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WEAK AND STRONG CONVERGENCE THEOREMS FOR THE MODIFIED ISHIKAWA ITERATION FOR TWO HYBRID MULTIVALUED MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce new iterative schemes by using the modified Ishikawa iteration for two hybrid multivalued mappings in a Hilbert space. We then obtain weak convergence theorem under suitable conditions. We use CQ and shrinking projection methods with Ishikawa iteration for obtaining strong convergence theorems. Furthermore, we give examples and numerical results for supporting our main results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed and convex subset of H. A subset $C \subset H$ is said to be *proximinal* if for each $x \in H$, there exists $y \in C$ such that

$$\|x - y\| = d(x, C) = \inf\{\|x - z\| : z \in C\}.$$

Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A, B) = \max\left\{ \sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A) \right\}$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b\in B} \|x - b\|$. A singlevalued mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A multivalued mapping $T : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|$$

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for all $x, y \in C$. An element $p \in C$ is called a *fixed point* of a mapping $T : C \to C$ (resp., a multivalued mapping $T : C \to CB(C)$) if $p = Tp$ (resp., $p \in Tp$). The fixed point set of T is denoted by $F(T)$. If $F(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\|$$

for all $x \in C$ and $p \in F(T)$, then T is said to be *quasi-nonexpansive*.

Since 1965, fixed point theorems and the existence of fixed points of singlevalued nonexpansive mappings have been intensively studied and considered by many authors (see, for example, [1, 3, 6–8, 11, 18–20, 22, 25]).

In 1953, Mann [14] introduced the following iterative procedure for approximating a fixed point of a nonexpansive mapping T in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where the initial point x_1 is taken in C arbitrarily and $\{\alpha_n\}$ is a sequence in [0,1]. We know that Mann's iteration has the only weak convergence theorem (see, for example, [2, 21]).

In 1974, Ishikawa [10] introduced the following iterative scheme which is a generalization of the Mann's iterative algorithm (1.1):

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tz_n, \quad n \geq 0, \\ z_n = \beta_n x_n + (1 - \beta_n)Tx_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate control sequences in [0, 1]. However, Ishikawa iteration processes also has only weak convergence even in a Hilbert space.

For obtaining strong convergence theorem, Nakajo and Takahashi [17] proposed the following modification of the Mann's iteration method (1.1) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above by 1, then the sequence $\{x_n\}$ converges strongly to $P_{Fix(T)}x_0$.

Recently, Takahashi et al. [27] introduced the following modification of the Mann's iteration method (1.1) which just involved one closed convex set for a family of nonexpansive mappings $\{T_n\}$:

$$\begin{cases} u_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C, u_1 = P_{c_1} x_0, \\ y_n = \alpha_n u_n + (1 - \alpha_n)T_n u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}} x_0. \end{cases}$$

They proved that if $\alpha_n \leq a$ for all $n \geq 1$ and for some $0 < a < 1$, then the sequence $\{u_n\}$ converges strongly to $P_{F(\tau)}x_0$.

In 2008, Kohsaka and Takahashi [12, 13] presented a new mapping which is called a nonspreading mapping and obtained fixed point theorems for a single nonspreading mapping and also a common fixed point theorems for a commutative family of nonspreading mapping in Banach spaces. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *nonspreading* if

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Recently, Iemoto and Takahashi [9] showed that $T : C \to C$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty\rangle \ \forall x, y \in C.$$

Further, Takahashi [26] defined a class of nonlinear mappings which is called *hybrid* as follows:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle$$

for all x, $y \in C$. It was shown that a mapping $T : C \to H$ is hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2$$

for all x, $y \in C$.

Inspired by Kohsaka and Takahashi [12,13], Iemoto and Takahashi [9], Takahashi [26], Cholamjiak and Cholamjiak [5] introduced a new concept of multivalued mappings in Hilbert spaces by using Hausdorff metric. A multivalued mapping $T : C \to CB(C)$ is said to be *hybrid* if

$$3H(Tx, Ty)^2 \leq \|x - y\|^2 + d(y, Tx)^2 + d(x, Ty)^2$$

for all x, $y \in C$. They showed that if T is hybrid and $F(T) \neq \emptyset$, then T is quasi-nonexpansive. Moreover, they gave an example of a hybrid multivalued mapping which is not nonexpansive.

Example 1.1 ([5])**.** Let $H = \mathbb{R}$ Consider $C = [0, 3]$ with the usual norm. Define a multivalued mapping $T : C \to CB(C)$ by

$$Tx = \begin{cases} \{0\}, & x \in [0, 2]; \\ \left[0, \frac{x}{x+1}\right], & x \in (2, 3]. \end{cases}$$

We now give other examples of hybrid multivalued mappings which are not nonexpansive.

Example 1.2. Let $H = \mathbb{R}$. Consider $C = [2, 5]$ with the usual norm. Define two hybrid multivalued mappings $T_1 : C \to K(C)$ by

$$T_1x = \begin{cases} \{5\}, & x \in [3, 5]; \\ \left[(x + 5)(\frac{\tan^{-1}(19x-65)}{2}) + x, 5\right], & x \notin [3, 5]. \end{cases}$$

To see that T_1 is hybrid, we observe the following cases.
Case 1. If $x, y \in [3,5]$, then $H(T_1x, T_1,y) = 0$.
Case 2. If $x \in [3,5]$ and $y \notin [3,5]$, then $T_1x = \{5\}$ and $T_1y = \big[(y+5)(\frac{\tan^{-1}(19y-65)}{2}) + y, 5\big]$. This implies that

$$\begin{aligned}3H(T_1x, T_1y)^2 &= 3\big((y+5)(\frac{\tan^{-1}(19y-65)}{2}) + y - 5\big)^2 < 3\\ &< \|x-y\|^2 + d(x, T_1y)^2 + d(y, T_1x)^2.\end{aligned}$$

Case 3. If $x, y \notin [3,5]$, then $T_1x = \big[(x+5)(\frac{\tan^{-1}(19x-65)}{2}) + x, 5\big]$ and $T_1y = \big[(y+5)(\frac{\tan^{-1}(19y-65)}{2}) + y, 5\big]$. This implies that

$$\begin{aligned}&3H(T_1x, T_1y)^2\\ &= 3\big((x+5)(\frac{\tan^{-1}(19x-65)}{2}) + x - \big((y+5)(\frac{\tan^{-1}(19y-65)}{2}) + y\big)\big)^2\\ &< 3\\ &< \|x-y\|^2 + d(x, T_1y)^2 + d(y, T_1x)^2.\end{aligned}$$

But T_1 is not nonexpansive since for $x = 2.94$ and $y = 3.42$, we have $T_1x = [4.45, 5]$ and $T_1y = \{5\}$. This implies that

$$H(T_1x, T_1y) = |5 - 4.45| = 0.55 > 0.48 = \|x - y\|.$$

Example 1.3. Let $H = \mathbb{R}$. Consider $C = [2,5]$ with the usual norm. Define two hybrid multivalued mappings $T_2 : C \to K(C)$ by

$$T_2x = \begin{cases} \{5\}, & x \in [3,5];\\ \big[(x-5)(\frac{-\cos(0.1x^{2.5}-0.98)}{1.29}) + x, 5\big], & x \notin [3,5].\end{cases}$$

To see that T_2 is hybrid, we observe the following cases.
Case 1. If $x, y \in [3,5]$, then $H(T_2x, T_2,y) = 0$.
Case 2. If $x \in [3,5]$ and $y \notin [3,5]$, then $T_2x = \{5\}$ and $T_2y = \big[(y+5)(\frac{-\cos(0.1y^{2.5}-0.98)}{1.29}) + y, 5\big]$. This implies that

$$\begin{aligned}3H(T_2x, T_2y)^2 &= 3\big((y+5)(\frac{-\cos(0.1y^{2.5}-0.98)}{1.29}) + y - 5\big)^2 < 3\\ &< \|x-y\|^2 + d(x, T_2y)^2 + d(y, T_2x)^2.\end{aligned}$$

Case 3. If $x, y \notin [3,5]$, then $T_2x = \big[(x+5)(\frac{-\cos(0.1x^{2.5}-0.98)}{1.29}) + x, 5\big]$ and $T_2y = \big[(y+5)(\frac{-\cos(0.1y^{2.5}-0.98)}{1.29}) + y, 5\big]$. This implies that

$$\begin{aligned}&3H(T_2x, T_2y)^2\\ &= 3\big((x+5)(\frac{-\cos(0.1x^{2.5}-0.98)}{1.29}) + x - \big((y+5)(\frac{-\cos(0.1y^{2.5}-0.98)}{1.29}) + y\big)\big)^2\\ &< 3\\ &< \|x-y\|^2 + d(x, T_2y)^2 + d(y, T_2x)^2.\end{aligned}$$

But T_2 is not nonexpansive since for $x = 2.97$ and $y = 3.55$, we have $T_2x = [4.28, 5]$ and $T_2y = \{5\}$. This implies that

$$H(T_2x, T_2y) = |5 - 4.28| = 0.72 > 0.58 = \|x - y\|.$$

Motivated and inspired by the above works, we introduce the iterative scheme for finding a common fixed point of two hybrid multivalued mappings by using the Ishikawa iteration. We also obtain weak convergence theorems. Moreover, we use CQ and shrinking projection methods with Ishikawa iteration for obtaining strong convergence theorems. As application, we give examples and numerical results for supporting our main results.

2. Preliminaries and lemmas

We now provide some basic results for the proof. In a Hilbert space H, let C be a nonempty closed and convex subset of H. Let $\{x_n\}$ be a sequence in H, we denote the weak convergence of $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and the strong convergence of $\{x_n\}$ to a point $x \in H$ by $x_n \to x$. For every point $x \in H$, there exists a unique nearest point of C, denoted by P_Cx, such that $\|x - P_Cx\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H on to C.

Lemma 2.1 ([7, 15])**.** *Let H be a real Hilbert space. Then for each $x, y \in H$ and each $t \in [0, 1]$*

(a) $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.
(b) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$.
(c) *If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2.$$

Lemma 2.2 ([16])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H. For each x, $y \in H$ and $a \in \mathbb{R}$, the set*

$$D = \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex.

Lemma 2.3 ([17])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H and $P_C : H \to C$ be the metric projection from H onto C. Then $\|y - P_Cx\|^2 + \|x - P_Cx\|^2 \leq \|x - y\|^2$ for all $x \in H$ and $y \in C$.*

Lemma 2.4 ([23])**.** *Let X be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that*

$$\lim_{n\to\infty} \|x_n - u\| \text{ and } \lim_{n\to\infty} \|x_n - v\| \text{ exist.}$$

If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Lemma 2.5 ([5]). *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for some $y_n \in Tx_n$. Then $p \in Tp$.*

Lemma 2.6 ([5]). *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$, then $F(T)$ is closed.*

Condition (A). Let H be a Hilbert space and C be a subset of H. A multivalued mapping $T : C \to CB(C)$ is said to satisfy *Condition* (A) if $\|x - p\| = d(x, Tp)$ for all $x \in H$ and $p \in F(T)$.

Lemma 2.7 ([5]). *Let C be a closed and convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a hybrid multivalued mapping with $F(T) \neq \emptyset$. If T satisfies Condition* (A)*, then $F(T)$ is convex.*

Remark 2.8. We see that T satisfies Condition (A) if and only if $Tp = \{p\}$ for all $p \in F(T)$. It is known that the best approximation operator P_T, which is defined by $P_T x = \{y \in Tx : \|y - x\| = d(x, Tx)\}$, also satisfies Condition (A) (see [4, 5, 24]).

3. Main results

In this section, we prove a weak convergence theorem for a modification of Ishikawa iteration for two hybrid multivalued mappings. Further, we use CQ and shrinking projection methods with Ishikawa iteration to obtain strong convergence theorems.

Theorem 3.1. *Let C be a closed and convex subset of a real Hilbert space H and $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n \in \beta_n x_n + (1 - \beta_n) T_1 x_n, \\ x_{n+1} \in \alpha_n x_n + (1 - \alpha_n) T_2 z_n, \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.

Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If T_1 and T_2 satisfy Condition (A)*, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2\}$.*

Proof. Let $p \in F(T_1) \cap F(T_2)$. By using Lemma 2.1(b) and T_1, T_2 satisfy Condition (A), for $v_n \in T_2 z_n$ and $w_n \in T_1 x_n$, we get

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n (x_n - p) + (1 - \alpha_n)\|v_n - p)\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n)\|v_n - p\|^2 \end{aligned}$$

$$= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(v_n - T_2p)^2$$
$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(T_2z_n, T_2p)^2$$
$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \tag{3.2}$$

and

$$\begin{aligned}\|z_n - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n(w_n - p)\|^2\\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2\\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)d(w_n, T_1p)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2\\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)H(T_1x_n, T_1p)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2\\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2. \end{aligned}\tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2]\\ &\leq \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2\\ &\leq \|x_n - p\|^2.\end{aligned}$$

This implies that

$$\|x_{n+1} - p\| \leq \|x_n - p\|. \tag{3.4}$$

Therefore, $\lim_{n\to\infty}\|x_n - p\|$ exists. From (3.3), we have

$$\beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $\limsup_{n\to\infty}\alpha_n < 1$, $0 < \liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$ and $\lim_{n\to\infty}\|x_n - p\|$ exists, we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)v_n - p\|^2\\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2\\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(v_n, T_2p)^2 - \alpha(1-\alpha_n)\|x_n - v_n\|^2\\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(T_2z_n, T_2p)^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2\\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2\\ &\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2.\end{aligned}\tag{3.6}$$

This implies that

$$\alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since $0 < \liminf_{n\to\infty}\alpha_n < \limsup_{n\to\infty}\alpha_n < 1$ and $\lim_{n\to\infty}\|x_n - p\|$ exists, we obtain

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{3.7}$$

From (3.5), we get

$$\begin{aligned}\|z_n - x_n\| &= \|\beta_n x_n + (1-\beta_n)w_n - x_n\| \\ &\le \|w_n - x_n\| \to 0 \end{aligned} \tag{3.8}$$

as $n \to \infty$.

It follows from (3.7) and (3.8) that

$$\|z_n - v_n\| \le \|z_n - x_n\| + \|x_n - v_n\| \to 0 \tag{3.9}$$

as $n \to \infty$.

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ for some $q \in C$. By Lemmas 2.5 and 3.5, we have $q \in T_1 q$. From (3.9), we also have $z_{n_k} \rightharpoonup q$. Again by Lemma 2.5, we can conclude that $q \in T_2 q$. This implies that $q \in F(T_1) \cap F(T_2)$. We next show that $\{x_n\}$ converges weakly to q. We take another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ converging weakly to some $q' \in F(T_1) \cap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F(T_1) \cap F(T_2)$, from Lemma 2.4, $q = q'$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, C_1 = C, \\ z_n \in \beta_n x_n + (1-\beta_n)T_1 x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n)T_2 z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \forall n \ge 1, \end{cases} \tag{3.10}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$.

Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If T_1 and T_2 satisfy Condition (A)*, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2\}$.*

Proof. We split the proof into four steps.

Step 1. Show that $\{x_n\}$ is well-defined. Since T_1 and T_2 satisfy Condition (A), from Lemmas 2.6-2.7, $F(T_1) \cap F(T_2)$ is close and convex. Now, we show that C_n is closed and convex for all $n \ge 1$. For this end, we prove by induction on n that C_n is closed and convex. For $n = 1$, $C_1 = C$ is closed and convex. Assume that C_n is closed and convex for some $n \in \mathbb{N}$. From the definition C_{n+1} and Lemma 2.4, we have that C_{n+1} also closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. Next, we show that $F(T_1) \cap F(T_2) \subset C_n$ for each $n \ge 1$. By using Lemma 2.1(b) and T_1, T_2 satisfy Condition (A), for each $p \in F(T_1) \cap F(T_2)$, $v_n \in T_2 z_n$ and $w_n \in T_1 x_n$, we have

$$\|y_n - p\|^2 = \|\alpha_n(x_n - p) + (1-\alpha_n)(v_n - p)\|^2$$

$$\begin{aligned} &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(v_n, T_2p)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(T_2z_n, T_2p)^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \|z_n - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(w_n - p)\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)d(w_n, T_1p)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)H(T_1x_n, T_1p)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2. \end{aligned} \tag{3.12}$$

Substituting (3.12) in (3.11), we have

$$\begin{aligned} \|y_n - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2] \\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\quad - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \\ &\le \|x_n - p\|^2. \end{aligned} \tag{3.13}$$

Therefore, $p \in C_n$, $n \ge 1$. This implies that $F(T_1) \cap F(T_2) \subseteq C_n$ for each $n \ge 1$ and so $C_n \ne \emptyset$. Hence the sequence $\{x_n\}$ is well-defined.

Step 2. Show that $x_n \to w \in C$ as $n \to \infty$. From $x_n = P_{C_n}x_1$, $C_{n+1} \subseteq C_n$ and $x_{n+1} \in C_n$, we have

$$\|x_n - x_1\| \le \|x_{n+1} - x_1\|, \quad \forall n \ge 1. \tag{3.14}$$

On the other hand, since $F(T_1) \cap F(T_2) \subseteq C_n$, we obtain

$$\|x_n - x_{1\|} \le \|z - x_1\|, \quad \forall n \ge 1 \tag{3.15}$$

for all $z \in F(T_1) \cap F(T_2)$. The inequalities (3.14) and (3.15) imply that the sequence $\{x_n - x_1\}$ is bound and nondecreasing, hence $\lim_{n\to\infty} \|x_n - x_1\|$ exists. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m}x_1 \in C_m \subseteq C_n$. By Lemma 2.3, we obtain that

$$\|x_m - x_n\|^2 \le \|x_m - x_1\|^2 - \|x_n - x_1\|^2. \tag{3.16}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, it follows from (3.16) that $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C and so $x_n \to w \in C$ as $n \to \infty$.

Step 3. Show that $\lim_{n\to\infty} \|x_n - w_n\| = 0 = \lim_{n\to\infty} \|z_n - v_n\|$ where $w_n \in T_1x_n$ and $v_n \in T_2z_n$.

From Step 2, we know that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Since $x_{n+1}\in C_n$, we have

$$\begin{aligned}\|y_n-x_n\| &\le \|y_n-x_{n+1}\|+\|x_{n+1}-x_n\|\\ &\le 2\|x_{n+1}-x_n\|\to 0 \qquad (3.17)\end{aligned}$$

as $n\to\infty$.
From (3.12) and T_2 satisfies Condition (A), we have

$$\begin{aligned}\|y_n-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)v_n-p\|^2, \quad \forall v_n\in T_2z_n\\ &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-v_n\|^2\\ &= \alpha_n\|x_n-p\|^2+(1-\alpha_n)d(v_n,T_2p)^2-\alpha(1-\alpha_n)\|x_n-v_n\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)H(T_2z_n,T_2p)^2-\alpha_n(1-\alpha_n)\|x_n-v_n\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|z_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-v_n\|^2\\ &\le \|x_n-p\|^2-\alpha_n(1-\alpha_n)\|x_n-v_n\|^2. \qquad (3.18)\end{aligned}$$

This implies that

$$\alpha_n(1-\alpha_n)\|x_n-v_n\|\le\|x_n-p\|^2-\|y_n-p\|^2.$$

Since $0<\liminf_{n\to\infty}\alpha_n<\limsup_{n\to\infty}\alpha_n<1$, it follows from (3.17) that

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \qquad (3.19)$$

From (3.13), we have

$$\|y_n-p\|^2\le\|x_n-p\|^2-\beta_n(1-\alpha_n)(1-\beta_n)\|x_n-w_n\|^2.$$

This implies that

$$\beta_n(1-\alpha_n)(1-\beta_n)\|x_n-w_n\|^2\le\|x_n-p\|^2-\|y_n-p\|^2.$$

Since $\limsup_{n\to\infty}\alpha_n<1$ and $0<\liminf_{n\to\infty}\beta_n<\limsup_{n\to\infty}\beta_n<1$, we obtain

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \qquad (3.20)$$

From (3.20), we have

$$\begin{aligned}\|z_n-x_n\| &= \|\beta_nx_n+(1-\beta_n)w_n-x_n\|\\ &= (1-\beta_n)\|w_n-x_n\|\to 0 \qquad (3.21)\end{aligned}$$

as $n\to\infty$.
From (3.19) and (3.21), so

$$\|z_n-v_n\|\le(\|z_n-x_n\|+\|x_n-v_n\|)\to 0 \qquad (3.22)$$

as $n\to\infty$.
Step 4. Show that $w=P_{F(T_1)\cap F(T_2)}x_1$. From Steps 2-3 and Lemma 2.5, we obtain $w\in F(T_1)\cap F(T_2)$. Form (3.15), we have $\|w-x_1\|\le\|x_1-z\|$, $\forall z\in F(T_1)\cap F(T_2)$. By the definition of the projection operator, we can conclude that $w=P_{F(T_1)\cap F(T_2)}x_1$. This completes the proof. □

Theorem 3.3. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n \in \beta_n x_n + (1-\beta_n)T_1 x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n)T_2 z_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \forall n \geq 1, \end{cases} \tag{3.23}$$

where $\{\alpha_n\}$, $\{\beta_n\} \subset (0,1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If T_1 and T_2 satisfy Condition (A)*, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2\}$.*

Proof. We split the proof into four steps.
Step 1. Show that $\{x_n\}$ is well-defined. From Lemmas 2.6-2.7, we know that $F(T_1) \cap F(T_2)$ is a closed and convex subset of C. From the definition of Q_n and Lemma 2.2, it is obvious that Q_n is closed and convex for each $n \geq 1$. As the same proof in Step 1 of Theorem 3.2, we have C_n is closed and convex for each $n \geq 1$. Next, show that $F(T_1) \cap F(T_2) \subseteq C_n \cap Q_n$. By using Lemma 2.1(b) and T_1, T_2 satisfy Condition (A), for each $p \in F(T_1) \cap F(T_2)$, $v_n \in T_2 z_n$ and $w_n \in T_1 x_n$, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)v_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(v_n, T_2 p)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(T_2 z_n, T_2 p)^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \|z_n - p\|^2 &= \|\beta_n(x_n - p) + (1-\beta_n)(w_n - p)\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|w_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &= \beta_n\|x_n - p\|^2 + (1-\beta_n)d(w_n, T_1 p) - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)H(T_1 x_n, T_1 p)^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2. \end{aligned} \tag{3.25}$$

Substituting (3.25) in (3.24), we have

$$\begin{aligned}\|y_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - w_n\|^2] \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\ &\quad - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \\ &\leq \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.26}$$

Therefore, $p \in C_n$, $n \geq 1$. This implies that $F(T_1) \cap F(T_2) \subseteq C_n$ for each $n \geq 1$. Next, we show that $F(T_1) \cap F(T_2) \subseteq Q_n$ for all $n \in \mathbb{N}$. For $n = 1$, we have $F(T_1) \cap F(T_2) \subseteq C = Q_1$. Assume that $F(T_1) \cap F(T_2) \subseteq Q_n$. Sine x_{n+1} is the projection of x_1 onto $C_n \subseteq Q_n$, we have

$$\langle x_1 - x_{n+1}, x_{n+1} - z\rangle \geq 0, \forall\ z \in C_n \cap Q_n.$$

Thus $F(T_1) \cap F(T_2) \subseteq Q_{n+1}$. This implies that $\{x_n\}$ is well-defined.

Step 2. Show that $x_n \to w \in C$ as $n \to \infty$. From the definition of Q_n, we get $x_n = P_{Q_n}x_1$. Since $x_{n+1} \in Q_n$, we have

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \qquad \forall n \geq \mathbb{N}. \tag{3.27}$$

On the other hand, we obtain

$$\|x_n - x_1\| \leq \|z - x_1\|, \qquad \forall z \in F(T_1) \cap F(T_2). \tag{3.28}$$

The inequalities (3.27) and (3.28) imply that the sequence $\{x_n - x_1\}$ is bounded and nondecreasing, hence $\lim_{n\to\infty}\|x_n - x_1\|$ exists. For $m > n$, by definition of Q_n, we have $x_m = P_{Q_m}x_1 \in Q_m \subseteq Q_n$. By Lemma 2.3, we obtain that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2. \tag{3.29}$$

Since $\lim_{n\to\infty}\|x_n - x_1\|$ exists, it follows from (3.29) that $\lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C and so $x_n \to w \in C$ as $n \to \infty$. In particular, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3. Show that $\lim_{n\to\infty}\|x_n - w_n\| = 0 = \lim_{n\to\infty}\|z_n - v_n\|$ where $w_n \in T_1x_n$ and $v_n \in T_2z_n$.

Since $x_{n+1} \in C_n$, from Step 2, we have

$$\begin{aligned}\|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq 2\|x_{n+1} - x_n\| \to 0\end{aligned} \tag{3.30}$$

as $n \to \infty$.

From (3.25) and T_2 satisfies Condition (A), we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)v_n - p\|^2, \quad \forall v_n \in T_2z_n \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)d(v_n, T_2p)^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)H(T_2z_n, T_2p)^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2\end{aligned}$$

$$\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2$$
$$\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - v_n\|^2. \tag{3.31}$$

This implies that

$$\alpha_n(1-\alpha_n)\|x_n - v_n\| \leq \|x_n - p\|^2 - \|y_n - p\|^2.$$

Since $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$, it follows from (3.30) that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.32}$$

From (3.26), we have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2.$$

This implies that

$$\beta_n(1-\alpha_n)(1-\beta_n)\|x_n - w_n\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2.$$

Since $\limsup_{n\to\infty} \alpha_n < 1$ and $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$, we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.33}$$

From (3.33), we have

$$\|z_n - x_n\| = \|\beta_n x_n + (1-\beta_n)w_n - x_n\|$$
$$= (1-\beta_n)\|w_n - x_n\| \to 0 \tag{3.34}$$

as $n \to \infty$.
From (3.32) and (3.34), so

$$\|z_n - v_n\| \leq (\|z_n - x_n\| + \|x_n - v_n\|) \to 0 \tag{3.35}$$

as $n \to \infty$.
Step 4. Show that $w = P_{F(T_1)\cap F(T_2)}x_1$. From Steps 2-3 and Lemma 2.5, we obtain $w \in F(T_1) \cap F(T_2)$. Form (3.28), we have $\|w - x_1\| \leq \|x_1 - z\|$, $\forall z \in F(T_1) \cap F(T_2)$. By the definition of the projection operator, we can conclude that $w = P_{F(T_1)\cap F(T_2)}x_1$. This completes the proof. □

If $Tp = \{p\}$ for all $p \in F(T)$, T satisfies Condition (A), then we obtain the following results.

Corollary 3.4. *Let C be a closed and convex subset of a real Hilbert space H and $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1)\cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily,} \\ z_n \in \beta_n x_n + (1-\beta_n)T_1 x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n)T_2 z_n, \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If $T_1 p = \{p\}$, $T_2 q = \{q\}$ for all $p \in F(T_1)$ and $q \in F(T_2)$, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2\}$.

Corollary 3.5. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in H, C_1 = C, \\ z_n \in \beta_n x_n + (1-\beta_n) T_1 x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n) T_2 z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If $T_1 p = \{p\}$, $T_2 q = \{q\}$ for all $p \in F(T_1)$ and $q \in F(T_2)$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2\}$.

Corollary 3.6. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to K(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n \in \beta_n x_n + (1-\beta_n) T_1 x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n) T_2 z_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If $T_1 p = \{p\}$, $T_2 q = \{q\}$ for all $p \in F(T_1)$ and $q \in F(T_2)$, then the sequence $\{x_n\}$ converge strongly to a common fixed point of $\{T_1, T_2\}$.

Since P_T satisfies Condition (A), then we obtain the results.

Corollary 3.7. *Let C be a closed and convex subset of a real Hilbert space H and $T_1, T_2 : C \to P(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n \in \beta_n x_n + (1-\beta_n) P_{T_1} x_n, \\ x_{n+1} \in \alpha_n x_n + (1-\alpha_n) P_{T_2} z_n, \end{cases}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If P_{T_1}, P_{T_2} are hybrid multivalued mappings, then the sequence $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2\}$.

Proof. By the same proof in Theorem 3.1, we have $x_n \to w_n \in P_{T_1}x_n \subseteq T_1x_n$ and we have $z_n \to v_n \in P_{T_2}z_n \subseteq T_2z_n$. From Lemma 2.5, we obtain this results. □

Corollary 3.8. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to P(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in H, C_1 = C, \\ z_n \in \beta_n x_n + (1-\beta_n)P_{T_1}x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n)P_{T_2}z_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If P_{T_1}, P_{T_2} are hybrid multivalued mappings, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2\}$.

Proof. By the same proof in Theorem 3.2, we have $x_n \to w_n \in P_{T_1}x_n \subseteq T_1x_n$ and we have $z_n \to v_n \in P_{T_2}z_n \subseteq T_2z_n$. From Lemma 2.5, we obtain this results. □

Corollary 3.9. *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to P(C)$ be hybrid multivalued mappings with $F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ z_n \in \beta_n x_n + (1-\beta_n)P_{T_1}x_n, \\ y_n \in \alpha_n x_n + (1-\alpha_n)P_{T_2}z_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_1 - x_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$.
Assume that the following hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

If P_{T_1}, P_{T_2} are hybrid multivalued mappings, then the sequence $\{x_n\}$ converge strongly to a common fixed point of $\{T_1, T_2\}$.

Proof. By the same proof in Theorem 3.3, we have $x_n \to w_n \in P_{T_1}x_n \subseteq T_1x_n$ and we have $z_n \to v_n \in P_{T_2}z_n \subseteq T_2z_n$. From Lemma 2.5, so we obtain the results. □

4. Example and numerical results

In this section, we give examples with numerical results for supporting our theorem.

Example 4.1. Let $H = \mathbb{R}$ and $C = [2,5]$. Define two hybrid multivalued mappings $T_1, T_2 : C \to K(C)$ by

$$T_1x = \begin{cases} \{5\}, & x \in [3,5]; \\ \big[(x+5)(\frac{\tan^{-1}(19x-65)}{2}) + x, 5\big], & x \notin [3,5] \end{cases}$$

and

$$T_2x = \begin{cases} \{5\}, & x \in [3,5]; \\ \big[(x-5)(\frac{-\cos(0.1x^{2.5}-0.98)}{1.29}) + x, 5\big], & x \notin [3,5] \end{cases}$$

for all $x \in C$. Choose $\alpha_n = \frac{n}{2n+1}$ and $\beta_n = \frac{2n}{5n+1}$.

It is easy to check that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy all conditions in Theorems 3.2-3.3. From Examples 1.2-1.3, we see that T_1 and T_2 are hybrid. Choosing $x_1 = 2$ and taking randomly $w_n \in T_1x_n$ and $v_n \in T_2z_n$, we obtain the numerical results of iteration (3.10) as follows:

TABLE 1. Numerical results of iteration (3.10) being randomized in the first time.

n	Randomized in the 1st				
	w_n	z_n	v_n	y_n	x_n
1	4.989956	3.993304	5.000000	4.000000	2.000000
2	4.915212	4.218772	5.000000	4.200000	3.000000
3	5.000000	4.475000	5.000000	4.400000	3.600000
4	5.000000	4.619048	5.000000	4.555556	4.000000
5	5.000000	4.722222	5.000000	4.671717	4.277778
6	5.000000	4.796676	5.000000	4.757576	4.474747
7	5.000000	4.850730	5.000000	4.820875	4.616162
8	5.000000	4.890154	5.000000	4.867538	4.718519
9	5.000000	4.919011	5.000000	4.901961	4.793028
10	5.000000	4.940194	5.000000	4.927378	4.847495
...	...	...	...	...	...
43	5.000000	4.999996	5.000000	4.999996	4.999991

Choosing $x_1 = 2$ and taking randomly $w_n \in T_1x_n$ and $v_n \in T_2z_n$, we obtain the numerical results of iteration (3.23) as follows:

TABLE 2. Numerical results of iteration (3.10) being randomized in the second time.

n	Randomized in the 2nd				
	w_n	z_n	v_n	y_n	x_n
1	4.605374	3.736916	5.000000	4.000000	2.000000
2	4.979731	4.259829	5.000000	4.200000	3.000000
3	5.000000	4.475000	5.000000	4.400000	3.600000
4	5.000000	4.619048	5.000000	4.555556	4.000000
5	5.000000	4.722222	5.000000	4.671717	4.277778
6	5.000000	4.796676	5.000000	4.757576	4.474747
7	5.000000	4.850730	5.000000	4.820875	4.616162
8	5.000000	4.890154	5.000000	4.867538	4.718519
9	5.000000	4.919011	5.000000	4.901961	4.793028
10	5.000000	4.940194	5.000000	4.927378	4.847495
...	...	...	...	...	...
43	5.000000	4.999996	5.000000	4.999996	4.999991

TABLE 3. Numerical results of iteration (3.23) being randomized in the first time.

n	Randomized in the 1st				
	w_n	z_n	v_n	y_n	x_n
1	4.637906	3.758604	5.000000	4.000000	2.000000
2	5.000000	4.272727	5.000000	4.200000	3.000000
3	5.000000	4.475000	5.000000	4.400000	3.600000
4	5.000000	4.619048	5.000000	4.555556	4.000000
5	5.000000	4.722222	5.000000	4.671717	4.277778
6	5.000000	4.796676	5.000000	4.757576	4.474747
7	5.000000	4.850730	5.000000	4.820875	4.616162
8	5.000000	4.890154	5.000000	4.867538	4.718519
9	5.000000	4.919011	5.000000	4.901961	4.793028
10	5.000000	4.940194	5.000000	4.927378	4.847495
...	...	...	...	...	...
43	5.000000	4.999996	5.000000	4.999996	4.999991

Remark 4.2. According to the investigations of our numerical results under the same conditions, we can see that

(i) the sequences $\{x_n\}$ are the same in each step of randomize.

(ii) the sequences $\{x_n\}$ of the shrinking projection method in Tables 1-2 and the CQ method in Tables 3-4 are also the same.

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TABLE 4. Numerical results of iteration (3.23) being randomized in the second time.

n	Randomized in the 2nd				
	w_n	z_n	v_n	y_n	x_n
1	4.453659	3.635773	5.000000	4.000000	2.000000
2	5.000000	4.272727	5.000000	4.200000	3.000000
3	5.000000	4.475000	5.000000	4.400000	3.600000
4	5.000000	4.619048	5.000000	4.555556	4.000000
5	5.000000	4.722222	5.000000	4.671717	4.277778
6	5.000000	4.796676	5.000000	4.757576	4.474747
7	5.000000	4.850730	5.000000	4.820875	4.616162
8	5.000000	4.890154	5.000000	4.867538	4.718519
9	5.000000	4.919011	5.000000	4.901961	4.793028
10	5.000000	4.940194	5.000000	4.927378	4.847495
...	...	...	...	...	...
43	5.000000	4.999996	5.000000	4.999996	4.999991

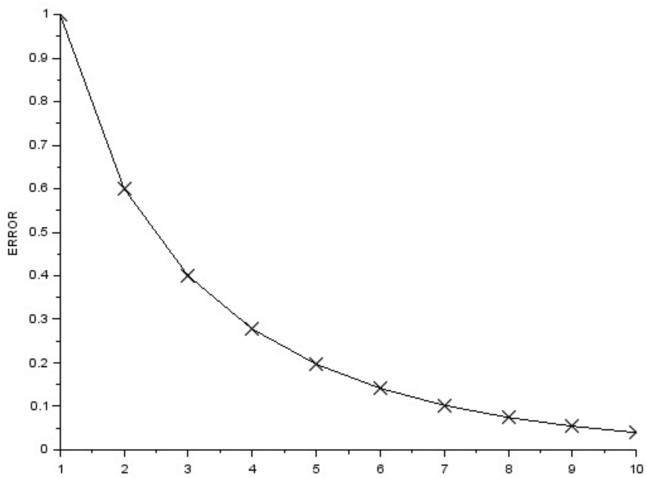

FIGURE 1. Error plots for all sequences $\{x_n\}$ in Tables 1-4.

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