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BRÜCK CONJECTURE AND A LINEAR DIFFERENTIAL POLYNOMIAL

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Abstract. In the paper we consider the uniqueness of a meromorphic function and a linear differential polynomial when they share a small function.

1. Introduction, definitions and results

Let f, g be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ the functions f, g are said to share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities.

The standard definitions and notations of the value distribution theory are available in [5]. We need the following in the paper.

Definition 1.1. For a meromorphic function f and for $a \in \mathbb{C} \cup \{\infty\}$ and for a positive integer k

(i) $N_{(k}(r, a; f)$ $(\overline{N}_{(k}(r, a; f))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than k;

(ii) $N_{k)}(r, a; f)$ $(\overline{N}_{k)}(r, a; f))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than k;

(iii) $N_k(r, a; f)$ denotes the sum $\overline{N}(r, a; f) + \sum_{j=2}^{k} \overline{N}_{(j}(r, a; f)$.

Clearly $N_k(r, a; f) \leq k\overline{N}(r, a; f)$.

Considering the uniqueness problem of an entire function sharing a single value CM with its first derivative, R. Brück [3] proved the following theorem.

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Theorem A ([3])**.** *Let f be a nonconstant entire function. If f and f' share the value 1 CM and $N(r,0;f') = S(r,f)$, then $f-1 = c(f'-1)$, where c is a nonzero constant.*

For a finite order entire function, L. Z. Yang [9] proved the following theorem.

Theorem B ([9])**.** *Let f be a nonconstant entire function of finite order and let $a(\neq 0,\infty)$ be a constant. If f and $f^{(k)}$ share the value a CM, then $f-a = c(f^{(k)}-a)$, where c is a nonzero constant and $k(\geq 1)$ is an integer.*

R. Brück [3] proposed the following conjecture.

Brück Conjecture. *Let f be a nonconstant entire function of finite nonintegral hyper-order. If f and f' share one finite value a CM, then $f'-a = c(f-a)$ for some constant $c(\neq 0)$.*

Apart from Theorem A and Theorem B a number of results on Brück's conjecture are available in the literature. H. L. Qiu [8] extended Theorem A to a linear differential polynomial.

A meromorphic function a is called a small function of a meromorphic function f if $T(r,a) = S(r,f)$.

A. H. H. Al-khaladi [1] pointed out that in Theorem A one cannot replace the value 1 by a small function by considering $f(z) = 1+\exp(e^z)$ and $a(z) = \frac{e^z}{e^z-1}$. He proved the following result.

Theorem C ([1])**.** *Let f be a nonconstant entire function satisfying $N(r,0;f') = S(r,f)$ and let $a(\not\equiv 0,\infty)$ be a small function of f. If $f-a$ and $f'-a$ share the value 0 CM, then $f-a = (1+\frac{c}{a})(f'-a)$, where $1+\frac{c}{a} = e^{\beta}$, c is a constant and β is an entire function.*

Extending Theorem C to a linear differential polynomial, J. F. Chen and G. R. Wu [4] proved the following result.

Theorem D ([4])**.** *Let f be a nonconstant entire function satisfying $N(r,0;f') = S(r,f)$, $a(\not\equiv 0,\infty)$ be a small function of f and $L = L(f) = \sum_{j=1}^{k} a_j f^{(j)}$, where k is a positive integer and $a_1, a_2, \ldots, a_k(\not\equiv 0)$ are small entire functions of f. If $f-a$ and $L-a$ share 0 CM, then $f-a = (1+\frac{c}{a})(L-a)$, where $1+\frac{c}{a} = e^{\beta}$, c is a constant and β is an entire function.*

A similar result of Theorem D is proved in [7] for meromorphic functions. In the paper we investigate the following problem: *Under which situation $f-a$ becomes a constant multiple of $L-a$ even if $a(\not\equiv 0,\infty)$ is a small function of f?*

Throughout the paper we denote by $L = L(f)$ a linear differential polynomial of the following form:

$$L = L(f) = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_k f^{(k)}, \tag{1.1}$$

where f is a nonconstant meromorphic function, $a_1, a_2, \ldots, a_k(\neq 0)$ are constants and k is a positive integer.

We prove in the paper the following theorems.

Theorem 1.1. *Let f be a nonconstant meromorphic function with $\overline{N}(r,0;f')+\overline{N}(r,\infty;f)=S(r,f)$. Suppose that L, as defined by (1.1), is nonconstant and $k(\geq 2)$ is a positive integer. Let $a(\not\equiv 0,\infty)$ be a small function of f such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{k}$. If $f-a$ and $L-a$ share 0 CM, then $f\equiv L$.*

Theorem 1.2. *Let f be a nonconstant meromorphic function with $\overline{N}(r,0;f^{(2)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$. Suppose that L, as defined by (1.1), is nonconstant, where $a_1=0$ and $k(\geq 2)$ is a positive integer. Let $a(\not\equiv 0,\infty)$ be a small function of f such that $N(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{k}$. If $f-a$ and $L-a$ share 0 CM, then $f-a\equiv c(L-a)$, where $c(\neq 0)$ is a constant.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1 ([2])**.** *Let $k(\geq 2)$ be a positive integer and f be a nonconstant meromorphic function. If $\overline{N}(r,0;f^{(k)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$, then either $N_{1)}(r,\infty;f)=S(r,f)$ or $f(z)=\frac{-(k+1)^{k+1}}{k!c\{z+d(k+1)\}}+p_{k-1}(z)$, where $c(\neq 0)$, d are constants and $p_{k-1}(z)$ is a polynomial of degree at most $k-1$.*

Lemma 2.2. *Let f be a nonconstant meromorphic function and $k(\geq 2)$ be a positive integer. Suppose that $a(\not\equiv 0,\infty)$ is a small function of f, and L, as given in Theorem 1.2, is nonconstant. If $\overline{N}(r,0;f^{(2)})+\overline{N}_{(2}(r,\infty;f)=S(r,f)$ and $f-a$, $L-a$ share 0 CM, then $\overline{N}(r,\infty;f)=S(r,f)$.*

Proof. If $f(z)=\frac{-27}{2c(z+3d)}+p_1(z)$, then a becomes a constant. Clearly, in this case, $f-a$ and $L-a$ cannot share 0 CM. Therefore by Lemma 2.1 we get $\overline{N}(r,\infty;f)=S(r,f)$. □

Lemma 2.3 ([5, p. 47, Th. 2.5])**.** *Let f be a nonconstant meromorphic function and a_1,a_2,a_3 be distinct meromorphic small functions of f. Then*

$$T(r,f)\leq\sum_{j=1}^{3}\overline{N}(r,0;f-a_j)+S(r,f).$$

Lemma 2.4 ([6])**.** *Given a transcendental meromorphic function f and a constant $\Gamma>1$. Then there exists a set $M(\Gamma)$ whose upper logarithmic density is at most*

$$\delta(\Gamma)=\min\{(2e^{\Gamma-1}-1)^{-1},(1+e(\Gamma-1))\exp(e(1-\Gamma))\}$$

such that for every positive integer k,

$$\limsup_{r\to\infty,r\notin M(\Gamma)}\frac{T(r,f)}{T(r,f^{(k)})}\leq 3e\Gamma.$$

3. Proofs of the theorems

We prove Theorem 1.2 only, as Theorem 1.1 can be proved similarly.

Proof of Theorem 1.2. If f is not transcendental, then f must be a polynomial because by Lemma 2.2 we have $\overline{N}(r,\infty;f) = S(r,f)$. If $\deg(f) > 2$, then $\deg(L) = \deg(f)-2$ and if $\deg(f) \le 2$, then $\deg(L) = 0$, which is impossible as L is nonconstant. Since in this case a is a constant, we see that $f-a$ and $L-a$ cannot share the value 0 CM, a contradiction. Therefore f is a transcendental meromorphic function.

Let $h = \frac{f-a}{L-a}$. Then h is entire and the poles of f are precisely the zeros of h so that by the hypothesis and Lemma 2.2 we get

$$\overline{N}(r,0;h) \le \overline{N}(r,\infty;f) = S(r,f). \tag{3.1}$$

Now differentiating

$$f - a = hL - ah \tag{3.2}$$

twice we get

$$f^{(2)} - a^{(2)} = (hL)^{(2)} - (ah)^{(2)}. \tag{3.3}$$

We now consider the following cases.

Case I. Let $a^{(2)} \not\equiv 0$. We put

$$W = \frac{(hL)^{(2)}}{hf^{(2)}} - \frac{(ha)^{(2)}}{ha^{(2)}}. \tag{3.4}$$

First we suppose that $W \not\equiv 0$. Let z_0 be a zero of $f^{(2)} - a^{(2)}$ and $a^{(2)}(z_0) \ne 0,\infty$. Then from (3.3) we see that z_0 is a zero of $(hL)^{(2)} - (ha)^{(2)}$. Hence $W(z_0) = 0$. We see that

$$\begin{aligned} m(r,W) &\le m\left(r,\frac{(hL)^{(2)}}{hf^{(2)}}\right) + m\left(r,\frac{(ha)^{(2)}}{ha^{(2)}}\right) \\ &\le m\left(r,\frac{(hL)^{(2)}}{hL}\right) + m\left(r,\frac{L}{f^{(2)}}\right) + m\left(r,\frac{(ha)^{(2)}}{ha}\right) + m\left(r,\frac{a}{a^{(2)}}\right) \\ &= S(r,f). \end{aligned}$$

Therefore

$$\begin{aligned} \overline{N}(r,0;f^{(k)} - a^{(k)}) &\le N(r,0;W) + S(r,f) \\ &\le T(r,W) + S(r,f) \\ &= N(r,W) + S(r,f). \end{aligned} \tag{3.5}$$

Let z_1 be a pole of f with multiplicity p such that $a(z_1) \ne 0,\infty$ and $a^{(2)}(z_1) \ne 0$. Then z_1 is a zero of h with multiplicity k. Hence z_1 is a pole of $(hL)^{(2)}$ with multiplicity $p+2$. Also $hf^{(2)}$ has a pole at z_1 of multiplicity $p+2-k$. Therefore z_1 is a pole of $\frac{(hL)^{(2)}}{hf^{(2)}}$ with multiplicity $(p+2)-(p+2-k) = k$. Also

z_1 is a pole of $\frac{(ha)^{(2)}}{ha^{(2)}}$ with multiplicity $2 \leq k$. Therefore z_1 is a pole of W with multiplicity at most k.

Let z_2 be a zero of $f^{(2)}$ with multiplicity q and $a(z_2) \neq 0, \infty$, $a^{(2)}(z_2) \neq 0$. If $q > k$, then z_2 is a zero of hL with multiplicity $q - k + 2$. So z_2 is a zero of $(hL)^{(2)}$ with multiplicity $(q - k + 2) - 2 = q - k$. Hence z_2 is a pole of W with multiplicity at most $q - (q - k) = k$.

Therefore in view of Lemma 2.2 we get

$$\begin{aligned} N(r, W) &\leq k\overline{N}(r, \infty; f) + N_k(r, 0; f^{(2)}) + \overline{N}(r, 0; f^{(2)}) + S(r, f) \\ &\leq k\overline{N}(r, \infty; f) + (1 + k)\overline{N}(r, 0; f^{(2)}) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we get $\overline{N}(r, 0; f^{(2)} - a^{(2)}) = S(r, f)$. So by Lemma 2.3 and Lemma 2.2 we obtain

$$\begin{aligned} T(r, f^{(2)}) &\leq \overline{N}(r, \infty; f^{(2)}) + \overline{N}(r, 0; f^{(2)}) + \overline{N}(r, 0; f^{(2)} - a^{(2)}) + S(r, f^{(2)}) \\ &= S(r, f). \end{aligned} \tag{3.7}$$

Let $M(\Gamma)$ be defined as in Lemma 2.4. Then by (3.7) there exists a sequence $r_n \to \infty$, $r_n \notin M(\Gamma)$ such that $\frac{T(r_n, f^{(2)})}{T(r_n, f)} \to 0$ as $n \to \infty$. This contradicts Lemma 2.4. Therefore $W \equiv 0$ and so from (3.3) and (3.4) we get $(f^{(2)} - a^{(2)})a^{(2)} = (ha)^{(2)}(f^{(2)} - a^{(2)})$. Since $f^{(2)} \not\equiv a^{(2)}$, we obtain $(ha)^{(2)} = a^{(2)}$. Integrating twice we get $ha = a + \alpha z + \beta$ and so $h = 1 + \frac{\alpha z + \beta}{a}$, where α, β are constants.

We again note that h is entire and the zeros of h are precisely the poles of f. Also each zero of h is of multiplicity k. Let $\alpha \neq 0$. Then $T(r, h) = T(r, a) + O(\log r)$. Also $\overline{N}(r, 1; h) = \overline{N}(r, \infty; a) + O(\log r)$ and $\overline{N}(r, 0; h) = \frac{1}{k}N(r, 0; h)$. Therefore by the second fundamental theorem and the hypothesis we get

$$\begin{aligned} T(r, h) &\leq \overline{N}(r, 1; h) + \overline{N}(r, 0; h) + S(r, h) \\ &= \overline{N}(r, \infty; a) + \frac{1}{k}N(r, 0; h) + O(\log r) + S(r, h) \\ &\leq \lambda T(r, a) + \frac{1}{k}T(r, h) + O(\log r) + S(r, h) \\ &= (\lambda + \frac{1}{k})T(r, h) + O(\log r) + S(r, h) \end{aligned}$$

and so $T(r, h) = O(\log r) + S(r, h)$. This implies that $h - 1$ is a polynomial, say $P(z)$. If $P(z) \equiv 0$, then $h \equiv 1$ and we get the result. We suppose that $P(z) \not\equiv 0$. Then $h = 1 + \frac{\alpha z + \beta}{a}$ implies $a = \frac{\alpha z + \beta}{P(z)}$.

We suppose that $\alpha z + \beta$ is a factor of $P(z)$. Then $a = \frac{1}{Q(z)}$, where $P(z) = (\alpha z + \beta)Q(z)$. This implies that $T(r, a) = (\deg Q) \log r + O(1) = N(r, \infty; a) + O(1)$, a contradiction. So $\alpha z + \beta$ is not a factor of $P(z)$. Then $T(r, a) =$

$\max\{\deg P, 1\} \log r + O(1)$ and $N(r, \infty; a) = (\deg P) \log r + O(1)$. By the hypothesis we get $\deg P \leq \lambda \max\{\deg P, 1\}$. This implies $\deg P = 0$ and so $a = \frac{\alpha z+\beta}{d}$, where $d(\neq 0)$ is a constant. Hence $h = 1 + d$, a constant.

Let $\alpha = 0$. Then $h = \frac{a+\beta}{a}$. Since h is entire and each zero of h is of multiplicity k, we have $\overline{N}(r, 0; a) \equiv 0$ and $\overline{N}(r, 0; a + \beta) \leq \frac{1}{k} N(r, 0; a + \beta)$. Therefore, if $\beta \neq 0$, we get by the second fundamental theorem

$$\begin{aligned} T(r, a) &\leq \overline{N}(r, \infty; a) + \overline{N}(r, 0; a) + \overline{N}(r, 0; a + \beta) + S(r, a) \\ &\leq (\lambda + \frac{1}{k}) T(r, a) + S(r, a), \end{aligned}$$

a contradiction. So $\beta = 0$ and $h \equiv 1$.

Case II. Let $a^{(2)} \equiv 0$. Then a is a polynomial of degree at most 1. From (3.3) we get $f^{(2)} = (hL)^{(2)} - (ah)^{(2)}$, which implies

$$\frac{1}{h} = \frac{(hL)^{(2)}}{hf^{(2)}} - \frac{(ah)^{(2)}}{hf^{(2)}}. \tag{3.8}$$

We put $F = f^{(2)}$, $G = \frac{(hL)^{(2)}}{hf^{(2)}}$ and $b = \frac{(ah)^{(2)}}{h}$. So from (3.8) we get

$$\frac{1}{h} = G - \frac{b}{F}. \tag{3.9}$$

Differentiating (3.9) we obtain

$$-\frac{1}{h} \cdot \frac{h'}{h} = G' - \frac{b'}{F} + \frac{b}{F} \cdot \frac{F'}{F}. \tag{3.10}$$

From (3.9) and (3.10) we have

$$\frac{A}{F} = G' + G\frac{h'}{h}, \tag{3.11}$$

where $A = b\frac{h'}{h} + b' - b\frac{F'}{F}$.

First we suppose that $G \equiv 0$. Then on integration we get $hL = Q_1$, where $Q_1 = Q_1(z)$ is a polynomial of degree at most 1. Putting $h = \frac{f-a}{L-a}$ we get

$$(f - a)L = (L - a)Q_1. \tag{3.12}$$

Since a is a polynomial, from (3.12) we see that f is an entire function. Hence h is an entire function having no zero. We put $h = e^{\alpha}$, where α is an entire function.

Now $f = a + h(L - a) = a + Q_1 - ae^{\alpha}$ and $L = Q_1 e^{-\alpha}$. Also we see from the definition of L that $L = R(\alpha')e^{\alpha}$, where $R(\alpha')(\not\equiv 0)$ is a differential polynomial in α' with polynomial coefficients. Therefore $R(\alpha')e^{\alpha} = Q_1 e^{-\alpha}$ and so $e^{2\alpha} = \frac{Q_1}{R(\alpha')}$. This shows that $T(r, e^{\alpha}) = S(r, e^{\alpha})$, a contradiction. Hence $G \not\equiv 0$.

If h is constant, then we achieve the result. So we suppose that h is nonconstant.

Let $b \equiv 0$. Then on integration we get $ah = P_1$, where $P_1 = P_1(z)$ is a polynomial of degree at most 1. Since h is entire and a is a polynomial of degree at most 1, $h = \frac{P_1}{a}$ implies that a is a factor of P_1 and hence

$$h = Q_1^*, \tag{3.13}$$

where $Q_1^* = Q_1^*(z)$ is a polynomial of degree at most 1. Since each pole of f is a zero of h with multiplicity $k(\geq 2)$, by (3.13) we see that f is entire. So h is an entire function having no zero, which by (3.13) implies that h is a constant, a contradiction. So $b \not\equiv 0$.

Let $A \equiv 0$. Then from (3.11) we get $\frac{G'}{G} + \frac{h'}{h} \equiv 0$ and so on integration we obtain $Gh \equiv K$ so that

$$(hL)^{(2)} = Kf^{(2)}, \tag{3.14}$$

where K is a nonzero constant.

Again $\frac{A}{b} = \frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F} = 0$ implies on integration that $hb = MF$ and so

$$(ah)^{(2)} = Mf^{(2)}, \tag{3.15}$$

where M is a nonzero constant.

Since a is a polynomial and h is entire, from (3.15) we see that f is entire and so $h = e^{\alpha}$, where α is an entire function.

Integrating (3.14) twice we get

$$hL = Kf + P_1^*, \tag{3.16}$$

where $P_1^* = P_1^*(z)$ is a polynomial of degree at most 1.

Since $hL = f - a + ah$, we get from (3.16)

$$(1 - K)f = a(1 - e^{\alpha}) + P_1^*. \tag{3.17}$$

If $K = 1$, from (3.17) we get $e^{\alpha} = 1 + \frac{P_1^*}{a}$, a contradiction. Hence $K \neq 1$ and from (3.17) we obtain

$$f = \frac{ae^{\alpha}}{K-1} - \frac{a + P_1^*}{K-1}. \tag{3.18}$$

From the definition of L we get by (3.18)

$$L = R_1(\alpha')e^{\alpha}, \tag{3.19}$$

where $R_1(\alpha')(\not\equiv 0)$ is a differential polynomial in α' with polynomial coefficients.

From (3.16) and (3.18) we get

$$L = \frac{Ka}{K-1} - \frac{a + (2-K)P_1^*}{K-1}e^{-\alpha}. \tag{3.20}$$

From (3.19) and (3.20) we obtain

$$R_1(\alpha')e^{2\alpha} = \frac{Kae^{\alpha}}{K-1} - \frac{a + (2-K)P_1^*}{K-1},$$

which implies $T(r, e^{\alpha}) = S(r, e^{\alpha})$, a contradiction. Therefore $A \not\equiv 0$.

Now $A = b\left(\frac{h'}{h} + \frac{b'}{b} - \frac{F'}{F}\right)$ implies $m(r, A) = S(r, f)$. Also the poles of A are contributed by (i) the poles of $b = \frac{(ah)^{(2)}}{h}$, (ii) the poles of $\frac{h'}{h}$ and (iii) the poles of $\frac{F'}{F} = \frac{f^{(3)}}{f^{(2)}}$. Since h is entire and the zeros of h are precisely the poles of f and each zero of h is of multiplicity k, we get

$$N(r, A) \leq (k+1)\overline{N}(r, \infty; f) + \overline{N}(r, 0; f^{(2)}) + S(r, f) = S(r, f),$$

by the hypothesis and Lemma 2.2. Therefore $T(r, A) = S(r, f)$.

Now by (3.11) we get

$$\begin{aligned} m(r, \frac{1}{F}) &\leq m(r, \frac{1}{A}) + m(r, G' + G\frac{h'}{h}) \\ &\leq T(r, A) + m(r, G) + m(r, \frac{G'}{G} + \frac{h'}{h}) \\ &= m(r, G) + S(r, f) \\ &= m(r, \frac{(hL)^{(2)}}{hL} \cdot \frac{L}{f^{(2)}}) + S(r, f) \\ &\leq m(r, \frac{(hL)^{(2)}}{hL}) + m(r, \frac{L}{f^{(2)}}) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.21}$$

Again in view of (3.1) we get

$$\begin{aligned} T(r, b) &= N(r, b) + S(r, f) \\ &= N(r, \frac{(ah)^{(2)}}{h}) + S(r, f) \\ &\leq 2\overline{N}(r, 0; h) + S(r, f) \\ &= S(r, f). \end{aligned} \tag{3.22}$$

Let z_3 be a zero of $F = f^{(2)}$ with multiplicity $q \geq k+1$ such that $a(z_0) \neq 0$. Then z_3 is a zero of $(hL)^{(2)}$ with multiplicity at least $q - (k-2) - 2 = q - k$. So z_3 is a zero of $FG = \frac{(hL)^{(2)}}{h}$ with multiplicity at least $q - k$. Hence z_3 is a zero of $b = FG - \frac{F}{h}$ with multiplicity at least $q - k$.

Therefore by (3.22) we get

$$N_{(k+1}(r, 0; f^{(2)}) \leq N(r, 0; b) + k\overline{N}_{(k+1}(r, 0; f^{(2)}) = k\overline{N}_{(k+1}(r, 0; f^{(2)}) + S(r, f).$$

Therefore

$$\begin{aligned} N(r, \frac{1}{F}) &= N(r, 0; f^{(2)}) \\ &= N_{k)}(r, 0; f^{(2)}) + N_{(k+1}(r, 0; f^{(2)}) \\ &\leq k\overline{N}_{k)}(r, 0; f^{(2)}) + k\overline{N}_{(k+1}(r, 0; f^{(2)}) + S(r, f) \\ &= k\overline{N}(r, 0; f^{(2)}) + S(r, f) \end{aligned}$$

$$= S(r,f). \tag{3.23}$$

From (3.21), (3.23) and the first fundamental theorem we get $T(r, f^{(2)}) = S(r,f)$, which is (3.7), and likewise we arrive at a contradiction. □

4. The counter-example of Al-Khaladi

As mentioned in the introduction A. H. H. Al-Khaladi, considering $f(z) = 1 + \exp(e^z)$ and $a(z) = \frac{e^z}{e^z-1}$, established that in Theorem A, the shared value cannot be replaced by a shared small function. In stead, he proved Theorem C.

In fact, the poles of $a(z) = \frac{e^z}{e^z-1}$ play the most crucial role. Here we note that $\overline{N}(r,\infty;a) = T(r,a) + S(r,a)$. On the other hand, we see that a small function with relatively less number of poles can yield a rather impressive output. For example, let $\overline{N}(r,\infty;a) \leq \lambda T(r,a) + S(r,a)$, where $0 < \lambda < 1$. Since by Theorem C, $e^\beta = 1 + \frac{c}{a}$, clearly a and $a + c$ have no zero. So if $c \neq 0$, by the second fundamental theorem we get

$$T(r,a) \leq \overline{N}(r,\infty;a) + S(r,a) \leq \lambda T(r,a) + S(r,a),$$

a contradiction. Therefore $c = 0$ and $f \equiv f'$.

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