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SOME MODELS FOR PROGRESSIVE TAXATION

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Abstract. We define *progressive tax rate functions*, study their properties, and describe some smooth models. The key requirement, defining the progressive nature of the taxation model, is that the progressive tax rate functions should have infinite contact with the zero function at the origin, in order to care the poor. In constructing a wide array of such functions, *assisting functions* are introduced.

1. Progressive tax rate functions

Historically, progressive taxation has been used as early as 6th century BC [2]. In practice, it is a rather complicated problem and there are many philosophical questions. We deal only with simple and ideal situations. A *progressive tax rate function*, which will be defined soon, has the graph which looks like Fig. 1.

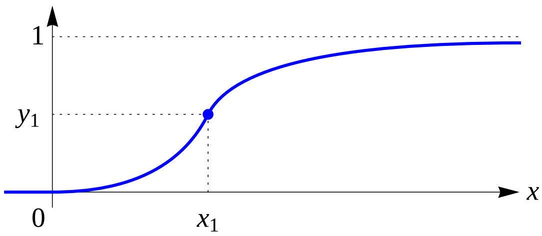

Fig. 1: The tax rate approaches 1 as income (x) approaches ∞.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is a *progressive tax rate function*, or simply a *T-function*, if it has the following properties: (0) $0 \leq f(x) < 1$.

(i) $f(x) = 0$ when $x \leq 0$, and $f(x) > 0$ when $x > 0$.
(ii) $f(x)$ is an increasing function when $x > 0$.
(iii) $\lim_{x\to\infty} f(x) = 1$.
(iv) $f(x)$ is convex when $0 < x < 1$, and $f(x)$ is concave when $x > 1$.

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(v) The *net income function*

$$g(x) := x(1 - f(x)) \qquad (x > 0)$$

is an increasing function.

In the definition of a T-function, the variable x means the total income of a person for a certain period. The property (0) follows from other properties (i)∼(iii). The property (i) says that "No positive income, no tax. Any positive income, some tax." If we want "Small income, no tax", then we may adjust the property (i) by *translating* the function $f(x)$. In (ii) and (v), "increasing" means "strictly increasing". The property (iv) fixes the unit $x_1 = 1$ as the *inflection point* of f. The *inflection value*

$$y_1 := f(x_1)$$

depends on various models. We will discuss some *tithing models* which use the traditional value $y_1 = 1/10$. The next proposition is trivial.

Proposition 1.1. *The space of T-functions is convex, i.e., if $f_0(x)$ and $f_1(x)$ are T-functions, then*

$$f_t(x) := (1 - t)f_0(x) + tf_1(x) \qquad (0 \le t \le 1)$$

are all T-functions.

Many countries use step functions, which depend on sequences

$$0 = y_0 < y_1 < y_2 < y_3 < \cdots \to 1$$

and *tax brackets* $x_1, x_2, x_3, \ldots$ of discontinuity. These discontinuous functions are not suitable because they have sudden changes which cause some trouble [Fig. 2].

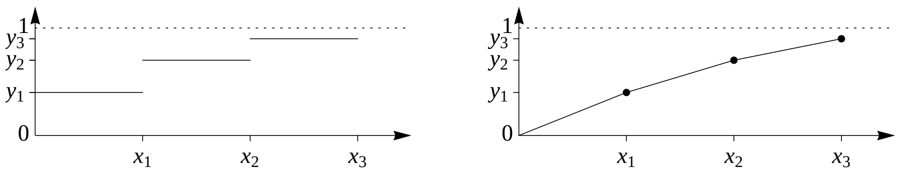

FIG. 2: A step function and a piecewise-linear function

One may consider piecewise-linear *continuous* models. But such models do not care much for persons with low income. For these reasons, we consider *smooth models*:

(vi) $f(x)$ is (continuous and) smooth ($\mathcal{C}^\infty$).

Since we are assuming that $f(x) = 0$ for $x \le 0$, this smoothness condition is equivalent to the condition

(vi)′ $f|_{(0,\infty)}$ is smooth, and $\displaystyle\lim_{x\to 0^+} \frac{f^{(n)}(x)}{x} = 0$ for all $n = 0, 1, 2, \ldots$.

In particular, the derivatives of f of all orders at the origin are zero:

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0, \quad \ldots.$$

This property implies [1]

$$\lim_{x\to 0} \frac{f(x)}{x^n} = 0 \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

which says that the tax rate for persons with low income is very low as in Fig. 1. Our T-functions are non-analytic smooth functions. If we throw away the smoothness condition (vi), it is still desirable to keep the "infinitely-contact-to-zero property" (1), because this is the condition "to care the poor infinitely."

Since we assume sufficient smoothness, the properties (ii), (iv), (v) are guaranteed by the following conditions:

(ii)′ $f'(x) > 0$ when $x > 0$.

(iv)′ If $0 < x < 1$, then $f''(x) > 0$. If $x > 1$, then $f''(x) < 0$.

(v)′ For $x > 0$, let $g(x) := x(1 - f(x))$. Then $g'(x) > 0$, i.e.,

$$(xf(x))' = f(x) + xf'(x) < 1.$$

Since we have conditions (ii) and (iii), it is easy to see that the condition (iv)′ is equivalent to the condition

(iv)″ $x = 1$ is the unique solution of the equation $f''(x) = 0$ for $x > 0$.

We call this condition the *inflection condition*.

1.1. Assisting functions

A smooth function

$$h : (0, \infty) \to (0, \infty)$$

will be called the *assisting function* of a T-function f, if

$$f(x) = e^{-1/h(x)}, \qquad \text{i.e.,} \quad h(x) = \frac{1}{\log(1/f(x))}$$

for $x > 0$. In particular, h is an increasing function such that

$$\lim_{x\to 0^+} h(x) = 0, \qquad \lim_{x\to\infty} h(x) = \infty.$$

For the inflection value $y_1 := f(1) \in (0, 1)$, we have

$$h_1 := h(1) = \frac{1}{\log(1/y_1)}.$$

If we use the tithing system $y_1 = 1/10$, then $h_1 = 1/\log 10 \approx 0.43$.

Proposition 1.2. *Suppose that a function $h : (0, \infty) \to (0, \infty)$ has the Hölder-continuity at the origin, i.e., for some $\alpha > 0$,*

$$\overline{\lim_{x\to 0^+}} \frac{h(x)}{x^\alpha} < \infty. \tag{2}$$

Then the function $f(x) := e^{-1/h(x)}$ has the property (1).

Proof. Take $N \in \mathbb{R}$ so that $\alpha N > n$. Then

$$0 \le \lim_{x \to 0} \frac{f(x)}{x^n} = \lim_{x \to 0^+} \frac{e^{-1/h(x)}}{x^n} = \lim_{x \to 0^+} \frac{e^{-1/h(x)}}{h(x)^N} \frac{h(x)^N}{x^n}$$
$$\le \lim_{t \to 0^+} \frac{e^{-1/t}}{t^N} \cdot \lim_{x \to 0^+} \left(\frac{h(x)}{x^\alpha} \right)^N \frac{x^{\alpha N}}{x^n} = 0.$$

Thus we have the property (1). $\square$

Theorem 1.3. *Suppose that a positive differentiable function $h : (0, \infty) \to \mathbb{R}$ satisfies a differential equation*

$$x h'(x) = P(x, h(x))$$

for some polynomial function $P(x, y)$. Then the function $f(x) := e^{-1/h(x)}$ has the property

$$f^{(n)}(x) = \frac{P_n(x, h(x))}{x^{2^{n-1}} h(x)^{2^n}} f(x) \qquad (n = 1, 2, 3, \dots) \tag{3}$$

for some polynomial $P_n(x, y)$. Moreover, if $h(x)$ has the Hölder continuity at the origin, then $f(x)$ is smooth at the origin in the sense of (vi)$'$.

Proof. Smoothness of $h(x)$ and $f(x)$ for $x > 0$ is trivial. Note that

$$f'(x) = \frac{h'(x)}{h(x)^2} f(x) = \frac{P(x, h(x))}{x\, h(x)^2} f(x).$$

Thus when $n = 1$, the relation (3) holds with $P_1(x, y) = P(x, y)$. General case follows from the induction. Finally, suppose h has the Hölder continuity (2). Then given a positive integer n, take a positive integer N large enough so that

$$\lim_{x \to 0^+} \frac{f^{(n)}(x)}{x} = \lim_{x \to 0^+} \frac{P_n(x, h(x)) e^{-1/h(x)}}{x^{2^{n-1}+1} h(x)^{2^n}}$$
$$= \lim_{x \to 0^+} P_n(x, h(x)) \left(\frac{h(x)}{x^\alpha} \right)^N \frac{x^{\alpha N}}{x^{2^{n-1}+1}} \lim_{x \to 0^+} \frac{e^{-1/h(x)}}{h(x)^{2^n+N}} = 0.$$

This completes the proof. $\square$

1.1.1. *Inflection condition for the assisting function.* Note that

$$f''(x) = \left(\left(\frac{h'(x)}{h(x)^2} \right)' + \left(\frac{h'(x)}{h(x)^2} \right)^2 \right) f(x)$$
$$= \frac{h'(x)^2 (1 - 2h(x)) + h(x)^2 h''(x)}{h(x)^4} f(x).$$

Thus the "inflection condition" (iv) (or (iv)$'$, (iv)$''$) means that the equation

$$h'(x)^2 (1 - 2h(x)) + h(x)^2 h''(x) = 0 \qquad (x > 0) \tag{4}$$

has a solution $x = 1$ and there are no other solutions.

1.1.2. *Net income condition for the assisting function.* Since

$$f(x) + xf'(x) = \left(1 + \frac{xh'(x)}{h(x)^2}\right) f(x).$$

the condition (v)$'$ is equivalent to the condition:

$$\text{(v)}'' \qquad 1 + \frac{xh'(x)}{h(x)^2} < e^{1/h(x)} \quad (x > 0).$$

Lemma 1.4. *Suppose that a differentiable function $h(x)$ satisfies the inequality*

$$0 < xh'(x) \le h(x) \qquad (x > 0).$$

Then the inequality (v)$''$ *holds.*

A proof of the above lemma follows from the observation

$$1 + \frac{xh'(x)}{h(x)^2} \le 1 + \frac{1}{h(x)} \le e^{1/h(x)}.$$

2. The least upper bound of personal net income

Given a T-function f, the net income function is $g(x) = x(1 - f(x))$. Let M be the limit

$$M := \lim_{x\to\infty} g(x).$$

Then using the assisting function $h(x)$, we have

$$M = \lim_{x\to\infty} x(1 - e^{1/h(x)}) = \lim_{x\to\infty} \frac{x}{h(x)} \frac{1 - e^{-1/h(x)}}{1/h(x)} = \lim_{x\to\infty} \frac{x}{h(x)}. \tag{5}$$

We will consider both the unbounded model ($M = \infty$) and the bounded model ($M < \infty$).

2.1. Two unbounded models

In this case, the assisting function $h(x)$ satisfies the condition:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x}{h(x)} = M = \infty.$$

2.1.1. *High Tax model.* Here we consider a model with high tax for high income persons. For real numbers $k > 2$ and $0 < \alpha < 1$, let

$$f(x) := e^{-k/x^\alpha} \qquad (x > 0).$$

In other words $h(x) := \frac{1}{k}x^\alpha$. We will soon determine the values of k and α. When $x > 0$, $f'(x) = \alpha k x^{-1-\alpha} f(x)$, and hence

$$f''(x) = \alpha k x^{-2}(\alpha k x^{-2\alpha} - (1+\alpha)x^{-\alpha}) f(x). \tag{6}$$

In general, one can see easily by induction that for each nonnegative integer n, there exists a polynomial $p_n(x)$ of degree n, such that

$$f^{(n)}(x) = x^{-n} p_n(x^{-\alpha}) f(x) \qquad (n = 0, 1, 2, \dots).$$

Thus

$$\lim_{x \to 0^+} \frac{f^{(n)}(x)}{x} = 0 \qquad (n = 0, 1, 2, \dots).$$

This implies that $f(x)$ is smooth at $x = 0$. In fact, we may use Theorem 1.3 directly to see the smoothness of f at the origin.

Moreover, from (6), $x = 1$ is the unique inflection point of $f(x)$ if

$$\alpha = \frac{1}{k - 1} \in (0, 1).$$

On the other hand

$$xh'(x) < h(x)$$

and hence Lemma 1.4 implies that the net income function $g(x)$ is increasing.

Now with $y_1 := f(1) = \frac{1}{10}$,

$$k = -\log y_1 = \log 10 \approx 2.3, \qquad \alpha = \frac{1}{\log 10 - 1} \approx 0.8.$$

In this case, the graph of $f(x)$ looks like Figure 3.

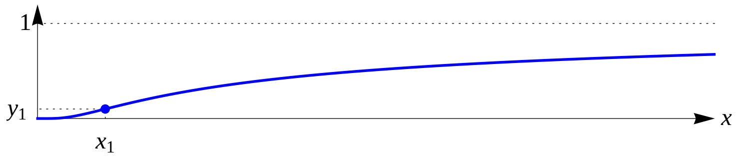

FIG. 3: An unbounded model $f(x) = e^{-k/x^\alpha}$

2.1.2. *Low tax model.* We discuss another model which pays less tax than the previous model for high income persons.

For positive real numbers k and a, let

$$h(x) = k \log(1 + ax)$$

be the assisting function. This is clearly an unbounded model. The values of k and a will be soon determined. Then for $x > 0$,

$$f^{(n)}(x) = \frac{p_{2n}(1/\log(1 + ax))}{(1 + ax)^n} f(x) \qquad (n = 0, 1, 2, \dots)$$

for some polynomial $p_{2n}(x)$ of degree $2n$. Thus $f(x)$ is smooth everywhere. Now

$$h_1 = k \log(1 + a). \tag{7}$$

We follow tithing condition: $h_1 = 1/\log 10 \simeq 0.43$. On the other hand

$$h'(x) = \frac{ka}{1 + ax}, \qquad h''(x) = -\frac{ka^2}{(1 + ax)^2}.$$

Since $xh'(x) < h(x)$, Lemma 1.4 implies that the net income function $g(x)$ is increasing. The inflection condition (i.e., the condition that $x = 1$ is the unique solution of the equation (4)) means now that the equation

$$k(\log(1 + ax))^2 + 2k \log(1 + ax) - 1 = 0$$

has a unique solution $x = 1$. Then

$$k(\log(1 + a))^2 + 2k \log(1 + a) - 1 = 0$$

and from the condition (7), we have

$$\log(1 + a) = (1 - 2h_1)/h_1 \quad \text{(i.e., } a \approx 0.35\text{)}, \qquad k = \frac{h_1^2}{1 - 2h_1} \approx 1.44.$$

In this model the graph of $f(x)$ is in Fig. 4.

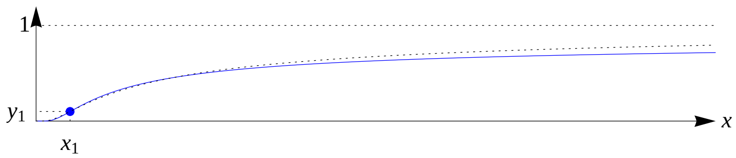

FIG. 4: $f(x) = e^{-/k \log(1+ax)}$ (Dotted curve is from Fig. 3.)

2.2. A bounded model

If the least upper bound M of personal net income is finite, then from (5), we have the asymptotic relation

$$h(x) \sim \frac{x}{M} \qquad (x \to \infty).$$

This M is meaningful only when it is sufficiently large. In particular M is larger than the inflection point $x = 1$: $M \gg 1$.

We will find an assisting function $h(x)$ for large M, which obeys the tithing rule: $y_1 = \frac{1}{10}$, i.e., $h_1 = \frac{1}{\log 10} \approx 0.43$.

For positive real numbers k and $\alpha \in (\frac{2}{3}, 1)$, we consider the assisting function

$$h(x) := \frac{x}{M} + kx^\alpha.$$

We now explain how to determine the values of k and α, when given M is large. Note that it is easy to check that the T-function $f(x)$ is smooth everywhere. In fact, smoothness follows from Theorem 1.3. Moreover, we have the equality $h_1 = \frac{1}{M} + k$, i.e.,

$$k = h_1 - 1/M. \tag{8}$$

Thus if M is very large, then k is very close to h_1. The value of α will be soon determined in the interval $(\frac{2}{3}, 1)$. On the other hand

$$h'(x) = \frac{1}{M} + k\alpha x^{\alpha - 1}.$$

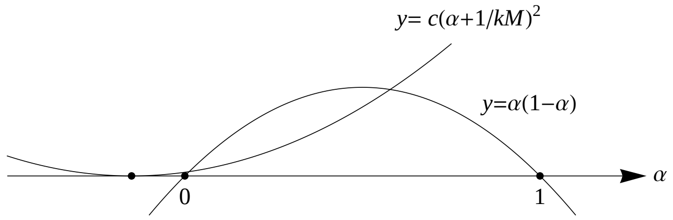

FIG. 5: Two parabolas

Thus, for $x > 0$,

$$xh'(x) = \frac{x}{M} + k\alpha x^{\alpha} < \frac{x}{M} + kx^{\alpha} = h(x).$$

Now from Lemma 1.4 again, the condition (v)$''$, or the condition (v) holds. Moreover

$$h''(x) = k\alpha(\alpha-1)x^{\alpha-2}.$$

Now from (4), the "inflection condition" is that the equation

$$\left(\frac{1}{M} + k\alpha x^{\alpha-1}\right)^2 \left(1 - 2\left(\frac{x}{M} + kx^{\alpha}\right)\right) + \left(\frac{x}{M} + kx^{\alpha}\right)^2 k\alpha(\alpha-1)x^{\alpha-2} = 0$$

has a unique solution $x = 1$. The left hand side of the above equation is

$$\frac{1}{M^2} + c_1 x^{\alpha-1} + c_2 x^{2(\alpha-1)} - c_3 x - c_4 x^{\alpha} - c_5 x^{2\alpha-1} - c_6 x^{3\alpha-2} \tag{9}$$

for some *positive* numbers $c_1, \ldots, c_6$. Therefore if $\frac{2}{3} < \alpha < 1$, then (9) is a (strictly) decreasing function and hence has a unique, if exists, zero. Now if $x = 1$ is a solution, then

$$\left(\frac{1}{M} + k\alpha\right)^2 (1 - 2h_1) + h_1^2 k\alpha(\alpha-1) = 0.$$

In other words

$$\frac{k(1-2h_1)}{h_1^2}\left(\alpha + \frac{1}{kM}\right)^2 = \alpha(1-\alpha),$$

where k is given by (8). From Fig. 5, we see immediately that there are two α's between 0 and 1 satisfying the above equation, when M is sufficiently large. We take the larger solution as the value of α:

$$\alpha := \frac{2-4h_1+h_1^3M^2+h_1M\left(3h_1-2+\sqrt{4h_1-7h_1^2+2h_1^2(3h_1-2)M+h_1^4M^2}\right)}{2\left(1-2h_1+h_1(3h_1-2)M+h_1^2(1-h_1)M^2\right)}.$$

From this when M approaches infinity, the limit of α is $h_1/(1-h_1) \simeq 0.77$. Thus $\frac{2}{3} < \alpha < 1$ for large M. This gives a bounded model.

3. Conclusion

Various smooth models for progressive tax rate functions are discussed. The key requirement of the progressive tax rate functions is that they should have infinite contact with the zero function at the origin. We regard these models better than any discontinuous or piecewise-linear continuous models, since they care the poor more.

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