) \leq \max\{(n-1)N + nkt, (n-1)M - nks + n\deg(g_1)\}. \tag{3.22}$$

Since $D(f)(z) - z^m$ has exactly one zero at z_0 with multiplicity l, we have

$$D(f(z)) = z^m + \frac{B(z-z_0)^l}{(z-\beta_1)^{n(n_1+k)} \cdots (z-\beta_t)^{n(n_t+k)}}, \tag{3.23}$$

where B is a non-zero constant and l is a positive integer. On differentiating (3.21) and (3.23) $m+1$ times, we get

$$(D(f))^{(m+1)} = \frac{(z-\alpha_1)^{m_1-m-1} \cdots (z-\alpha_s)^{m_s-m-1} h_1(z)}{(z-\beta_1)^{n(n_1+k)+m+1} \cdots (z-\beta_t)^{n(n_t+k)+m+1}}, \tag{3.24}$$

where h_1 is a polynomial with $\deg h_1 \leq (m+1)(s+t-1) + \deg(g)$. And

$$(D(f))^{(m+1)} = \frac{(z-z_0)^{l-m-1} h_2(z)}{(z-\beta_1)^{n(n_1+k)+m+1} \cdots (z-\beta_t)^{n(n_t+k)+m+1}}, \tag{3.25}$$

where h_2 is a polynomial with $\deg h_2 \leq (m+1)t$.

Since $\alpha_i \neq z_0$ for $i = 1, 2, \ldots, s$, it follows from (3.24) and (3.25) that

$$M - (m+1)s \leq \deg(h_2) \leq (m+1)t,$$

which implies that

$$M \leq (m+1)(s+t) < (k+1)(s+t) \leq (k+1)\left(\frac{M}{2k+2} + \frac{N}{2k+3}\right).$$

Hence we deduce that

$$M < N. \tag{3.26}$$

Now we discuss the following two subcases.

Case 1.1.1 If $l \neq m + nN + nkt$. It follows from (3.21) that

$$nN + nkt \leq M + \deg(g). \tag{3.27}$$

If $\deg(g) \leq (n-1)N + nkt$, we thus from (3.21) obtain that $nq + nkt \leq M + (n-1)N + nkt$, which implies that $N \leq M < N$ by (3.26). This is impossible.

If $\deg(g) \leq (n-1)M - nks + n\deg(g_1)$, since $\deg(g_1) \leq k(s+t-1)$, hence $nN + nkt \leq (n-1)M - nks + nk(s+t-1)$, we have $N \leq M - 1 < N - 1$ by (3.26). We thus arrive at a contradiction.

Case 1.1.2 When $l = m + nN + nkt$. It is obtained from (3.24) and (3.25) that

$$l - m - 1 \leq \deg(h_1) \leq (m+1)(s+t-1) + \deg(g). \tag{3.28}$$

If $\deg(g) \leq (n-1)N + nkt$, we thus from (3.28) obtain that

$$l \leq (m+1)(s+t) + \deg(g),$$

which implies that

$$m + nN + nkt \leq (m+1)(s+t) + (n-1)N + nkt.$$

We have,

$$N \leq (m+1)(s+t) < (k+1)\left(\frac{M}{2k+2} + \frac{N}{2k+3)}\right) < M,$$

which is a contradiction.

If

$$\deg(g) \leq (n-1)M - nks + n\deg(g_1) \leq (n-1)M - nks + nk(s+t-1).$$

By (3.28), we obtain that

$$m + nN + nkt \leq (m+1)(s+t) + (n-1)M - nks + nk(s+t-1).$$

This gives that

$$nN \le (m+1)(s+t)+(n-1)M-nk < (k+1)\left(\frac{M}{2k+2}+\frac{N}{2k+3}\right)+(n-1)M-nk.$$

Which gives $N < M-1 < N-1$, this is a contradiction.

Case 2. Let $D(f)(z)-z^m$ has no zero. Then f can not be a polynomial. Hence f is non polynomial rational function. Now putting $l=0$ in (3.23) and proceeding as in Case 1.1 of lemma, we have a contradiction. $\square$

4. Proof of main results

First we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Since normality is a local property, we assume that $D = \Delta$. Suppose that $\mathcal{F}$ is not normal in Δ. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0 in Δ. Without loss of generality, we may assume that $z_0 = 0$. By Lemma 3.1, there exist

(1) a sequence of complex numbers $z_j \to 0$, $|z_j| < r < 1$,
(2) a sequence of functions $f_j \in \mathcal{F}$,
(3) a sequence of positive numbers $\rho_j \to 0$,

such that $g_j(\zeta) = \rho_j^{-k} f_j(z_j+\rho_j\zeta)$ converges to a non-constant meromorphic function $g(\zeta)$ on $\mathbb{C}$ with $g^{\#}(\zeta) \le g^{\#}(0) = 1$. Moreover, g is of order at most two.

We see that

$$\begin{aligned} &f_j(z_j+\rho_j\zeta)+a(z_j+\rho_j\zeta)\left(f_j^{(k)}(z_j+\rho_j\zeta)\right)^n-\alpha(z_j+\rho_j\zeta)\\ &\to a(0)\left(g^{(k)}(\zeta)\right)^n-\alpha(0),\end{aligned} \tag{4.1}$$

locally uniformly with respect to spherical metric on every compact subset of $\mathbb{C}$ which contains no poles of g.

Clearly $a(0)\left(g^{(k)}(\zeta)\right)^n-\alpha(0) \not\equiv 0$. Therefore by Lemma 3.3 and Lemma 3.4, we know that $a(0)\left(g^{(k)}(\zeta)\right)^n-\alpha(0)$ has at least two distinct zeros. Now we claim that $a(0)\left(g^{(k)}(\zeta)\right)^n-\alpha(0)$ has only one zero.

Contrary to this, let $a(0)\left(g^{(k)}(\zeta)\right)^n-\alpha(0)$ has two distinct zeros at ζ_0 and ζ_1. Now choose a small positive number δ such that $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_1,\delta)=\emptyset$ and $a\left(g^{(k)}(\zeta)\right)^n-b$ has no other zeros in $\Delta(\zeta_0,\delta)\cup\Delta(\zeta_1,\delta)$. By Hurwitz's theorem, there exist two sequences $\{\zeta_j\}\subset\Delta(\zeta_0,\delta)$, $\{\zeta_{1_j}\}\subset\Delta(\zeta_1,\delta)$ converging to ζ_0, and ζ_1 respectively and from (4.1), for sufficiently large j, we have

$$f_j(z_j+\rho_j\zeta_j)+a(z_j+\rho_j\zeta_j)\left(f_j^{(k)}(z_j+\rho_j\zeta_j)\right)^n-\alpha(z_j+\rho_j\zeta_j)=0,$$
$$f_j(z_j+\rho_j\zeta_{1_j})+a(z_j+\rho_j\zeta_{1_j})\left(f_j^{(k)}(z_j+\rho_j\zeta_{1_j})\right)^n-\alpha(z_j+\rho_j\zeta_{1_j})=0.$$

Since $f + a\left(f^{(k)}\right)^n$ and $g + a\left(g^{(k)}\right)^n$ share α in Δ, therefore for any positive integer m, we have

$$f_m(z_j + \rho_j\zeta_j) + a(z_j + \rho_j\zeta_j)\left(f_m^{(k)}(z_j + \rho_j\zeta_j)\right)^n - \alpha(z_j + \rho_j\zeta_j) = 0,$$

$$f_m(z_j + \rho_j\zeta_{1_j}) + a(z_j + \rho_j\zeta_{1_j})\left(f_m^{(k)}(z_j + \rho_j\zeta_{1_j})\right)^n - \alpha(z_j + \rho_j\zeta_{1_j}) = 0.$$

Fix m and take $j \to \infty$, then we get $z_j + \rho_j\zeta_j \to 0,\ z_j + \rho_j\zeta_{1_j} \to 0$ and

$$f_m(0) + a(0)\left(f_m^{(k)}(0)\right)^n - \alpha(0) = 0. \tag{4.2}$$

Since the zeros are isolated, so for large values of j, we have $z_j + \rho_j\zeta_j = 0 = z_j + \rho_j\zeta_{1_j}$. Hence

$$\zeta_j = -\frac{z_j}{\rho_j}, \qquad \zeta_{1_j} = -\frac{z_j}{\rho_j}. \tag{4.3}$$

Which contradicts the fact that $\Delta(\zeta_0, \delta) \cap \Delta(\zeta_1, \delta) = \emptyset$. □

Proof of Theorem 1.1. As in the proof of Theorem 1.2 assume $D = \Delta$ and $z_0 = 0$.

Case 1. When $\alpha(0) \neq 0$, then there exists $r > 0$ such that $\mathcal{F}$ is normal in $|z| < r$ by Theorem 1.2.

Case 2. When $\alpha(0) = 0$. So, we can write $\alpha(z) = z^m\beta(z)$, where m is a positive integer and $\beta(z)$ is a holomorphic function in D such that $\beta(0) \neq 0$. Again assuming that $\mathcal{F}$ is not normal at 0, then by Lemma 3.1, there exist

(1) a sequence of complex numbers $z_j \to 0$, $|z_j| < r < 1$,
(2) a sequence of functions $f_j \in \mathcal{F}$,
(3) a sequence of positive numbers $\rho_j \to 0$,

such that $g_j(\zeta) = \rho_j^{-\frac{nk}{n-1}} f_j(z_j + \rho_j\zeta)$ converges compactly to a non-constant meromorphic function $g(\zeta)$. Again, we have two cases to consider:

Subcase 2.1. If $z_j/\rho_j \to \infty$. Then we consider the following family

$$\mathcal{G} := \left\{G_j(\zeta) = z_j^{-\frac{nk}{n-1}} f_j(z_j(1+\zeta)) : f_j \in \mathcal{F}\right\} \tag{4.4}$$

defined on D. From (4.4), we obtain

$$G_j^{(k)} = z_j^{-\frac{k}{n-1}} f_j^{(k)}(z_j(1+\zeta)).$$

Let us define $D(f)(z) = f(z) + a\left(f^{(k)}\right)^n(z)$, then we have

$$\begin{aligned} D(f_j)(z_j(1+\zeta)) &= f_j(z_j(1+\zeta)) + a\left\{f_j^{(k)}(z_j(1+\zeta))\right\}^n \\ &= z_j^{\frac{nk}{n-1}} G_j(\zeta) + z_j^{\frac{nk}{n-1}} a\left\{G_j^{(k)}(\zeta)\right\}^n \\ &= z_j^{\frac{nk}{n-1}} D(G_j)(\zeta). \end{aligned}$$

Now, by the hypothesis for each pair f^1, f^2 in $\mathcal{F}$,

$$(D(f^1) - \alpha)(z_j(1+\zeta)) = 0 \text{ if and only if } (D(f^2) - \alpha)(z_j(1+\zeta)) = 0.$$

This gives that

$$z_j^{\frac{nk}{n-1}} D(G^1)(\zeta) - \alpha(z_j(1+\zeta)) = 0 \text{ if and only if } z_j^{\frac{nk}{n-1}} D(G^2)(\zeta) - \alpha(z_j(1+\zeta)) = 0.$$

This means

$$D(G^1)(\zeta) = z_j^{m-\frac{nk}{n-1}}(1+\zeta)^m \beta(z_j(1+\zeta))$$
$$\text{if and only if}$$
$$D(G^2)(\zeta) = z_j^{m-\frac{nk}{n-1}}(1+\zeta)^m \beta(z_j(1+\zeta)).$$

Since $z_j^{m-\frac{nk}{n-1}}(1+\zeta)^m \beta(z_j(1+\zeta)) \neq 0$ at the origin therefore by the previous case $\mathcal{G}$ is normal in D, hence there exists a subsequence $\{G_j\}$ (after renumbering) in $\mathcal{G}$ such that $G_j \to G$, compactly in D.

Now, if $G(0) \neq 0$, then we have

$$g_j(\zeta) = \rho_j^{-\frac{nk}{n-1}} f_j(z_j + \rho_j \zeta) = \left(\frac{z_j}{\rho_j}\right)^{\frac{nk}{n-1}} G_j\left(\frac{\rho_j}{z_j}\zeta\right)$$

which converges to ∞ compactly on $\mathbb{C}$, which is a contradiction. Thus we must have $G(0) = 0$ and $G^{(2k+1)}(0) \neq \infty$.

And for each $\zeta \in \mathbb{C}$, we have

$$g_j^{(2k+1)}(\zeta) = \left(\frac{\rho_j}{z_j}\right)^{\frac{(n-2)k+n-1}{n-1}} G_j^{(2k+1)}\left(\frac{\rho_j}{z_j}\zeta\right) \to 0.$$

This implies $g^{(2k+1)}(\zeta) \to 0$, since all zeros of g have multiplicity at least $2k+2$, so g is a constant.

Subcase 2.2. If $z_j/\rho_j \to w_0$, where w_0 is a finite complex number. Then we see that

$$H_j(\zeta) = \rho_j^{-\frac{nk}{n-1}} f_j(\rho_j \zeta) = g_j\left(\zeta - \frac{z_j}{\rho_j}\right) \to g(\zeta - w_0) := H(\zeta)$$

compactly on $\mathbb{C}$.

Also from Lemma 3.7 and Lemma 3.8, we have $D(H)(\zeta) - \zeta^m$ has at least two zeros. Now we proceed as in the proof of Theorem 1.2. □

Proof of Theorem 1.3. We again assume that $D = \Delta$. Suppose that $\mathcal{F}$ is not normal in Δ. Then there exists at least one point z_0 such that $\mathcal{F}$ is not normal at the point z_0 in Δ. Without loss of generality, we may assume that $z_0 = 0$. Then by Lemma 3.1, there exist

(1) a sequence of complex numbers $z_j \to 0$, $|z_j| < r < 1$,
(2) a sequence of functions $f_j \in \mathcal{F}$,
(3) a sequence of positive numbers $\rho_j \to 0$,

such that $g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j\zeta)$ converges to a non-constant meromorphic function g on $\mathbb{C}$ with $g^{\#}(\zeta) \le g^{\#}(0) = 1$. Moreover, g is of order at most two.

We see that

$$f_j(z_j + \rho_j\zeta) + a(z_j + \rho_j\zeta)\left(f_j^{(k)}(z_j + \rho_j\zeta)\right)^n - b \to a(0)\left(g^{(k)}(\zeta)\right)^n - b, \tag{4.5}$$

locally uniformly with respect to spherical metric on every compact subsets of $\mathbb{C}$ which contains no poles of g.

Now we claim that $a(0)\left(g^{(k)}(\zeta)\right)^n - b$ has at most one zero IM. Suppose on contrary, let $a(0)\left(g^{(k)}(\zeta)\right)^n - b$ has two distinct zeros at ζ_0 and ζ_1. Now choose a small positive number δ such that $\Delta(\zeta_0,\delta)\cap\Delta(\zeta_1,\delta) = \emptyset$ and $a(0)\left(g^{(k)}(\zeta)\right)^n - b$ has no other zeros in $\Delta(\zeta_0,\delta)\cup\Delta(\zeta_1,\delta)$. By Hurwitz's theorem, there exist two sequences $\{\zeta_j\} \subset \Delta(\zeta_0,\delta), \{\zeta_{1_j}\} \subset \Delta(\zeta_1,\delta)$ converging to ζ_0 and ζ_1 respectively and from (4.1), for sufficiently large j, we have

$$f_j(z_j + \rho_j\zeta_j) + a(z_j + \rho_j\zeta_j)\left(f_j^{(k)}(z_j + \rho_j\zeta_j)\right)^n - b = 0,$$
$$f_j(z_j + \rho_j\zeta_{1_j}) + a(z_j + \rho_j\zeta_{1_j})\left(f_j^{(k)}(z_j + \rho_j\zeta_{1_j})\right)^n - b = 0.$$

For large values of j, $z_j + \rho_j\zeta_j \in \Delta(\zeta_0,\delta)$ and $z_j + \rho_j\zeta_{1_j} \in \Delta(\zeta_1,\delta)$, so $f_j + a\left(f_j^{(k)}\right)^n - b$ has two distinct zeros, which contradicts the fact that $f_j + a\left(f_j^{(k)}\right)^n - b$ has at most one zero. But Lemma 3.3 and Lemma 3.4 confirm the non-existence of such non-constant meromorphic function g. This contradiction shows that $\mathcal{F}$ is normal in Δ and this proves the theorem. □

The proof of Theorem 1.4 is same as the proof of Theorem E with little changes in the last lines. For completeness we give the proof of Theorem 1.4.

Proof of Theorem 1.4. Suppose $f + a\left(f^{(k)}\right)^n$ assumes each value $b \in \mathbb{C}$ only finitely many times. This means

$$N\left(r, \frac{1}{f + a\left(f^{(k)}\right)^n - b}\right) = o(\log r) = S(r,f). \tag{4.6}$$

Let us define

$$F := f + a\left(f^{(k)}\right)^n - b, \tag{4.7}$$

$$\phi := \frac{F'}{F}, \tag{4.8}$$

$$\psi := n\frac{f^{(k+1)}}{f^{(k)}} - \frac{F'}{F}. \tag{4.9}$$

Now, we claim that $\phi\psi \not\equiv 0$. If $\phi \equiv 0$, then $F' \equiv 0$. We can deduce that $F \equiv c$, where c is finite complex number. From (4.7) and Lemma 3.5, we get that f must be a polynomial, which is a contradiction.

Next, if $\psi \equiv 0$, from (4.9), we get

$$c\left(f^{(k)}\right)^n = f + a\left(f^{(k)}\right)^n - b, \tag{4.10}$$

where $c \in \mathbb{C}$. From (4.10), we get

$$(a-c)\left(f^{(k)}\right)^n + f = b. \tag{4.11}$$

If $a - c = 0$, we get that $f \equiv b$, which is contradiction. Otherwise, we conclude from (4.11) and Lemma 3.5 that f must be a polynomial, which is a contradiction.

From (4.7), we have $T(r,F) = O(T(r,f))$, thus from (4.8) and (4.9), we have

$$m(r,\phi) = S(r,f) \text{ and } m(r,\psi) = S(r,f). \tag{4.12}$$

From (4.6), (4.8), (4.9) and Nevanlinna's First Fundamental Theorem, we get

$$\begin{aligned} N\left(r,\frac{1}{\phi}\right) &\leq m(r,\phi) + N(r,\phi) - m\left(r,\frac{1}{\phi}\right) + O(1) \\ &\leq N(r,\phi) + S(r,f) \leq \overline{N}(r,f) + S(r,f), \end{aligned} \tag{4.13}$$

$$\begin{aligned} N\left(r,\frac{1}{\psi}\right) &\leq m(r,\psi) + N(r,\psi) - m\left(r,\frac{1}{\psi}\right) + O(1) \\ &\leq N(r,\psi) + S(r,f) \leq \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + S(r,f). \end{aligned} \tag{4.14}$$

Again, by (4.8) and (4.9), we get

$$(f-b)\phi - f' = a\left(f^{(k)}\right)^n \psi. \tag{4.15}$$

So, we get from (4.6), (4.12) and (4.13)

$$\begin{aligned} T(r,(f-b)\phi - f') &= T\left(r,(f-b)\left(\phi - \tfrac{f'}{f-b}\right)\right) \\ &\leq T(r,f-b) + T\left(r,\phi - \tfrac{f'}{f-b}\right) + S(r,f) \\ &\leq m(r,f-b) + N(r,f-b) + m(r,\phi - \tfrac{f'}{f-b}) \\ &\quad + N\left(r,\phi - \tfrac{f'}{f-b}\right) + S(r,f) \\ &\leq m(r,f) + N(r,f) + m(r,\phi) + m(r,\tfrac{f'}{f-b}) \\ &\quad + N\left(r,\phi - \tfrac{f'}{f-b}\right) + S(r,f) \\ &\leq T(r,f) + \overline{N}(r,f) + S(r,f). \end{aligned} \tag{4.16}$$

It follows from (4.12)-(4.16) that

$$nT\left(r,f^{(k)}\right) \leq T(r,\psi) + T(r,\phi(f-b) - f') + S(r,f)$$

$$
\begin{aligned}
&\le m(r,\psi)+N(r,\psi)+T(r,f)+\overline{N}(r,f)+N\left(r,\frac{1}{F}\right)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{F}\right)+m\left(r,\frac{1}{f}\right)\\
&\quad +N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&\le \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+2N\left(r,\frac{1}{F}\right)+m\left(r,\frac{1}{f^{(k)}}\right)+N\left(r,\frac{1}{f}\right)\\
&\quad +\overline{N}(r,f)+S(r,f)\\
&\le T\left(r,\frac{1}{f^{(k)}}\right)+2N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&\le T\left(r,f^{(k)}\right)+2N\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f).
\end{aligned}
\tag{4.17}
$$

Therefore, we have from (4.6),

$$
(n-1)T\left(r,f^{(k)}\right)\le N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f). \tag{4.18}
$$

Also, we have

$$
(n-1)T\left(r,f^{(k)}\right)\ge (n-1)N\left(r,f^{(k)}\right)\ge (n-1)N(r,f)+(n-1)\overline{N}(r,f). \tag{4.19}
$$

Thus by (4.18) and (4.19), we have

$$
\begin{aligned}
(n-1)N(r,f)+(n-1)\overline{N}(r,f)&\le N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f)\\
&\le T\left(r,f\right)+\overline{N}(r,f)+S(r,f).
\end{aligned}
$$

This gives

$$
(n-2)N(r,f)+(n-2)\overline{N}(r,f)\le S(r,f).
$$

So, we get

$$
N(r,f)=S(r,f). \tag{4.20}
$$

Therefore from (4.18), we get

$$
\begin{aligned}
(n-1)T\left(r,f^{(k)}\right)&\le N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+S(r,f),\\
&\le T(r,f)+S(r,f).
\end{aligned}
$$

And this gives

$$
(n-2)T\left(r,f^{(k)}\right)+T\left(r,f^{(k)}\right)-T(r,f)\le S(r,f).
$$

Now, by (4.20), we get

$$
T\left(r,f^{(k)}\right)\le S(r,f), \tag{4.21}
$$

which is a contradiction. So $f + a\left(f^{(k)}\right)^n$ assumes each value $b \in \mathbb{C}$ infinitely often. $\square$

Acknowledgement. The last author expresses her gratitude to Prof. N. Raghavendra, Harish-Chandra Research Institute, Allahabad, India for his warm hospitality during the preparation of this paper. The last author thanks CSIR India for research fellowship.

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