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ON WEIGHTED AND PSEUDO-WEIGHTED SPECTRA OF BOUNDED OPERATORS

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Abstract. In the present paper, we extend the main results of Jeribi in [6] to weighted and pseudo-weighted spectra of operators in a non-separable Hilbert space $\mathcal{H}$. We investigate the characterization, the stability and some properties of these weighted and pseudo-weighted spectra.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space, $\mathcal{L}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the closed ideal of all compact operators. Let $\Phi(\mathcal{H})$ be the set of all Fredholm operators. We denote the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ by $\mathcal{C}(\mathcal{H})$. Let $\pi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{C}(\mathcal{H})$ be the quotient map. It is well known (Atkinson's theorem) that

$$T \in \Phi(\mathcal{H}) \Longleftrightarrow \pi(T) \text{ is invertible in } \mathcal{C}(\mathcal{H}). \tag{1}$$

This implies that

$$\sigma(\pi(T)) = \sigma_e(T) \text{ for all } T \in \mathcal{L}(\mathcal{H}), \tag{2}$$

where $\sigma(\pi(T))$ is the spectrum of $\pi(T)$ and $\sigma_e(T)$ is the essential spectrum of T. The structure of closed, two-sided ideals of the Banach algebra of operators of Hilbert space $\mathcal{H}$ seems to be a little-understood area. Indeed, in [13], I. C. Gohberg; A. S. Markus and I. A. Feldman showed that, in the special case where $\mathcal{H} = \ell^2$, $\mathcal{K}(\mathcal{H})$ is the only (non-trivial) closed ideal in $\mathcal{L}(\mathcal{H})$. It seems to be unknown if this is true for any other Banach space. In 1968, E. Luft established ([22]) a new concept called the α-compact operator, where α is a cardinal number. Let $\mathcal{H}$ be an arbitrary Hilbert space, $\aleph_0$ be a cardinality of the naturel numbers $\mathbb{N}$ and α be a cardinal number. Accordingly ([22, Definitions 3.1 and 4.1]), a subset $\mathcal{K} \subset \mathcal{H}$ is called α-bounded, if for each $\varepsilon > 0$, there exists a set of points $\{h_i\}_{i\in I}$, with card(I)$< \alpha$, and with $\mathcal{K} \subset \cup_{i\in I} B(h_i, \varepsilon)$, where $B(h_i, \varepsilon)$ is the open ball in $\mathcal{H}$ with center h_i and radius ε. A linear map $T \in \mathcal{L}(H)$ is called α-compact, if the image $T(B)$ of each bounded subset

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$B \subset \mathcal{H}$ is α-bounded. Let $\mathcal{K}_\alpha(\mathcal{H})$ be the set of all α-compact operators. E. Luft in [22, Theorem 4.1] proved that $\mathcal{K}_\alpha(\mathcal{H})$ is a closed two-sided ideal in the algebra $\mathcal{L}(\mathcal{H})$.

Let us return to Eq. (1). We denote the algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}_\alpha(\mathcal{H})$ by $\mathcal{C}_\alpha(\mathcal{H})$. Let $\pi_\alpha : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{C}_\alpha(\mathcal{H})$ denotes the natural homomorphism. The usual question that arises is: Can we have a relation between the spectrum $\sigma(\pi(T))$ and a part analogous to the essentiel spectrum? In [10], the authors answer positively to this question by giving a new concept called α-closed subspace, where a subset A of a Hilbert space $\mathcal{H}$ is said to be α-closed if there is a closed subspace B of $\mathcal{H}$ such that $B \subset A$ and $\dim(A \cap B^\perp) < \alpha$. Therefore, the notion of Fredholm operators can be extended to an arbitrary dimension of the null space of bounded linear operator T on Hilbert space $\mathcal{H}$ using the α-closedness. Accordingly [10, Definition 2.7], $T \in \mathcal{L}(\mathcal{H})$ is α-Fredholm operator (in notation $T \in \Phi_\alpha(\mathcal{H})$), if $\mathcal{R}(T)$ is α-closed and $\max\{\dim(\mathcal{N}(T)),\ \dim(\mathcal{N}(T^*))\} < \alpha$, where $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, are the null-space and the range of T. For α-Fredholm operators we have an Atkinson type of characterization (see, [10, Theorem 2.8]):

$$T \in \Phi_\alpha(\mathcal{H}) \Longleftrightarrow \pi_\alpha(T) \text{ is invertible in } \mathcal{C}_\alpha(\mathcal{H}). \tag{3}$$

Consequently, we obtain $\sigma(\pi_\alpha(T)) = \sigma_\alpha(T)$, where

$$\sigma_\alpha(T) = \{\lambda \in \mathbb{C} \text{ such that } T - \lambda \notin \Phi_\alpha(\mathcal{H})\}$$

is the weighted spectrum. Note that

$$\sigma_\alpha(T) \subset \sigma_e(T) = \sigma_{\aleph_0}(T) \subset \sigma(T)/\sigma_d(T),$$

where $\sigma(T)$ is the spectrum of T and $\sigma_d(T)$ is the discrete spectrum of T. Therefore, the weighted spectrum helps us to study the non-discrete part of the spectrum. In 2016, Djordjević and Hernández-Díaz (see, [8]) studied the Fredholm and α-Fredholm operators and the relation between them. The purpose of the present paper is to extend some results and some properties of types given by Jeribi [21] to the weighted spectrum. One of the central questions consists in characterizations the relation between the weighted spectrum of the sum of two bounded linear operators and weighted spectrum of each of the two operators. In the second place, we work with the notion of pseudo-weighted spectra of linear operator. We start by giving the definition and we study the stability and some properties of these pseudo-weighted spectra.

We organize the paper in the following way. Section 2 contains preliminary and auxiliary properties that prove the main results of other sections. The main aim of Section 3 is to characterize the weighted spectrum and the weighted Weyl spectrum of the sum of two bounded operators defined on a arbitrary Hilbert space. In Section 4, we investigate the stability of the weighted spectrum. Finally, in Section 5, we introduce the concept of pseudo-weighted spectra. We extend the main results in [1] to pseudo-weighted spectra.

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2. Preliminaries

The goal of this section consists in establishing some preliminary results which will be needed in the sequel.

Let $\mathcal{H}$ be a Hilbert space. We do not assume $\mathcal{H}$ to be separable. Let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. For each $T \in \mathcal{L}(\mathcal{H})$, we will denote the kernel and the range of T by $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively, and the adjoint of T by T^*. The nullity, $n(T)$, of T is defined as the dimension of $\mathcal{N}(T)$. Let h denote the dimension of the Hilbert space $\mathcal{H}$, where $h \geq \aleph_0$. A subset $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ is called α-closed if there is a closed subspace $\mathcal{L} \subset \mathcal{K}$ such that $\dim(\mathcal{K} \cap \mathcal{L}^{\perp}) < \alpha$, (see, [10]). Sets of upper and lower α-Fredholm operators, respectively, are defined as:

$$\Phi_\alpha^+(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \text{ such that } \mathcal{R}(T) \text{ is } \alpha\text{-closed and } n(T) < \alpha\}$$

and

$$\Phi_\alpha^-(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) \text{ such that } \mathcal{R}(T) \text{ is } \alpha\text{-closed and } n(T^*) < \alpha\}.$$

The set of α-Fredholm is defined by $\Phi_\alpha(\mathcal{H}) = \Phi_\alpha^+(\mathcal{H}) \cap \Phi_\alpha^-(\mathcal{H})$. Let

$$\Theta_+(\mathcal{H}) = \{\alpha \text{ be a cardinal number such that } \aleph_0 \leq \alpha \leq h\}.$$

We extend the sum operation from $\mathbb{Z}$ to $\Theta(\mathcal{H}) = \Theta_+(\mathcal{H}) \cup \{-\beta : \ \beta \in \Theta_+(\mathcal{H})\}$, by setting, for each $\alpha \in \Theta(\mathcal{H})$ such that $\alpha \geq \aleph_0$,

(i) $\alpha + \beta = \alpha$ if $-\alpha < \beta \leq \alpha$,

(ii) $-\alpha + \beta = -\alpha$ if $-\alpha \leq \beta < \alpha$,

(iii) $\alpha - \alpha = 0$.

For $T \in \mathcal{L}(\mathcal{H})$, we can extend the definition index of T (see, [5]) by

$$i(T) = n(T) - n(T^*).$$

Notice that $i(T) \in \Theta(\mathcal{H})$. Let $\alpha \in \Theta_+(\mathcal{H})$, we define

$$i_\alpha(T) = \begin{cases} i(T), & \text{if } \alpha = \aleph_0 \text{ or } \alpha > \aleph_0 \text{ and } \max(n(T), n(T^*)) \geq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $i_\alpha \in \Theta(\mathcal{H})$ and $i_\alpha(T) = -i_\alpha(T^*)$. Let α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$. In the way of Schechter definition of the Weyl operators, the α-Weyl operators is defined by ([9])

$$\Phi_\alpha^0(\mathcal{H}) = \Big\{T \in \Phi_\alpha(\mathcal{H}) : i_\gamma(T) = 0 \text{ for all cardinal } \gamma, \ \aleph_0 \leq \gamma < \alpha\Big\}.$$

In this paper we are concerned with the following spectra:

$\sigma_\alpha(T) = \{\lambda \in \mathbb{C} \text{ such that } T - \lambda \notin \Phi_\alpha(\mathcal{H})\}$, the weighted spectrum of T of weight α.

$\sigma^0_\alpha(T) = \{\lambda \in \mathbb{C}$ such that $T - \lambda \notin \Phi^0_\alpha(\mathcal{H})\}$, the weighted Weyl spectrum of T of weight α.

They can be ordered as

$$\sigma_\alpha(T) \subseteq \sigma^0_\alpha(T).$$

In the case when $\alpha = \aleph_0$, the subsets $\sigma_{\aleph_0}(\cdot)$ is the Wolf essential spectrum [14–16, 20, 25] and $\sigma^0_{\aleph_0}(\cdot)$ is the Schechter essential spectrum [17–19].

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space and α be a cardinal number. A subset $\mathcal{S} \subset \mathcal{H}$ is called α-bounded, if for each $\varepsilon > 0$ there exists a set of points $(h_m)_{m\in I}$, $h_m \in \mathcal{S}$, $| I | < \alpha$, and with

$$\mathcal{S} \subset \cup_{m\in I} B(h_m, \varepsilon),$$

where $| I |$ is the cardinal of I and $B(h_m, \varepsilon)$ is the open ball in $\mathcal{H}$ with center h_m and radius ε.

Remark 2.2. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S} \subset \mathcal{H}$. Then $\mathcal{S}$ is totally bounded if and only if the closure of $\mathcal{S}$ is compact. Then, if $\alpha = \aleph_0$, Definition 2.1 coincides with the definition of S to be totally bounded.

Definition 2.3. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert space and let α be a cardinal number. An operator $K \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is said to be α-compact if $K(B)$ is α-bounded in $\mathcal{K}$ for every bounded subset $B \subset \mathcal{H}$.

The family of α-compact operators from $\mathcal{H}$ to $\mathcal{K}$ is denoted by $\mathcal{K}_\alpha(\mathcal{H}, \mathcal{K})$. If $\mathcal{H} = \mathcal{K}$, the family of α-compact operators on $\mathcal{H}$, $\mathcal{K}_\alpha(\mathcal{H}) = \mathcal{K}_\alpha(\mathcal{H}, \mathcal{H})$ is a closed two-sided ideal of $\mathcal{L}(\mathcal{H})$ (see, [22]). In the case when $\alpha = \aleph_0$, we have $\mathcal{K}_{\aleph_0}(\mathcal{H}) = \mathcal{K}(\mathcal{H})$, the ideal of all compact operators. Let $\mathcal{F}_{\aleph_0}(\mathcal{H})$ denote the two sided ideal in $\mathcal{L}(\mathcal{H})$ of all bounded linear operators of rank less than α.

In the next theorem, we recall the Atkinson type characterization of α-Fredholm operators.

Theorem 2.4 ([10, Theorem 2.8])**.** *Let $\mathcal{H}$ be a Hilbert space of infinite dimension h and let α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. An operator T on $\mathcal{L}(\mathcal{H})$ is an α-Fredholm operator if and only if it is invertible modulo $\mathcal{K}_\alpha(\mathcal{H})$, or alternatively, if and only if it is an invertible modulo $\mathcal{F}_\alpha(\mathcal{H})$.*

Sets of left and right α-Fredholm operators, respectively, are defined as

$$\Phi^l_\alpha(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : T \text{ is left invertible modulo } \mathcal{K}_\alpha(\mathcal{H})\}$$

and

$$\Phi^r_\alpha(\mathcal{H}) = \{T \in \mathcal{L}(\mathcal{H}) : T \text{ is right invertible modulo } \mathcal{K}_\alpha(\mathcal{H})\}.$$

Let $\Phi^\alpha(\mathcal{H}) = \Phi^l_\alpha(\mathcal{H}) \cup \Phi^r_\alpha(\mathcal{H})$. In the next theorem, we will recall some well-known properties of the α-Fredholm sets (see, [3, 6, 8, 9]).

Theorem 2.5. *Let $\mathcal{H}$ be a Hilbert space of infinite dimension h and let α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$. Let $S \in \mathcal{L}(\mathcal{H})$ and let $T \in \mathcal{L}(\mathcal{H})$.*

(i) *If* $TS \in \Phi_\alpha(\mathcal{H})$ *and* $TS = ST$, *then* $T \in \Phi_\alpha(\mathcal{H})$ *and* $S \in \Phi_\alpha(\mathcal{H})$.
(ii) *If* $T \in \Phi_\alpha(\mathcal{H})$ *and* $S \in \Phi_\alpha(\mathcal{H})$, *then* $TS \in \Phi_\alpha(\mathcal{H})$.
(iii) *If* $T \in \Phi^0_\alpha(\mathcal{H})$ *and* $S \in \Phi^0_\alpha(\mathcal{H})$, *then* $TS \in \Phi^0_\alpha(\mathcal{H})$.
(iv) *If* $T \in \Phi^\alpha(\mathcal{H})$ *and* $S \in \Phi^\alpha(\mathcal{H})$, *then* $i_\alpha(TS) = i_\alpha(T) + i_\alpha(S)$.

Proposition 2.6. *Let* $\mathcal{H}$ *be a Hilbert space of infinite dimension* h *and let* α *be a cardinal number such that* $\aleph_0 \le \alpha \le h$.

(i) ([6, Theorem 1]) *Two operators are in the same component of* $\Phi^\alpha(\mathcal{H})$ *if and only if they have the same index.*
(ii) ([6, Corollary 2]) *If* $T \in \Phi^\alpha(\mathcal{H})$ *and* $K \in \mathcal{K}_\alpha(\mathcal{H})$, *then* $i_\alpha(T + K) = i_\alpha(T)$.
(iii) ([8, Theorem 2.6]) *If* $T \in \Phi_\alpha(\mathcal{H})$ *and* $K \in \mathcal{K}_\alpha(\mathcal{H})$, *then* $T+K \in \Phi_\alpha(\mathcal{H})$.

3. Weighted Fredholm and Weyl spectra for the sum of bounded linear operators

In this section, we suppose that $\mathcal{H}$ is a Hilbert space of infinite dimension h and α be a cardinal number such that $\aleph_0 \le \alpha \le h$. We first prove the auxiliary assertion.

Lemma 3.1. *Let* $T \in \mathcal{L}(\mathcal{H})$ *and let* $\mathcal{H}$ *be a direct sum of closed subspaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *which are* T*-invarient. If* $T_1 = T_{|\mathcal{H}_1} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1$ *and* $T_2 = T_{|\mathcal{H}_2} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$, *then* T *is an* α*-Fredholm operator if and only if* T_1 *and* T_2 *are* α*-Fredholm operators.*

Proof. The operator T has the following matrix form with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}.$$

Suppose that $T_1 \in \Phi_\alpha(\mathcal{H}_1)$ and $T_2 \in \Phi_\alpha(\mathcal{H}_2)$. Using Theorem 2.4, there exist $S_1 \in \mathcal{L}(\mathcal{H}_1)$, $S_2 \in \mathcal{L}(\mathcal{H}_2)$, $F_1 \in \mathcal{F}_\alpha(\mathcal{H}_1)$, $F_1' \in \mathcal{F}_\alpha(\mathcal{H}_1)$, $F_2 \in \mathcal{F}_\alpha(\mathcal{H}_2)$ and $F_2' \in \mathcal{F}_\alpha(\mathcal{H}_2)$ such that

$$T_1S_1 = I - \mathcal{F}_1,\ S_1T_1 = I - \mathcal{F}_1',\ T_2S_2 = I - \mathcal{F}_2 \text{ and } S_2T_2 = I - \mathcal{F}_2'.$$

Let

$$F = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix},\ F' = \begin{bmatrix} F_1' & 0 \\ 0 & F_2' \end{bmatrix} \text{ and } S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}.$$

It is clear that F and F' are in $\mathcal{F}_\alpha(\mathcal{H})$. We get

$$TS = \begin{bmatrix} I - F_1 & 0 \\ 0 & I - F_2 \end{bmatrix} = I - F$$

and

$$ST = \begin{bmatrix} I - F_1' & 0 \\ 0 & I - F_2' \end{bmatrix} = I - F'.$$

Thus, T is α-Fredholm.

Conversely, suppose that T is α-Fredholm. Then, there exist $F \in \mathcal{F}_\alpha(\mathcal{H})$, $F' \in \mathcal{F}_\alpha(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ such that

$$TS = I - F \text{ and } ST = I - F'.$$

The operators $S,\ F$ and F' have the following matrix form with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \text{ and } F' = \begin{bmatrix} F'_{11} & F'_{12} \\ F'_{21} & F'_{22} \end{bmatrix}.$$

Therefore

$$TS = \begin{bmatrix} T_1S_{11} & T_1S_{12} \\ T_2S_{21} & T_2S_{22} \end{bmatrix} = \begin{bmatrix} I - F_{11} & -F_{12} \\ -F_{21} & I - F_{22} \end{bmatrix}$$

and

$$ST = \begin{bmatrix} S_{11}T_1 & S_{12}T_1 \\ S_{21}T_2 & S_{22}T_2 \end{bmatrix} = \begin{bmatrix} I - F'_{11} & -F'_{12} \\ -F'_{21} & I - F'_{22} \end{bmatrix},$$

which implies $T_1S_{11} = I - F_{11},\ \ S_{11}T_1 = I - F'_{11},\ \ S_{22}T_2 = I - F'_{22}$ and $T_2S_{22} = I - F_{22}$, where

$$F_{ii} \in \mathcal{F}_\alpha(\mathcal{H}_i), \quad F'_{ii} \in \mathcal{F}_\alpha(\mathcal{H}_i) \text{ and } S_{ii} \in \mathcal{L}(\mathcal{H}_i) \text{ for } i = 1, 2.$$

So, T_1 and T_2 are α-Fredholm operators. □

Let $T \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator, then $\sigma_\alpha(T) = \sigma^0_\alpha(T)$. Now, we consider the following example:

Example 3.2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. Let α and β be two cardinal numbers such that $\aleph_0 \le \alpha < \beta \le \dim(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Take $T = T_1 \oplus I$, where $T_1 \in \Phi_\beta(\mathcal{H}_1)$, $n(T_1) = \alpha$ and $n(T_1^*) = \aleph_0$. Using Lemma 3.1, we have $T \in \Phi_\beta(\mathcal{H})$ and $i_\alpha(T) = i_\alpha(T_1)$. Since $\alpha > \aleph_0$ and $\max(n(T_1), n(T_1^*)) \ge \alpha$. Then

$$i_\alpha(T) = \alpha - \aleph_0 = \alpha \neq 0.$$

So, $T \notin \Phi^0_\alpha(\mathcal{H})$. Consequently, we get

$$\sigma_\alpha(T) \subsetneq \sigma^0_\alpha(T).$$

Theorem 3.3. *Let $\mathcal{H}$ be a Hilbert space, α be a cardinal number such that $\aleph_0 \le \alpha \le \dim(\mathcal{H})$, and consider $T,\ S \in \mathcal{L}(\mathcal{H})$.*

If $TS \in \mathcal{K}_\alpha(\mathcal{H})$, then

$$\sigma_\alpha(T + S)\backslash\{0\} \subset [\sigma_\alpha(T) \cup \sigma_\alpha(S)]\backslash\{0\}$$

and

$$\sigma^0_\alpha(T + S)\backslash\{0\} \subset \left[\sigma^0_\alpha(T) \cup \sigma^0_\alpha(S)\right]\backslash\{0\}.$$

If, further, $ST = TS$, then

$$\sigma_\alpha(T + S)\backslash\{0\} = [\sigma_\alpha(T) \cup \sigma_\alpha(S)]\backslash\{0\}.$$

Proof. Let $\lambda \in \mathbb{C}^*$. We can write

$$(T-\lambda)(S-\lambda) = TS - \lambda(T+S-\lambda) \tag{4}$$

and

$$(S-\lambda)(T-\lambda) = ST - \lambda(T+S-\lambda). \tag{5}$$

Suppose that $(T-\lambda) \in \Phi_\alpha(\mathcal{H})$ and $(S-\lambda) \in \Phi_\alpha(\mathcal{H})$, then $(T-\lambda)(S-\lambda) \in \Phi_\alpha(\mathcal{H})$. Since $TS \in \mathcal{K}_\alpha(\mathcal{H})$, we can apply Eq. (4) and Proposition 2.6(iii), we infer that $(T+S-\lambda) \in \Phi_\alpha(\mathcal{H})$. Therefore

$$\sigma_\alpha(T+S)\backslash\{0\} \subset [\sigma_\alpha(T) \cup \sigma_\alpha(S)]\backslash\{0\}. \tag{6}$$

A similar proof as before, we have

$$\sigma^0_\alpha(T+S)\backslash\{0\} \subset \left[\sigma^0_\alpha(T) \cup \sigma^0_\alpha(S)\right]\backslash\{0\}. \tag{7}$$

To prove the inverse inclusion of Eq. (6).

Suppose that $(T+S-\lambda) \in \Phi_\alpha(\mathcal{H})$. Since $TS = ST \in \mathcal{K}_\alpha(\mathcal{H})$, then by Eqs. (4) and (5), we have

$$(T-\lambda)(S-\lambda) = (S-\lambda)(T-\lambda) \in \Phi_\alpha(\mathcal{H}).$$

By Theorem 2.5(i), it is clear that $(T-\lambda) \in \Phi_\alpha(\mathcal{H})$ and $(T-\lambda) \in \Phi_\alpha(\mathcal{H})$. This proved that

$$[\sigma_\alpha(T) \cup \sigma_\alpha(S)]\backslash\{0\} \subset \sigma_\alpha(T+S)\backslash\{0\}.$$

Therefore,

$$\sigma_\alpha(T+S)\backslash\{0\} = [\sigma_\alpha(T) \cup \sigma_\alpha(S)]\backslash\{0\}. \qquad \square$$

Remark 3.4. In the case when $\alpha = \aleph_0$ (see, [4]), if $\sigma_{\aleph_0}(TS) = \{0\}$ and $TS - ST \in \mathcal{K}_{\aleph_0}(\mathcal{H})$, then

$$\sigma_{\aleph_0}(T+S)\backslash\{0\} = [\sigma_{\aleph_0}(T) \cup \sigma_{\aleph_0}(S)]\backslash\{0\}.$$

The case of $\alpha > \aleph_0$ is more complicated. We leave this question open.

4. Invariance of the weighted spectrum

We first prove the following useful stability result.

Theorem 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$. If there exists $\lambda \in \rho(T) \cap \rho(S)$ such that $(T-\lambda)^{-1} - (S-\lambda)^{-1} \in \mathcal{K}_\alpha(\mathcal{H})$. Then*

$$\sigma_\alpha(T) = \sigma_\alpha(S).$$

Proof. Without loss of generality, we assume that $\lambda = 0$. For $\mu \in \mathbb{C}^*$, we can write

$$\mu - T = -\mu(\mu^{-1} - T^{-1})T.$$

Since T is one-to-one and onto, then

$$\mathcal{N}(\mu - T) = \mathcal{N}(\mu^{-1} - T^{-1}) \text{ and } \mathcal{R}(\mu - T) = \mathcal{R}(\mu^{-1} - T^{-1}).$$

This shows that

$$\mu - T \in \Phi_\alpha(\mathcal{H}) \text{ if and only if } \mu^{-1} - T^{-1} \in \Phi_\alpha(\mathcal{H}). \tag{8}$$

Since $T^{-1} - S^{-1} \in \mathcal{K}_\alpha(\mathcal{H})$, then using Proposition 2.6, we have

$$\sigma_\alpha(T^{-1}) = \sigma_\alpha(S^{-1}).$$

Therefore, by Eq. (8), we prove that $\sigma_\alpha(T) = \sigma_\alpha(S)$. □

Let (G_n) be a sequence of compact subsets of $\mathbb{C}$. The superior limit, $\limsup G_n$ is the set of all λ in $\mathbb{C}$ such that every neighborhood of λ intersects infinitely many G_n. A function f, defined on $\mathcal{L}(\mathcal{H})$, whose values are a non-empty compact subset of $\mathbb{C}$ said to be upper semi-continuous at T, when if $\| T_n - T \| \longrightarrow 0$, then $\limsup f(T_n) \subset f(T)$.

Theorem 4.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces, $T \in \mathcal{L}(\mathcal{H})$, $S \in \mathcal{L}(\mathcal{K})$ and $C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Let*

$$M_C = \begin{bmatrix} T & C \\ 0 & S \end{bmatrix}.$$

Then

$$\sigma_\alpha(M_C) \subset \sigma_\alpha(T) \cup \sigma_\alpha(S).$$

Proof. Let $M = \left[\begin{smallmatrix} T & 0 \\ 0 & S \end{smallmatrix}\right]$. Using Lemma 3.1, we have

$$\sigma_\alpha(M) = \sigma_\alpha(T) \cup \sigma_\alpha(S).$$

On the other hand, we have

$$M_{\frac{1}{k}C} = \begin{bmatrix} I & 0 \\ 0 & kI \end{bmatrix} \begin{bmatrix} T & C \\ 0 & S \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{k}I \end{bmatrix} \text{ for } k \in \mathbb{N}^*.$$

Observe that $\sigma_\alpha(M_C) = \sigma_\alpha(M_{\frac{1}{k}C})$ for all $k \in \mathbb{N}^*$ and $M_{\frac{1}{k}} \longrightarrow M$ as $k \longrightarrow N_0$. Using [8, Theorem 3.2], $\sigma_\alpha(\cdot)$ is upper semi-continuous function in $\mathcal{L}(\mathcal{H} \oplus \mathcal{K})$. Then

$$\sigma_\alpha(M_C) = \limsup \sigma_\alpha(M_{\frac{1}{k}C}) \subset \sigma_\alpha(M) = \sigma_\alpha(T) \cup \sigma_\alpha(S).$$ □

5. Pseudo-weighted spectrum

We refer the reader to Trefthen's article [24] for the definition of the ε-pseudo-spectrum $\sigma_\varepsilon(T)$ of a matrix or an operator $T \in \mathcal{L}(\mathcal{H})$:

$$\sigma_\varepsilon(T) = \sigma(A) \cup \Big\{ \lambda \in \mathbb{C} : \quad \|(\lambda - T)^{-1}\| \geq \frac{1}{\varepsilon} \Big\}.$$

By convention we write $\|(\lambda - T)^{-1}\| = \aleph_0$ if λ is in the spectrum $\sigma(T)$. This means that the pseudo-spectrum can be introduced as a zone of spectral instability. Davies in [7] defines another equivalent of pseudo-spectrum. For any closed operator T, we have

$$\sigma_\varepsilon(T) = \bigcup_{\|D\| < \varepsilon} \sigma(A + D).$$

In the way of the notion of pseudo-Fredholm and Weyl operator, we defined the α-pseudo-Fredholm and Weyl operators.

Definition 5.1. Let T be an operator an $\mathcal{L}(\mathcal{H})$ and let α be a cardinal number such that $\aleph_0 \leq \alpha \leq h$.

(i) T is called an α-pseudo-Fredholm operator if $A + D \in \Phi_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \varepsilon$.

(ii) T is called an α-pseudo-Weyl operator if $A+D \in \Phi^0_\alpha(\mathcal{H})$ for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \varepsilon$.

We denote by $\Phi^\varepsilon_\alpha(\mathcal{H})$ the set of all α-pseudo-Fredholm operators and by $\Phi^{0,\varepsilon}_\alpha(\mathcal{H})$ the set of all α-pseudo-Weyl operators. In this section, we are concerned with the following pseudo-spectra:

$\sigma^\varepsilon_\alpha(T) = \{\lambda \in \mathbb{C}$ such that $T - \lambda \notin \Phi^\varepsilon_\alpha(\mathcal{H})\}$, the pseudo-weighted spectrum.

$\sigma^{0,\varepsilon}_\alpha(T) = \{\lambda \in \mathbb{C}$ such that $T - \lambda \notin \Phi^{0,\varepsilon}_\alpha(\mathcal{H})\}$, the pseudo-weighted Weyl spectrum.

They can be ordered as

$$\sigma^\varepsilon_\alpha(T) \subseteq \sigma^{0,\varepsilon}_\alpha(T). \tag{9}$$

Note that if $\alpha = \aleph_0$, we recover the usual definitions of the pseudo-Fredholm and pseudo-Weyl spectra (see, [1, 2, 11, 12, 21]).

Remark 5.2. Let $T \in \mathcal{L}(\mathcal{H})$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$.

(i) $\Phi^\varepsilon_\alpha(\mathcal{H}) \subset \Phi^\varepsilon_\beta(\mathcal{H})$ and $\Phi^{0,\varepsilon}_\alpha(\mathcal{H}) \subset \Phi^{0,\varepsilon}_\beta(\mathcal{H})$ for all cardinal number such that $\alpha \leq \beta \leq \dim(\mathcal{H})$.

(ii) $\sigma^\varepsilon_\alpha(T) \cup \sigma^{0,\varepsilon}_\alpha(T) \subset \sigma_\varepsilon(T)$.

(iii) $\cap_{\varepsilon>0}\sigma^\varepsilon_\alpha(T) = \sigma_\alpha(T)$ and $\cap_{\varepsilon>0}\sigma^{0,\varepsilon}_\alpha(T) = \sigma^0_\alpha(T)$.

(iv) If $0 < \varepsilon_1 < \varepsilon_2$, then

$$\sigma^{\varepsilon_1}_\alpha(T) \subset \sigma^{\varepsilon_2}_\alpha(T) \text{ and } \sigma^{0,\varepsilon_1}_\alpha(T) \subset \sigma^{0,\varepsilon_2}_\alpha(T).$$

(v) $\sigma^\varepsilon_\alpha(T + K) = \sigma^\varepsilon_\alpha(T)$ and $\sigma^{0,\varepsilon}_\alpha(T + K) = \sigma^{0,\varepsilon}_\alpha(T)$ for all $K \in \mathcal{K}_\alpha(\mathcal{H})$.

The aim of the next result is to take the inverse inclusion of Eq. (9). Let $T \in \mathcal{L}(\mathcal{H})$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$. Let $\varepsilon > 0$ and set

$$\Phi^T_\alpha = \Big\{\lambda \in \mathbb{C} \text{ such that } T - \lambda \in \Phi_\alpha(\mathcal{H})\Big\}.$$

Theorem 5.3. *If $\Phi^{T+D}_{\aleph_0}$ is connected for all $D \in B(0, \varepsilon)$, then*

$$\sigma^{0,\varepsilon}_{\aleph_0}(T) = \sigma^\varepsilon_{\aleph_0}(T).$$

Proof. Since the inclusion $\sigma^\varepsilon_{\aleph_0}(T) \subset \sigma^{0,\varepsilon}_{\aleph_0}(T)$ is known, it suffices to show that $\sigma^{0,\varepsilon}_{\aleph_0}(T) \subset \sigma^\varepsilon_{\aleph_0}(T)$. Let $\lambda \notin \sigma^\varepsilon_\alpha(T)$. Then

$$T + D - \lambda \in \Phi_\alpha(\mathcal{H}) \text{ for all } D \in \mathcal{L}(\mathcal{H}), \quad \| D \| < \varepsilon.$$

Let $\lambda_D \in \rho(T+D)$, where $\rho(T+D)$ is the resolvent set of $T+D$. Then

$$T + D - \lambda_D \in \Phi_\alpha(\mathcal{H}) \text{ and } i(T + D - \lambda_D) = 0.$$

Since $\Phi_{\aleph_0}^{T+D}$ is connected, we can apply [23, Theorem 7.25], we infer that

$$i(T + D - \lambda) = i(T + D - \lambda_D) = 0.$$

In this way, we see that $\lambda \notin \sigma_{\aleph_0}^{0,\varepsilon}(T)$. □

Theorem 5.4. *Let $\mathcal{H}$ be a Hilbert space $\varepsilon > 0$, α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$, and consider $T,\ S \in \mathcal{L}(\mathcal{H})$.*

If for all $D \in B(0,\varepsilon)$, $T(D+S) \in \mathcal{K}_\alpha(\mathcal{H})$, then

$$\sigma_\alpha^\varepsilon(T+S)\backslash\{0\} \subset [\sigma_\alpha(T) \cup \sigma_\alpha^\varepsilon(S)]\backslash\{0\}$$

and

$$\sigma_\alpha^{0,\varepsilon}(T+S)\backslash\{0\} \subset \left[\sigma_\alpha^0(T) \cup \sigma_\alpha^{0,\varepsilon}(S)\right]\backslash\{0\}.$$

If, further, $T(D+S) = (D+S)T$, then

$$\sigma_\alpha^\varepsilon(T+S)\backslash\{0\} = [\sigma_\alpha(T) \cup \sigma_\alpha^\varepsilon(S)]\backslash\{0\}.$$

Proof. For $\lambda \in \mathbb{C}^*$ and $D \in \mathcal{L}(\mathcal{H})$, we can write

$$(\lambda - T)(\lambda - S - D) = T(S+D) + \lambda(\lambda - T - S - D) \tag{10}$$

and

$$(\lambda - S - D)(\lambda - T) = (S+D)T + \lambda(\lambda - T - S - D). \tag{11}$$

Suppose that $\lambda - T \in \Phi_\alpha(\mathcal{H})$ and $\lambda - S \in \Phi_\alpha^\varepsilon(\mathcal{H})$. Then, $\lambda - T \in \Phi_\alpha(\mathcal{H})$ and $\lambda - S - D \in \Phi_\alpha(\mathcal{H})$ for all $D \in B(0,\varepsilon)$. Using Theorem 2.5 and Eq. (10), we have

$$T(S+D) + \lambda(\lambda - T - S - D) \in \Phi_\alpha(\mathcal{H}) \text{ for all } D \in B(0,\varepsilon).$$

Since $T(S+D) \in \mathcal{K}_\alpha(\mathcal{H})$. Using Proposition 2.6, we infer that $\lambda - T - S - D \in \Phi_\alpha(\mathcal{H})$. This proved that

$$\sigma_\alpha^\varepsilon(T+S)\backslash\{0\} \subset [\sigma_\alpha(T) \cup \sigma_\alpha^\varepsilon(S)]\backslash\{0\}. \tag{12}$$

Similarly, we show that

$$\sigma_\alpha^{0,\varepsilon}(T+S)\backslash\{0\} \subset \left[\sigma_\alpha(T) \cup \sigma_\alpha^{0,\varepsilon}(S)\right]\backslash\{0\}. \tag{13}$$

Conversely, suppose that $\lambda - T - S - D \in \Phi_\alpha(\mathcal{H})$ for all $D \in B(0,\varepsilon)$. Since $T(S+D) = (S+D)T \in \mathcal{K}_\alpha(\mathcal{H})$. Then by Eq. (10) and (11), we have

$$(\lambda - T)(\lambda - S - D) = (\lambda - S - D)(\lambda - T) \in \Phi_\alpha(\mathcal{H}).$$

By Theorem 2.5(i), $\lambda - T \in \Phi_\alpha(\mathcal{H})$ and $\lambda - S \in \Phi_\alpha^\varepsilon(\mathcal{H})$. This proof is complete. □

Theorem 5.5. *Let $\mathcal{H}$ be a Hilbert space, $\varepsilon > 0$, $T \in \mathcal{L}(\mathcal{H})$ and α be a cardinal number such that $\aleph_0 \leq \alpha \leq \dim(\mathcal{H})$. Then*

$$\sigma_\alpha^{0,\varepsilon}(T) \subset \bigcap_{K \in \mathcal{K}_\alpha(\mathcal{H})} \sigma_\varepsilon(A+K).$$

Proof. Let β be a cardinal number such that $\aleph_0 \leq \beta < \alpha$. Let

$$\lambda \notin \bigcap_{K \in \mathcal{K}_\alpha(\mathcal{H})} \sigma_\varepsilon(A+K).$$

Then, there exists $K \in \mathcal{K}_\alpha(\mathcal{H})$ such that

$$\lambda \notin \sigma_\varepsilon(A+K) = \bigcup_{\|D\|<\varepsilon} \sigma(A+K+D).$$

This implies that, for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \varepsilon$, we have

$$A + D + K - \lambda \in \Phi_\alpha(\mathcal{H}) \text{ and } i_\beta(A+D+K-\lambda) = 0.$$

It comes from Proposition 2.6(ii) and (iii) that for all $D \in \mathcal{L}(\mathcal{H})$ such that $\|D\| < \varepsilon$ we have

$$A + D - \lambda \in \Phi_\alpha(\mathcal{H}) \text{ and } i_\beta(A+D-\lambda) = 0.$$

This prove that $\lambda \notin \sigma_\alpha^{0,\varepsilon}(T)$. □

We finish with the following question:

Question: Does it follow that

$$\sigma_\alpha^{0,\varepsilon}(T) = \bigcap_{K \in \mathcal{K}_\alpha(\mathcal{H})} \sigma_\varepsilon(A+K)?$$

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