Commun. Korean Math. Soc. **33** (2018), No. 3, pp. 787–798
https://doi.org/10.4134/CKMS.c170144
pISSN: 1225-1763 / eISSN: 2234-3024

HYPONORMAL SINGULAR INTEGRAL OPERATORS WITH CAUCHY KERNEL ON L^2

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Abstract. For $1 \le p \le \infty$, let H^p be the usual Hardy space on the unit circle. When α and β are bounded functions, a singular integral operator $S_{\alpha,\beta}$ is defined as the following: $S_{\alpha,\beta}(f+\bar{g}) = \alpha f + \beta \bar{g}$ $(f \in H^p, g \in zH^p)$. When $p = 2$, we study the hyponormality of $S_{\alpha,\beta}$ when α and β are some special functions.

1. Introduction

Let $\mathcal{K}$ be a Hilbert space and $\mathcal{B}(\mathcal{K})$ be the set of bounded linear operators on $\mathcal{K}$. For X in $\mathcal{B}(\mathcal{K})$, $[X^*, X] = X^*X - XX^*$ is called the selfcommutator of X. If $[X^*, X] = 0$, then X is called a normal operator and if $[X^*, X] \ge 0$, then X is called a hyponormal operator. When $\mathcal{H}$ is a closed subspace of $\mathcal{K}$, P is the orthogonal projection from $\mathcal{K}$ to $\mathcal{H}$ and I is the identity operator, if

$$P[X^*, X](I - P) = (I - P)[X^*, X]P = 0,$$

then X is called a D-operator. Of course, if X is a normal operator, then X is a D-operator. However a hyponormal operator is not necessary a D-operator. When a hyponormal X is a D-operator, we call it nearly normal.

There are a lot of papers about a normal operator and a hyponormal operator. We are interested in when a concrete operator is normal or hyponormal. There are many researches when X is a Toeplitz operator, for example, [1], [2], [3] and [11]. Recently Yamamoto and the author [12] started to study when X is some singular integral operator. In this paper we continue to study such a problem.

Let L^2 be the usual Lebesgue space and H^2 denotes the usual Hardy space on the unit circle. For x, y in L^2, put

$$\langle x, y\rangle = \int_0^{2\pi} x(e^{i\theta})\bar{y}(e^{i\theta})d\theta/2\pi.$$

Then by the inner product $\langle , \rangle$, L^2 and H^2 become Hilbert spaces.

Received April 7, 2017; Revised December 4, 2017; Accepted December 29, 2017.
2010 *Mathematics Subject Classification.* 45E10, 47B35, 47B20.
Key words and phrases. singular integral operator, Toeplitz operator, Hardy space, hyponormal operator.

For α, β in L^∞ put

$$S_{\alpha,\beta} = (\alpha - \beta)P + \beta I = \alpha P + \beta(I - P),$$

where I is the identity operator on L^2 and P denotes the orthogonal projection to H^2. Then $S_{\alpha,\beta}$ is called a singular integral operator on L^2 and

$$(S_{\alpha,\beta}f)(z) = \frac{\alpha(z) + \beta(z)}{2} f(z) + \frac{\alpha(z) - \beta(z)}{2}\ \frac{1}{\pi i} \int \frac{f(\xi)}{\xi - z} d\zeta,$$

where the integral is understood in the sense of Cauchy's principal value (cf. [5, Vol. I, p. 12]). Throughout this paper

$$\alpha = a + F + \bar{f}$$

and

$$\beta = b + G + \bar{g},$$

where $a, b \in \mathbb{C}$ and $F, G, f, g \in zH^2$. Moreover

$$\phi = \alpha - \beta = c + A + \bar{B},$$

where $c = a - b, A = F - G$ and $B = f - g$.

Yamamoto and the author [12] describe completely the symbols α and β for a normal $S_{\alpha,\beta}$. In this paper, we study when $S_{\alpha,\beta}$ is a hyponormal operator or a nearly normal operator.

In order to study $S_{\alpha,\beta}$, we need a few notations as the following:

$$T_\alpha = PM_\alpha P,\ \tilde{T}_\alpha = (I - P)M_\alpha(I - P)$$

and

$$H_\alpha = (I - P)M_\alpha P,\ \tilde{H}_\alpha = PM_\alpha(I - P).$$

Then T_α is called a Toeplitz operator and H_α is called a Hankel operator. Moreover $T_\alpha^* = T_{\bar{\alpha}},\ \tilde{T}_\alpha^* = \tilde{T}_{\bar{\alpha}}$ and $H_\alpha^* = \tilde{H}_{\bar{\alpha}}$.

2. Hyponormal operator

In this section, we show a necessary and sufficient condition for a hyponormal $S_{\alpha,\beta}$.

Theorem 1. *$S_{\alpha,\beta}$ is hyponormal if and only if $\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}} \geq 0$ and $H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_{\bar{\alpha}} \geq 0$, and*

$$|\langle(\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}})u, u\rangle| \cdot |\langle(H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_{\bar{\alpha}})v, v\rangle| \geq |\langle(\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}})u, v\rangle|^2,$$

where u is in H^2 and v is in $\bar{z}\bar{H}^2$.

Proof. By Lemma 3.1 in [12], $S_{\alpha,\beta}$ is hyponormal if and only if

$$\begin{aligned}&\langle(\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}})u, u\rangle + \langle(H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_{\bar{\alpha}})v, v\rangle \\ &+ \langle(\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}})u, v\rangle + \overline{\langle(\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha \tilde{T}_{\bar{\phi}})u, v\rangle} \geq 0.\ (u \in H^2, v \in \bar{z}\bar{H}^2)\end{aligned}$$

Hence if $S_{\alpha,\beta}$ is hyponormal, then as $v = 0$, $\langle(\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}})u, u\rangle \geq 0$ and as $u = 0$, $\langle(H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_\alpha)v, v\rangle \geq 0$. Moreover then for any real number t,

$$\langle(\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}})u, u\rangle t^2 - 2|\langle(\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_\phi)u, v\rangle| t + \langle(H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_{\bar{\alpha}})v, v\rangle \geq 0.$$

This shows the 'only if' part. The proof is reversible and so the 'if part' follows. □

The following lemma is known by [6, Theorem 7] or the proof of [7, Lemma 4]. However we give a proof for completeness.

Lemma 1. (1) $\tilde{H}_\alpha H_{\bar{\alpha}} \geq \tilde{H}_\beta H_{\bar{\beta}}$ *if and only if there exists* k *in* H^∞ *such that* $\|k\|_\infty \leq 1$ *and* $k\bar{\alpha} - \bar{\beta}$ *belongs to* H^∞.

(2) $H_\beta \tilde{H}_{\bar{\beta}} \geq H_\alpha \tilde{H}_{\bar{\alpha}}$ *if and only if there exists* h *in* H^∞ *such that* $\|h\|_\infty \leq 1$ *and* $h\beta - \alpha$ *belongs to* H^∞.

Proof. We show (2) because (1) can be shown similarly.

Suppose there exists a contractive function h in H^∞ such that $h\beta - \alpha \in H^\infty$. Then $H_{h\beta} = H_\alpha$ and so $\tilde{H}_{\overline{h\beta}} = \tilde{H}_{\bar{\alpha}}$. Hence

$$H_\alpha \tilde{H}_{\bar{\alpha}} = H_{h\beta} \tilde{H}_{\overline{h\beta}} = H_\beta T_h T_{\bar{h}} \tilde{H}_{\bar{\beta}} \leq H_\beta \tilde{H}_{\bar{\beta}}$$

because $T_h T_{\bar{h}} \leq I$, $H_{h\beta} = H_\beta T_h$ and $\tilde{H}_{\overline{h\beta}} = T_{\bar{h}} \tilde{H}_{\bar{\beta}}$.

Conversely suppose $H_\beta \tilde{H}_{\bar{\beta}} \geq H_\alpha \tilde{H}_{\bar{\alpha}}$. Then by a theorem of Douglas [4], there exists a contraction B such that $B\tilde{H}_{\bar{\beta}} = \tilde{H}_{\bar{\alpha}}$. Since $T_{\bar{z}} \tilde{H}_{\bar{\beta}} = \tilde{H}_{\bar{\beta}} \tilde{T}_z$ and $T_{\bar{z}} \tilde{H}_{\bar{\alpha}} = \tilde{H}_{\bar{\alpha}} \tilde{T}_{\bar{z}}, B\tilde{H}_{\bar{\beta}} \tilde{T}_z = BT_{\bar{z}} \tilde{H}_{\bar{\beta}}$ and $B\tilde{H}_{\bar{\beta}} \tilde{T}_z = \tilde{H}_{\bar{\alpha}} \tilde{T}_z = T_{\bar{z}} \tilde{H}_{\bar{\alpha}} = T_{\bar{z}} B\tilde{H}_{\bar{\beta}}$. Hence $BT_{\bar{z}} \tilde{H}_{\bar{\beta}} = T_{\bar{z}} B\tilde{H}_{\bar{\beta}}$. Since $T_{\bar{z}}$ $(Ran\tilde{H}_{\bar{\beta}}) \subseteq Ran\tilde{H}_{\bar{\beta}}, BS_{\bar{z}} = S_{\bar{z}} B$ where $S_{\bar{z}}$ is the restriction to of $T_{\bar{z}}$ to the closure $Ran\tilde{H}_{\bar{\beta}}$. By a theorem of Sarason [13], $B^* = S_h$ for some h in H^∞ with $\|h\|_\infty \leq 1$. Hence $\tilde{H}_{\bar{\alpha}} = S_{\bar{h}} \tilde{H}_{\bar{\beta}} = T_{\bar{h}} \tilde{H}_{\bar{\beta}} = \tilde{H}_{\bar{\beta}\bar{h}}$. Thus $h\beta - \alpha$ belongs to H^∞. □

Theorem 2. *If* $S_{\alpha,\beta}$ *is hyponormal, then there exist* k *and* h *are in* H^∞ *such that* $\|k\|_\infty \leq 1$ *and* $\|h\|_\infty \leq 1$, *and* $k\bar{\alpha} - \bar{\beta}$ *and* $h\beta - \alpha$ *belong to* H^∞. *Hence*

$$(k-1)\bar{G} + k\bar{A} \quad and \quad (h-1)\bar{f} - h\bar{B}$$

belong to H^∞.

Proof. It is clear by Theorem 1 and Lemma 1. □

Theorem 3. *Let* $\alpha = a + F + \bar{f}$, $\beta = b + G + \bar{g}$, $\psi_+ = F + \bar{G}$ *and* $\psi_- = f + \bar{g}$ *where* F, f, G *and* g *are in* zH^∞. *If* $S_{\alpha,\beta}$ *is hyponormal, then* T_{ψ_+} *and* T_{ψ_-} *are hyponormal.*

Proof. If $S_{\alpha,\beta}$ is hyponormal, then by Theorem 2, there exist contractions k and h in H^∞ such that $k\bar{\alpha} - \bar{\beta}$ and $h\beta - \alpha$ belong to H^∞. Hence $k\bar{F} - \bar{G}$ and $h\bar{g} - \bar{f}$ belong to H^∞. Therefore $k(\bar{F} + G) - (\bar{G} + F)$ and $h(\bar{g} + f) - (\bar{f} + g)$ belong to H^∞. Now by [11, Lemma 1] T_{ψ_+} and T_{ψ_-} are hyponormal. □

If $T_{F+\bar{G}}$ is hyponormal, then it is easy to see $[T_F^*, T_F] \geq [T_G^*, T_G]$. If F and G are inner, then $I - T_F T_F^* \geq I - T_G T_G^*$. Hence $FH^2 \supseteq GH^2$. Therefore $G = QF$ for some inner Q. If $S_{\alpha,\beta}$ is hyponormal, then by Theorem 3 $[T_F^*, T_F] \geq [T_G^*, T_G]$ and $[T_g^*, T_g] \geq [T_f^*, T_f]$. Moreover if F, G, f and g are inner, then $G = QF$ and $f = qg$ for some inner Q and q.

When $T_{F+\bar{G}}$ is hyponormal, if F and G are polynomials we have a lot of papers [8], [9], [10], [11] and [14].

3. Analytic symbol

In this section we give sufficient conditions for $S_{\alpha,\beta}$ to be a hyponormal operator.

Theorem 4. (1) *$S_{\alpha,\beta}$ is nearly normal if and only if $\tilde{H}_\alpha H_{\bar{\alpha}} - \tilde{H}_\beta H_{\bar{\beta}} \geq 0, H_\beta \tilde{H}_{\bar{\beta}} - H_\alpha \tilde{H}_{\bar{\alpha}} \geq 0$ and $\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}} = 0$.*

(2) *If $S_{\alpha,\beta}$ is hyponormal, and $\tilde{H}_\alpha H_{\bar{\alpha}} = \tilde{H}_\beta H_{\bar{\beta}}$ or $H_\beta \tilde{H}_{\bar{\beta}} = H_\alpha \tilde{H}_{\bar{\alpha}}$, then $S_{\alpha,\beta}$ is nearly normal.*

Proof. (1) It is clear by Theorem 1 and the definition of a nearly normal operator.

(2) It is clear by Theorem 1 and (1) and the definition of a nearly normal operator. □

Corollary 1. *If both α and β are in H^∞, then the following* (1), (2) *and* (3) *are equivalent.*

(1) *$S_{\alpha,\beta}$ is hyponormal.*

(2) *$S_{\alpha,\beta}$ is nearly normal.*

(3) *$T_{\alpha+\bar{\beta}}$ is hyponormal and $\tilde{T}_\phi H_{\bar{\beta}} = 0$.*

Proof. (1)⇒(2). Since $H_\beta \tilde{H}_{\bar{\beta}} = H_\alpha \tilde{H}_{\bar{\alpha}} = 0$, by (2) of Theorem 4 $S_{\alpha,\beta}$ is nearly normal.

(2)⇒(3). By (1) of Theorem 4 and by (1) of Lemma 1 there exists k in H^∞ such that $\|k\|_\infty \leq 1$ and $k\bar{\alpha} - \bar{\beta} \in H^\infty$. Hence $k(\bar{\alpha} + \beta) - (\alpha + \bar{\beta})$ belongs to H^∞. By [11, Lemma 1], $T_{\alpha+\bar{\beta}}$ is hyponormal and $\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}} = 0$ by (1) of Theorem 4 and $\alpha \in H^\infty$ by hypothesis.

(3)⇒(1). By [11, Lemma 1] there exists k in H^∞ such that $\|k\|_\infty \leq 1$ and $k(\bar{\alpha} + \beta) - (\alpha + \bar{\beta}) \in H^\infty$. Hence $k\bar{\alpha} - \bar{\beta}$ be longs to H^∞. By (1) and (2) of Lemma 1, and (1) of Theorem 4 $S_{\alpha,\beta}$ is nearly normal and so hyponormal. □

Corollary 2. *If both $\bar{\alpha}$ and $\bar{\beta}$ are in H^∞, then a result similar to Corollary 1 holds.*

Lemma 2. (1) *Put $\alpha = q_\alpha t_\alpha, \beta = q_\beta t_\beta$ and $\phi = \alpha - \beta = qt$ where q_α, q_β and q are inner, and t_α, t_β and t are outer. Suppose $Ker\tilde{H}_\beta = \bar{Q}_\beta \bar{z} \bar{H}^2$ and Q_β is inner. Then $\tilde{H}_\beta \tilde{T}_{\bar{\phi}} = 0$ if and only if $q = Q_\beta q_0$ where q_0 is inner.*

(2) *Put* $\bar{\alpha} = q_\alpha t_\alpha, \bar{\beta} = q_\beta t_\beta$, *and* $\bar{\phi} = \bar{\alpha} - \bar{\beta} = qt$ *where* q_α, q_β *and* q *are inner, and* t_α, t_β *and* t *are outer. Suppose* $KerH_\alpha = Q_\alpha H^2$. *Then* $H_\alpha T_{\bar{\phi}} = 0$ *if and only if* $q = Q_\alpha q_0$ *where* q_0 *is inner.*

Proof. (1) $\tilde{H}_\beta \tilde{T}_{\bar{\phi}} = 0$ if and only if $\tilde{T}_{\bar{\phi}}(\bar{z}\bar{H}^2) = \bar{q}\bar{z}\bar{t}\bar{H}^2 \subseteq \bar{Q}_\beta \bar{z}\bar{H}^2$ by the definition of Q_β. Hence this is equivalent to $q_0 = \bar{Q}_\beta q$ is inner.

(2) It can be proved as (1). □

Corollary 3. *Suppose* q_α, q_β *and* q *are inner, and* t_α, t_β *and* t *are outer.*

(1) *Put* $\alpha = q_\alpha t_\alpha, \beta = q_\beta t_\beta$ *and* $\phi = \alpha - \beta = qt$. *Suppose* $Ker\tilde{H}_\beta = \bar{Q}_\beta \bar{z}\bar{H}^2$ *and* Q_β *are inner. Then* $S_{\alpha,\beta}$ *is nearly normal if and only if there exists a contraction* k *in* H^∞ *such that* $k\bar{\alpha} - \bar{\beta} \in H^\infty$ *and* $q = Q_\beta q_0$ *where* q_0 *is inner.*

(2) *Put* $\bar{\alpha} = q_\alpha t_\alpha, \bar{\beta} = q_\beta t_\beta$ *and* $\bar{\phi} = \bar{\alpha} - \bar{\beta} = qt$. *Suppose* $KerH_\alpha = Q_\alpha H^2$ *and* Q_α *are inner. Then* $S_{\alpha,\beta}$ *is nearly normal if and only if there exists a contraction* h *in* H^∞ *such that* $h\beta - \alpha \in H^\infty$ *and* $q = Q_\alpha q_0$ *where* q_0 *is inner.*

Proof. (1) Since $H_\alpha T_{\bar{\phi}} = 0$, if $S_{\alpha,\beta}$ is nearly normal, then by (1) of Theorem 4 $\tilde{H}_\beta \tilde{T}_{\bar{\phi}} = 0$. Now (1) of Lemma 1 and (1) of Lemma 2 show (1). The converse is clear by (1) of Lemma 1 and (1) of Lemma 2 and (1) of Theorem 4.

(2) Since $\tilde{T}_\phi H_{\bar{\beta}} = 0$, if $S_{\alpha,\beta}$ is nearly normal, then by (1) of Theorem 4 $H_\alpha T_{\bar{\phi}} = 0$. Now (2) of Lemma 1 and (2) of Lemma 2 show (2). The converse is clear by (2) of Lemma 1 and (2) of Lemma 2 and (1) of Theorem 4. □

In (1) of Corollary 3, Ker$\tilde{H}_\beta = \{0\}$ if and only if $\beta = t_\beta$ is a cyclic vector of $T_{\bar{z}}$ in H^2. Similarly, in (2) of Corollary 3, Ker$H_\alpha = \{0\}$ if and only if $\alpha = \bar{t}_\alpha$ is a cyclic vector of $\tilde{T}_z$ in $\bar{z}\bar{H}^2$.

Corollary 4. (1) *Let* α *and* β *be in* H^∞. *Suppose* β *is a cyclic vector of* $T_{\bar{z}}$ *in* H^2. *Then* $S_{\alpha,\beta}$ *is nearly normal if and only if there exists a contraction* k *in* H^∞ *such that* $k\bar{\alpha} - \bar{\beta}$ *belongs to* H^∞.

(2) *Let* $\bar{\alpha}$ *and* $\bar{\beta}$ *be in* H^∞. *Suppose* α *is a cyclic vector of* $\tilde{T}_z$ *in* $\bar{z}\bar{H}^2$. *Then* $S_{\alpha,\beta}$ *is nearly normal if and only if there exists a contraction* h *in* H^∞ *such that* $h\beta - \alpha$ *belongs to* H^∞.

Example I. (1) If $\alpha = Q(cq_\beta + \bar{m})$ and $\beta = cq_\beta$ where Q and q_β are inner and $m \in H^2$, $q_\beta m \in H^2 \ominus QzH^2$ and $c \in \mathbb{C}$, then $S_{\alpha,\beta}$ is nearly normal.

(2) If $\alpha = \Sigma_{j=1}^n a_j z^j$ and $\beta = a_n z$, then $S_{\alpha,\beta}$ is nearly normal.

(3) Suppose $\alpha = a_0 + a_1 z$, $\beta = b_0 + b_1 z$ and $\alpha \neq \beta$. Then $S_{\alpha,\beta}$ is nearly normal if and only if $\alpha = a_0$ and $\beta = b_0$.

Proof. (1) Since $m \in H^2 \ominus QzH^2$, α belongs to H^∞. Moreover $Q\bar{\alpha} - \bar{\beta} = m$. On the other hand, since $q_\beta m \in H^2 \ominus QzH^2$,

$$\alpha - \beta = c(Q-1)q_\beta + Q\bar{m} = c(Q-1)q_\beta + q_\beta s,$$

where $s = Q\bar{q}_\beta \bar{m} \in H^\infty$. In (1) of Corollary 3, if q is the inner part of $q_\beta\{c(Q-1) + s\}$ and $Q_\beta = q_\beta$, then $S_{\alpha,\beta}$ is nearly normal.

(2) In (1), put $Q = z^{n-1}$, $q_\beta = z$ and $a_n = c$. Then $\alpha = Q(cz + \bar{m})$ and $m = \bar{a}_1 z^{n-2} + \cdots + \bar{a}_{n-2} z + \bar{a}_{n-1}$.

(3) By (1) of Corollary 3, it is clear. □

We can give a similar example to Example I using (2) of Corollary 3.

4. Polynomial symbol

In this section, we would like to study the hyponormality of $S_{\alpha,\beta}$ when $\alpha = \sum_{j=-n}^{n} \alpha_j z^j$ and $\beta = \sum_{j=-n}^{n} \beta_j z^j$. In this special case, it is still difficult to study the hyponormality. We study it essentially when $n = 1$ and $n = 2$.

Lemma 3. *Let $\alpha = a + F_n z^n + \bar{f}_n \bar{z}^n$, $\beta = b + G_n z^n + \bar{g}_n \bar{z}^n$ and $\phi = \alpha - \beta = c + A_n z^n + \bar{B}_n \bar{z}^n$ for $n \geq 1$. If $u = \Sigma_{j=0}^{\ell} u_j z^j$, $\ell \geq n$ and $v = \Sigma_{j=1}^{m} v_j \bar{z}^j$, $m \geq n+1$, then the followings hold.*

(1) $\langle \tilde{T}_\phi H_{\bar{\beta}} u, v\rangle = \bar{G}_n (c \Sigma_{j=0}^{n-1} u_j \bar{v}_{n-j} + \bar{B}_n \Sigma_{j=0}^{n-1} u_j \bar{v}_{2n-j})$.

(2) $\langle H_\alpha T_{\bar{\phi}} u, v\rangle = \bar{f}_n (\bar{c} \Sigma_{j=0}^{n-1} u_j \bar{v}_{n-j} + \bar{A}_n \Sigma_{j=n}^{2n-1} u_j \bar{v}_{2n-j})$.

(3) $\langle (\tilde{T}_\phi H_{\bar{\beta}} - H_\alpha T_{\bar{\phi}}) u, v\rangle = (\bar{G}_n c - \bar{f}_n \bar{c}) \Sigma_{j=0}^{n-1} u_j \bar{v}_{n-j} + \bar{G}_n \bar{B}_n \Sigma_{j=0}^{n-1} u_j \bar{v}_{2n-j} - \bar{f}_n \bar{A}_n \Sigma_{j=n}^{2n-1} u_j \bar{v}_{2n-j}$.

(4) $\|H_{\bar{\alpha}} u\|^2 = |F_n|^2 \Sigma_{j=0}^{n-1} |u_j|^2$.

(5) $\|\tilde{H}_{\bar{\beta}} v\|^2 = |g_n|^2 \Sigma_{j=1}^{n} |v_j|^2$.

Proof. It is easy to see this lemma by a calculation. □

Theorem 5. *Let $\alpha = a + F_n z^n + \bar{f}_n \bar{z}^n$, $\beta = b + G_n z^n + \bar{g}_n \bar{z}^n$ and $\alpha - \beta = c + A_n z^n + \bar{B}_n \bar{z}^n$. Then $S_{\alpha,\beta}$ is hyponormal if and only if for any complex sequences $\{u_j\}_{j=0}^{n}$ and $\{v_j\}_{j=1}^{n+1}$*

$$|(\bar{G}_n c - \bar{f}_n \bar{c})|^2 \leq (|F_n|^2 - |G_n|^2)(|g_n|^2 - |f_n|^2),$$

where $|F_n| \geq |G_n|$, $|g_n| \geq |f_n|$ and $G_n(\overline{f_n - g_n}) = f_n(F_n - G_n) = 0$.

Proof. By Theorem 1 and Lemma 3, $S_{\alpha,\beta}$ is hyponormal if and only if

$$|(\bar{G}_n c - \bar{f}_n \bar{c}) \sum_{j=0}^{n-1} u_j \bar{v}_{n-j} + \bar{G}_n \bar{B}_n \sum_{j=0}^{n-1} u_j \bar{v}_{2n-j} - \bar{f}_n \bar{A}_n \sum_{j=n}^{2n-1} u_j \bar{v}_{2n-j}|^2$$

$$\leq (|F_n|^2 - |G_n|^2)(|g_n|^2 - |f_n|^2) \left(\sum_{j=0}^{n-1} |u_j|^2 \right) \left(\sum_{j=1}^{n} |v_j|^2 \right),$$

where $|F_n| \geq |G_n|$ and $|g_n| \geq |f_n|$. If $\prod_{j=n+1}^{2n} v_j \neq 0$, then choosing $\{u_j\}_{j=0}^{n-1}$ as like $|\Sigma_{j=0}^{n-1} u_j \bar{v}_{2n-j}| \to \infty$, we can get $\bar{G}_n \bar{B}_n = 0$. If $\prod_{j=n}^{2n-1} u_j \neq 0$, then choosing $\{v_j\}_{j=1}^{n}$ as like $|\Sigma_{j=n}^{2n-1} u_j \bar{v}_{2n-j}| \to \infty$, we can get $f_n A_n = 0$. Hence $S_{\alpha,\beta}$ is hyponormal if and only if

$$|(\bar{G}_n c - \bar{f}_n \bar{c}) \sum_{j=0}^{n-1} u_j \bar{v}_{n-j}|^2$$

$$\leq (|F_n|^2 - |G_n|^2)(|g_n|^2 - |f_n|^2) \left(\sum_{j=0}^{n-1} |u_j|^2 \right) \left(\sum_{j=1}^{n} |v_j|^2 \right),$$

where $G_n B_n = f_n A_n$, $|F_n| \geq |G_n|$ and $|g_n| \geq |f_n|$. This shows this theorem because

$$|\sum_{j=0}^{n-1} u_j \bar{v}_{n-j}|^2 \leq \left(\sum_{j=0}^{n-1} |u_j|^2 \right) \left(\sum_{j=0}^{n-1} |\bar{v}_{n-j}|^2 \right).$$

□

Corollary 5. *Let $\alpha = a_{-1}\bar{z} + a_0 + a_1 z$ and $\beta = b_{-1}\bar{z} + b_0 + b_1 z$. Then $S_{\alpha,\beta}$ is hyponormal if and only if*

$$|\bar{b}_1(a_0 - b_0) - a_{-1}(\overline{a_0 - b_0})|^2 \leq (|a_1|^2 - |b_1|^2)(|b_{-1}|^2 - |a_{-1}|^2),$$

where $|a_1| \geq |b_1|$, $|b_{-1}| \geq |a_{-1}|$ and $b_1(a_{-1} - b_{-1}) = a_{-1}(a_1 - b_1) = 0$.

Corollary 6. *Let $\alpha = a + F_n z^n + \bar{f}_n \bar{z}^n$, $\beta = b + G_n z^n + \bar{g}_n \bar{z}^n$ and $\alpha - \beta = c + A_n z^n + \bar{B}_n \bar{z}^n$. Then $S_{\alpha,\beta}$ is hyponormal if and only if $S_{\alpha,\beta}$ is nearly normal.*

Proof. By Theorem 5, if $S_{\alpha,\beta}$ is hyponormal, then $G_n(\overline{f_n - g_n}) = f_n(F_n - G_n) = 0$. Hence if $F_n = G_n$, then $S_{\alpha,\beta}$ is nearly normal. If $f_n = 0$, then $G_n \bar{g}_n = 0$ and $|\bar{G}_n c|^2 \leq (|F_n|^2 - |G_n|^2)|g_n|^2$. This shows $S_{\alpha,\beta}$ is nearly normal. □

The following theorem is a generalization of Corollary 5. It is not beautiful but will be useful and important.

Theorem 6. *Let $\alpha = a + F_1 z + F_2 z^2 + \bar{f}_1 \bar{z} + \bar{f}_2 \bar{z}^2$, $\beta = b + G_1 z + G_2 z^2 + \bar{g}_1 \bar{z} + \bar{g}_2 \bar{z}^2$ and $\phi = \alpha - \beta = c + A_1 z + A_2 z^2 + \bar{B}_1 \bar{z} + \bar{B}_2 \bar{z}^2$. Then $S_{\alpha,\beta}$ is hyponormal if and only if*

$$\begin{aligned} &|\{(c\bar{G}_1 u_0 + c\bar{G}_2 u_1 + A_1 \bar{G}_2 u_0) - \bar{f}_1(\bar{c} u_0 + \bar{A}_1 u_1) - \bar{f}_2(\bar{c} u_1)\}\bar{v}_1 \\ &\quad + (c\bar{G}_2 u_0 + \bar{B}_1 \bar{G}_1 u_0 + \bar{B}_1 \bar{G}_2 u_1)\bar{v}_2|^2 \\ &\leq \{(|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 - |\bar{G}_1 u_0 + \bar{G}_2 u_1|^2) + (|\bar{F}_2|^2 - |\bar{G}_2|^2)|u_0|^2\} \\ &\quad \times \{(|g_1 v_1 + g_2 v_2|^2 - |f_1 v_1 + f_2 v_2|^2) + (|g_2|^2 - |f_2|^2)|v_1|^2\} \end{aligned}$$

and $|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 - |\bar{G}_1 u_0 + \bar{G}_2 u_1|^2 + (|\bar{F}_2|^2 - |\bar{G}_2|^2)|u_0|^2 \geq 0$, $|g_1 v_1 + g_2 v_2|^2 - |f_1 v_1 + f_2 v_2|^2 + (|g_2|^2 - |f_2|^2)|v_1|^2 \geq 0$ for $u_j \in \mathbb{C}$ $(j = 0, 1)$ and $v_j \in \mathbb{C}$ $(j = 1, 2)$, where $B_2 G_2 = 0$, $B_1 G_2 + \bar{c} B_2 G_1 = 0$, $A_2 f_2 = 0$, $A_2 f_1 = 0$, and $A_1 f_2 = 0$.

Proof. Note that

$$\|H_{\bar{\alpha}} u\|^2 - \|H_{\bar{\beta}} u\|^2 = (|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 - |\bar{G}_1 u_0 + \bar{G}_2 u_1|^2) + (|\bar{F}_2|^2 - |\bar{G}_2|^2)|u_0|^2$$

and

$$\|\tilde{H}_{\bar{\beta}} v\|^2 - \|\tilde{H}_{\bar{\alpha}} v\|^2 = (|g_1 v_1 + g_2 v_2|^2 - |f_1 v_1 + f_2 v_2|^2) + (|g_2|^2 - |f_2|^2)|v_1|^2.$$

Moreover

$$\begin{aligned} \langle \tilde{T}_\phi H_{\bar{\beta}} u, v \rangle &= (c\bar{G}_1 u_0 + c\bar{G}_2 u_1 + A_1 \bar{G}_2 u_0)\bar{v}_1 + (c\bar{G}_2 u_0 + \bar{B}_1 \bar{G}_1 u_0 + \bar{B}_1 \bar{G}_2 u_1)\bar{v}_2 \\ &\quad + (\bar{B}_1 \bar{G}_2 u_0 + c\bar{B}_2 \bar{G}_1 u_0 + c\bar{B}_2 \bar{G}_2 u_1)\bar{v}_3 + (\bar{B}_2 \bar{G}_2 u_0)\bar{v}_4 \end{aligned}$$

and

$$\langle H_\alpha T_{\bar\phi} u, v\rangle = \{\bar f_1(\bar c u_0 + \bar A_1 u_1 + \bar A_2 u_2) + \bar f_2(\bar c u_1 + \bar A_1 u_2 + \bar A_2 u_3)\}\bar v_1 + \bar f_2(\bar c u_0 + \bar A_1 u_1 + \bar A_2 u_2)\bar v_2.$$

Theorem 1 and the proof of Theorem 5 show the theorem. For example, as $|v_4| \to \infty$ we $B_2G_2 = 0$ and as $|v_3| \to \infty$ we get $B_1G_2 + \bar c B_2 G_1 = 0$. Similarly as $|u_3| \to \infty$ we get $f_2A_2 = 0$ and as $|u_2| \to \infty$ we get $f_1A_2 = f_2A_1 = 0$. □

Corollary 7. *When $c = 0, S_{\alpha,\beta}$ is hyponormal if and only if*

$$\begin{aligned}&|A_1(G_2u_0 - \bar f_1 u_1)\bar v_1 + \bar B_1(\bar G_1 u_0 + \bar G_2)v_2|^2 \\ &\le \{|F_1u_0 + \bar F_2 u_1|^2 - |\bar G_1 u_0 + \bar G_2 u_1|^2 + (|\bar F_2|^2 - |\bar G_2|^2)|u_0|^2\} \\ &\quad \times \{(|g_1v_1 + g_2v_2|^2 - |f_1v_1 + f_2v_2|^2 + (|g_2|^2|f_2|^2|v_1|^2\}\end{aligned}$$

and $|\bar F_1u_0 + \bar F_2u_1|^2 - |\bar G_1u_0 + \bar G_2u_1|^2 + (|\bar F_2|^2 - |\bar G_2|^2)|u_0|^2 \ge 0$, $|g_1v_1 + g_2v_2|^2 - |f_1v_1 + f_2v_2|^2 + (|g_2|^2 - |f_2|^2)|v_1|^2 \ge 0$ where $B_2G_2 = B_1G_2 = A_2f_2 = A_2f_1 = A_1f_2 = 0$ and $u_i, v_j \in \mathbb{C}$ $(i = 0, 1;\ j = 1, 2)$.

Corollaries 8 and 9 give examples that are hyponormal but not nearly normal and Corollary 10 give examples that are nearly normal.

Corollary 8. *When $A_1 = 0,\ A_2B_1B_2 \ne 0, S_{\alpha,\beta}$ is hyponormal if and only if*

$$|c\bar G_1 u_0 \bar v_1 + B_1 G_1 u_0 \bar v_2|^2 \le (|\bar F_1 u_0 + \bar F_2 u_1|^2 + |F_2|^2|u_0|^2)(|g_1v_1 + g_2v_2|^2 + |g_2|^2|v_1|^2),$$

where $G_2 = f_1 = f_2 = 0$ and $u_i, v_j \in \mathbb{C}$ $(i = 0, 1;\ j = 1, 2)$.

Proof. If $S_{\alpha,\beta}$ is hyponormal, then Theorem 6 shows and so $G_2 = f_1 = f_2 = 0$ the 'only if' part. Conversely the 'if' part is clear. □

Example II. Let $\alpha = a + F_1z + F_2z^2$ and $\beta = b + G_1z + \bar g_1 \bar z + \bar g_2 \bar z^2$.

(1) If $S_{\alpha,\beta}$ is hyponormal, then $|(a-b)\bar G_1 + g_1G_1|^2 \le (|F_1+F_2|^2 + |F_2|^2)(|g_1 + g_2|^2 + |g_2|^2)$. If $S_{\alpha,\beta}$ is nearly normal, then $(a-b)\bar G_1 + g_1G_1 = 0$.

(2) If $\alpha = 1 + z + z^2$ and $\beta = z + \bar z + \bar z^2$, then $S_{\alpha,\beta}$ is hyponormal but not nearly normal.

Proof. (1) In Corollary 8, as $u_0 = u_1 = v_1 = v_2 = 1$, we can get (1).

(2) In Corollary 8, $c = B_1 = 1$ and $F_1 = F_2 = G_1 = g_1 = g_2 = 1$. Hence $S_{\alpha,\beta}$ is hyponormal if and only if $|u_0\bar v_1 + u_0\bar v_2|^2 \le (|u_0 + u_1|^2 + |u_0|^2)(|v_1 + v_2|^2 + |v_1|^2)$. Hence $S_{\alpha,\beta}$ is hyponormal and by (1) $S_{\alpha,\beta}$ is not nearly normal because $(a - b)\bar G_1 + g_1G_1 = 2$. □

Corollary 9. *Suppose $A_1A_2B_1B_2 \ne 0$. Then $S_{\alpha,\beta}$ is hyponormal if and only if*

$$|B_1G_1|^2|v_2|^2 \le (|F_2|^2 - |G_1|^2)(|g_1v_1 + g_2v_2|^2 + |g_2|^2|v_1|^2),$$

where $G_2 = cG_1 = f_2 = f_1 = 0$ and v_j $(j = 1, 2)$.

Proof. If $S_{\alpha,\beta}$ is hyponormal, by Theorem 6, $G_2 = cG_1 = f_2 = f_1 = 0$. Hence by Theorem 6,

$$|B_1G_1|^2|u_0|^2|v_2|^2 \le \{|\bar{F}_1u_0 + \bar{F}_2u_1|^2 + (|F_2|^2 - |G_1|^2)|u_0|^2\} \times (|g_1v_1 + g_2v_2|^2 + |g_2|^2|v_1|^2).$$

Since $F_2 \ne 0$, if we choose u_1 as $-\bar{F}_1u_0/\bar{F}_2$, then the 'only if' part of corollary follows. The 'if' part is clear. □

Example III. Let $\alpha = a + F_1z + F_2z^2$ and $\beta = a + G_1z + \bar{g}_1\bar{z} + \bar{g}_2\bar{z}^2$ where $F_2 \ne 0$, $g_1g_2 \ne 0$, $|F_2| \ne |G_1|$ and $F_1 \ne G_1$. Then

(1) $S_{\alpha,\beta}$ is hyponormal if and only if

$$\frac{|G_1|^2}{|F_2|^2 - |G_1|^2} \le \left|t + \frac{g_2}{g_1}\right|^2 + |t|^2\left|\frac{g_2}{g_1}\right|^2$$

for any $t \in \mathbb{C}$. Moreover $S_{\alpha,\beta}$ is not nearly normal.

(2) When $g_1 = g_2, S_{\alpha,\beta}$ is hyponormal if and only if $2|G_1|^2 \le |F_2|^2$.

Proof. (1) By Theorem 6, $S_{\alpha,\beta}$ is hyponormal if and only if

$$|g_1G_1|^2 \le (|F_2|^2 - |G_1|^2)\left(\left|g_1\frac{v_1}{v_2} + g_2\right|^2 + |g_2|^2\left|\frac{v_1}{v_2}\right|^2\right).$$

Put $t = v_1/v_2$. Then

$$\frac{|G_1|^2}{|F_2|^2 - |G_1|^2} \le \left|t + \frac{g_2}{g_1}\right|^2 + |t|^2\left|\frac{g_2}{g_1}\right|^2$$

for any $t \in \mathbb{C}$ and $S_{\alpha,\beta}$ is not nearly normal.

(2) When $g_1 = g_2$, by (1) $S_{\alpha,\beta}$ is hyponormal if and only if

$$\frac{|G_1|^2}{|F_2|^2 - |G_1|^2} \le |t+1|^2 + |t|^2$$

for any $t \in \mathbb{C}$. Since $\inf(|t+1|^2 + |t|^2) = 1/4, S_{\alpha,\beta}$ is hyponormal if and only if $4\ |G_1|^2 \le |F_2|^2 - |G_1|^2$. □

Lemma 4. *Let $\alpha = a+F_1z+F_2z^2+\bar{f}_1\bar{z}+\bar{f}_2\bar{z}^2$, $\beta = b+G_1z+G_2z^2+\bar{g}_1\bar{z}+\bar{g}_2\bar{z}^2$ and $\phi = \alpha - \beta = c + A_1z + A_2z^2 + \bar{B}_1\bar{z} + \bar{B}_2\bar{z}^2$. Suppose $S_{\alpha,\beta}$ is hyponormal. Then the following hold.*

(1) *If $A_1 = A_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*
(2) *If $B_1 = B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*
(3) *If $A_1 = B_1 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*
(4) *If $A_2 = B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*
(5) *If $A_1 = B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*
(6) *If $A_2 = B_1 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*

Proof. (1) Since $A_1 = A_2 = 0$, $|\bar{F}_1u_0 + \bar{F}_2u_1|^2 - |\bar{G}_1u_0 + \bar{G}_2u_1|^2 + (|F_2|^2 - |G_2|^2)|u_0|^2 = 0$. Hence by Theorem 6, $S_{\alpha,\beta}$ is nearly normal.

(2) Since $B_1 = B_2 = 0$, $|g_1v_1 + g_2v_2|^2 - |f_1v_1 + f_2v_2|^2 + (|g_2|^2 - |f_2|^2)|v_1|^2 = 0$. Hence by Theorem 6, $S_{\alpha,\beta}$ is nearly normal.

(3) When $A_1 = B_1 = 0$, by (1) and (2) we may assume $A_2B_2 \neq 0$. By Theorem 6, $G_2 = \bar{c}G_1 = f_2 = f_1 = 0$. By Theorem 6

$$\begin{aligned}&\{(c\bar{G}_1u_0 + c\bar{G}_2u_1 + A_1\bar{G}_2u_0) - \bar{f}_1(\bar{c}u_0 + \bar{A}_1u_1) - \bar{f}_2(\bar{c}u_1)\}v_1\\&+ (cG_2u_0 + \bar{B}_1\bar{G}_1u_0 + \bar{B}_1\bar{G}_2u_1)\bar{v}_2 = 0\end{aligned}$$

and hence $S_{\alpha,\beta}$ is nearly normal.

(4) When $A_2 = B_2 = 0$ by (1) and (2) we may assume $A_1B_1 \neq 0$. By Theorem 6 $B_1G_2 = A_1f_2 = 0$ and so $G_2 = f_2 = 0$. By Theorem 6

$$|\{(c\bar{G}_1 - \bar{f}_1\bar{c})u_0 - \bar{f}_1\bar{A}_1u_1\}\bar{v}_1 + \bar{B}_1\bar{G}_1u_0\bar{v}_2|^2 \leq (|F_1|^2 - |G_1|^2)(|g_1|^2 - |f_1|^2)|u_0|^2|v_1|^2.$$

As $|v_2| \to \infty$, $B_1G_1 = 0$ and so $G_1 = 0$. Hence $|(\bar{f}_1\bar{c}u_0 + \bar{f}_1\bar{A}_1u_1)\bar{v}_1|^2 \leq |F_1|^2(|g_1|^2 - |f_1|^2)|u_0|^2|v_1|^2$. As $|u_1| \to \infty$, $f_1A_1 = 0$ and so $f_1 = 0$. Therefore by the definition $S_{\alpha,\beta}$ is nearly normal.

(5) When $A_1 = B_2 = 0$, by (1) and (2) we may assume $A_2B_1 \neq 0$. By Theorem 6 $G_2 = f_2 = f_1 = 0$. By Theorem 6

$$|c\bar{G}_1u_0\bar{v}_1 + \bar{B}_1\bar{G}_1u_0\bar{v}_2|^2 \leq (|\bar{F}_1u_0 + \bar{F}_2u_1|^2 - |\bar{G}_1u_0|^2)|g_1v_1|^2.$$

As $|v_2| \to \infty$, $B_1G_1 = 0$ and so $G_1 = 0$. Hence by the definition $S_{\alpha,\beta}$ is nearly normal.

(6) When $A_2 = B_1 = 0$ by (1), (2) and (4) we may assume $A_1B_2 \neq 0$. By Theorem 6 $G_2 = \bar{c}G_1 = f_2 = 0$.

By Theorem 6,

$$|\bar{f}_1(\bar{c}u_0 + \bar{A}_1u_1)\bar{v}_1|^2 \leq |F_1|^2|g_2|^2|u_0|^2|v_1|^2.$$

As $|u_1| \to \infty$, $f_1A_1 = 0$ and so $f_1 = 0$. By the definition $S_{\alpha,\beta}$ is nearly normal. □

Corollary 10. *Let $\alpha = a + F_1z + F_2z^2 + \bar{f}_1\bar{z} + \bar{f}_2\bar{z}^2$, $\beta = b + G_1z + G_2z^2 + \bar{g}_1\bar{z} + g_2\bar{z}^2$ and $\phi = \alpha - \beta = c + A_1z + A_2z^2 + \bar{B}_1\bar{z} + \bar{B}_2\bar{z}^2$. Suppose $S_{\alpha,\beta}$ is hyponormal. When $c \neq 0$, if $A_1B_1A_2B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal. When $c = 0$, if $B_1A_2B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal.*

Proof. We will show that if $B_1A_2B_2 = 0$, then $S_{\alpha,\beta}$ is nearly normal. When $B_1 = 0$, by (2), (3) and (6) of Lemma 4 we may assume $A_1A_2B_2 \neq 0$. By Theorem 6, $G_2 = \bar{c}G_1 = f_1 = f_2 = 0$. This shows that $S_{\alpha,\beta}$ is nearly normal by Theorem 6.

When $A_2 = 0$, by (1), (4) and (6) of Lemma 4 we may assume $A_1B_1B_2 \neq 0$. By Theorem 6, $G_2 = \bar{c}G_1 = f_2 = 0$ and so Theorem 6 shows that

$$|f_1(\bar{c}u_0 + A_1u_1)\bar{v}_1 + B_1G_1u_0\bar{v}_2|^2 \leq (|F_1|^2 - |G_1|^2)|u_0|^2(|g_1v_1 + g_2v_2|^2 - |f_1v_1|^2).$$

As $|u_1| \to \infty, f_1 A_1 = 0$ and so $f_1 = 0$. Hence

$$|B_1 G_1|^2 |v_2|^2 \leq (|F_1|^2 - |G_1|^2)|g_1 v_1 + g_2 v_2|^2.$$

Choosing v_1 and v_2, we can assume $g_1 v_1 + g_2 v_2 = 0$. This shows $S_{\alpha,\beta}$ is nearly normal.

When $B_2 = 0$, by (2), (4) and (5) of Lemma 4 we may assume $A_1 A_2 B_1 \neq 0$. By Theorem 6 $G_2 = f_1 = f_2 = 0$ and so Theorem 6 shows that

$$\begin{aligned} &|c\bar{G}_1 u_0 \bar{v}_1 + \bar{B}_1 \bar{G}_1 u_0 \bar{v}_2|^2 \\ &\leq \{|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 + (|F_2|^2 - |G_1|^2)|u_0|^2\}(|g_1 v_1 + g_2 v_2|^2 + |g_2||v_1|^2). \end{aligned}$$

Since $B_2 = 0, g_2 = 0$ and so

$$|c\bar{G}_1 u_0 \bar{v}_1 + \bar{B}_1 \bar{G}_1 u_0 \bar{v}_2|^2 \leq \{|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 + (|F_2|^2 - |G_1|^2)|u_0|^2\}|g_1|^2|v_1|^2.$$

As $|v_2| \to \infty$, $B_1 G_1 = 0$ and so $G_1 = 0$ because $B_1 \neq 0$. This shows $S_{\alpha,\beta}$ is nearly normal.

We will show that if $A_1 = 0$ and $c \neq 0$, then $S_{\alpha,\beta}$ is nearly normal. By (1), (3) and (5) of Lemma 4 we may assume $A_2 B_1 B_2 \neq 0$. By Theorem 6 $G_2 = \bar{c} G_1 = f_1 = f_2 = 0$ and so Theorem 6 shows that

$$|B_1 G_1 u_0 v_2|^2 \leq (|\bar{F}_1 u_0 + \bar{F}_2 u_1|^2 - |\bar{G}_1|^2 |u_0|^2 + |F_2|^2 |u_0|^2)|g_2|^2 (v_1|^2 + |v_2|^2).$$

Since $c \neq 0, G_1 = 0$ and so by Theorem 6 $S_{\alpha,\beta}$ is nearly normal. □

Remark. In this section, we consider only very special case, that is, α and β are polynomials. However we can prove a few results only when $\alpha - \beta$ is a polynomial.

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