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CHARACTERISTIC POLYNOMIAL OF THE HYPERPLANE ARRANGEMENTS $\mathcal{J}_n$ VIA FINITE FIELD METHOD

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Abstract. We use the finite method developed by C. Athanasiadis based on Crapo-Rota's theorem to give a complete formula for the characteristic polynomial of hyperplane arrangements $\mathcal{J}_n$ consisting of the hyperplanes $x_i + x_j = 1$, $x_k = 0$, $x_l = 1$, $1 \le i, j, k, l \le n$.

1. Introduction and preliminaries

In this paper, we shall revisit the hyperplane arrangement problem investigated in [2–5] from a different point of view. A hyperplane arrangement in $\mathbb{R}^n$ is a finite collection $\mathcal{A}$ of hyperplanes in $\mathbb{R}^n$, and the particular hyperplane arrangement $\mathcal{J}_n$ we considered consists of

(1) the walls or hyperplanes of **type I**: $H_{\alpha\beta} = \{x \in \mathbb{R}^n : x_\alpha + x_\beta = 1\} = H_{\beta\alpha}$, $1 \le \alpha, \beta \le n$;
(2) the walls of **type II**: $0_i := \{x \in \mathbb{R}^n : x_i = 0\}$, and $1_i := \{x \in \mathbb{R}^n : x_i = 1\}$, $\forall i \in [n] := \{1, 2, \dots, n\}$.

The main question about an arrangement $\mathcal{A}$ is the number of (relatively bounded) chambers of the complement

$$\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

The number of chambers can be computed via the *characteristic polynomial*

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{B}} (-1)^{|\mathcal{B}|} t^{n - \operatorname{rank}(\mathcal{B})},$$

where $\mathcal{B}$ runs through all subarrangements of $\mathcal{A}$ such that the intersection of all hyperplanes in $\mathcal{B}$ is nonempty, and $\operatorname{rank}(\mathcal{B})$ denotes the *rank* of $\mathcal{B}$ which is the dimension of the space spanned by the normal vectors to the hyperplanes in $\mathcal{B}$.

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Theorem ([6]). *Let $\mathcal{A}$ be a hyperplane arrangement in an n-dimensional real vector space. Let $r(\mathcal{A})$ be the number of chambers and $b(\mathcal{A})$ be the number of relatively bounded chambers. Then we have*

(1) $b(\mathcal{A}) = (-1)^n \chi(+1)$.
(2) $r(\mathcal{A}) = (-1)^n \chi(-1)$.

In the aforementioned papers, we gave a generating function for the coefficients of the characteristic polynomial of $\mathcal{J}_n$ by associating 3-colored graphs with subarrangements of $\mathcal{J}_n$ and enumerating the 3-colored graphs corresponding to central subarrangements of given rank.

Theorem 1 ([3]). *Let $\bar{\gamma}^{(0)}_{r,c}$ denote the number of connected, non-colored, bipartite graphs without isolated vertices whose rank and cardinality are r and c. Let $\bar{b}_{n,k}$ be the number of connected bipartite graphs of order n and size k. The characteristic polynomial of $\mathcal{J}_n$ is given by*

$$\chi_{\mathcal{J}_n}(t) = \sum_{r=0}^{n} \left(\sum_{c\geq 1} \sum_{r+\nu\leq n} \binom{n}{r+\nu} (-1)^c \, \Gamma_{r,c,\nu} \right) t^{n-r},$$

where $\Gamma_{r,c,\nu}$ is determined by

$$\begin{aligned}
&\sum_{r,c,\nu\geq 0} \frac{\Gamma_{r,c,\nu}}{(r+\nu)!} x^r y^c z^\nu \\
&= \exp\left[\left(\frac{1}{2} \log \left(1 + \sum_{n\geq 1, k\geq 0} \sum_{i=0}^{n} \binom{n}{i} \binom{i(n-i)}{k} \frac{1}{n!} x^n y^k \right) - x \right) \frac{z}{x} \right] \cdot \\
&\quad \left(\sum_{r=0}^{\infty} \frac{2^r}{r!} x^r y^r \right) \cdot \left(\sum \left(\sum_{t=1}^{c-r} 2\overline{\gamma}^{(0)}_{r-1,c-t} \binom{r}{t} \right) \frac{1}{r!} x^r y^c \right) \cdot \\
&\quad \left(\exp\left(\log \left(1 + \sum_{n\geq 1, k\geq 0} \binom{\binom{n}{2}}{k} \frac{1}{n!} x^n y^k \right) - x \right) - \sum_{n\geq 2, k\geq 1} \bar{b}_{n,k} \frac{1}{n!} x^n y^k \right).
\end{aligned}$$

While it sports an admittedly complicated look, this gives a relatively efficient way of computing $\chi_{\mathcal{J}_n}$, and we computed it for n up to 10 fairly readily by using Mathematica.

There is a very powerful and elegant method for computing the characteristic polynomial of hyperplane arrangements: Cristos Athanasiadis [1] revised a theorem of Carpo and Rota and proved that the characteristic polynomial can be obtained simply by counting the number of points in a finite field (thus named *finite field method*) that miss all the hyperplanes! In Section 2, we shall briefly discuss his method and work out an example as a warmup to our main analysis. In Section 3, we shall apply the finite field method to $\mathcal{J}_n$ and obtain a formula for $\chi_{\mathcal{J}_n}$:

Theorem. *Let q be a large prime, and $m = \frac{q-1}{2} - 1$. Then $\chi_{\mathcal{J}_n}(q)$ is equal to*

$$\chi_{\mathcal{J}_n}(q) = \sum \binom{n}{k} \binom{k}{k_1, k_2, \ldots, k_{s+1}} \binom{n-k}{j_1, j_2, \ldots, j_s} \prod_{i=1}^{s+1} \binom{k_i}{p_i} \binom{j_i}{q_i} p_i^{k_i - p_i} (q_i + 1)^{j_i - q_i},$$

where the sum runs over all choices of

(1) *an integer s between $m-n$ and $m-1$,*
(2) *an integer k between 0 and n,*
(3) *all partitions $m - s = \sum_{i=1}^{s+1} (p_i + q_i)$,*
(4) *all partitions $k = \sum_{i=1}^{s+1} k_i$ such that $k_i \geq p_i$,*
(5) *all partitions $n - k = \sum_{i=1}^{s} j_i$ such that $j_i \geq q_i$.*

This is not quite strong as the generating function formula in [3], and it is similar to the formula we obtained in [4] using graph theory in that they both require summing over partitions. In a forthcoming paper, we plan to show that the two formulae are equivalent both numerically for small n and symbolically/combinatorially for general n, although the latter seems to be an exceedingly difficult problem.

2. Finite field method

We say that a hyperplane arrangement $\mathcal{A}$ is *defined over the integers* if the equations of the hyperplanes in $\mathcal{A}$ have integer coefficients. The theorem of Crapo-Rota revisited by Athanasiadis is as follows:

Theorem 2 ([1, Theorem 2.2])**.** *Let $\mathcal{A}$ be any subspace arrangement in $\mathbb{R}^n$ defined over the integers and q be a large enough prime number. Then we have*

$$\chi(\mathcal{A}, q) = \#(\mathbb{F}_q^n - \cup\mathcal{A}).$$

Here, the union class $\cup\mathcal{A}$ means the union of the hyperplanes in $\mathcal{A}$.

Although the statement of the theorem is direct and easy to understand, we shall demonstrate one example carefully below, before we dive the $\mathcal{J}_n$ case that proves to be far more complicated due to inherent disparities from the hyperplane arrangements considered in [1].

Example 1. This example is the content of [1, Theorem 3.3]. We present it here to lay the ground work for the proof of our main theorem in the next section. Let $\mathcal{A} \subset \mathbb{R}^n$ consist of hyperplanes satisfying

$$x_i = x_j, \ \forall\ i \neq j \text{ and}$$

$$x_i - x_j = 1, \ 1 \leq i < j \leq n.$$

We will be counting $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$ such that x never satisfies any linear equation defining a hyperplane in $\mathcal{A}$. For this end, we regard $\mathbb{F}_q$ as a circle of q boxes, and regard x as a function $x : [n] \to \mathbb{F}_q$, given by $x(i) = x_i$. So, we need to enumerate functions $x : [n] \to \mathbb{F}_q$ such that

$$x_i \neq x_j \text{ and } x_i \neq x_j + 1, \ \forall\, i < j. \tag{†}$$

To construct such a function, we use a three-step strategy:

Step 1. Fix a partition

$$\sum_{i=1}^{q-n} s_i = n,$$

and group n numbers into $q - n$ groups

$$\{a_{i1}, a_{i2}, \ldots, a_{is_i}\},\ i = 1, \ldots, q - n$$

Step 2. We may assume that for any ℓ, $i_\ell < i_{\ell+1}$. Now, put $a_{11}, a_{12}, \ldots, a_{1s_1}$ into the first s_1 boxes of $\mathbb{F}_q$ in the given order $a_{11} < a_{12} < \cdots < a_{1s_1}$. Leave the $(s_1 + 1)$st box blank, and then put $a_{21}, a_{22}, \ldots, a_{2s_2}$ in the next s_2 boxes, followed by another blank box at $(s_2 + 1)$st position.

Step 3. Repeat the process. Define $x(i_k)$ to be the box in $\mathbb{F}_q$ in which i_k resides.

In the end, there will $q - n$ blank boxes and n boxes with a number. Such constructed x satisfies the two conditions of (†): Within a group, the function values satisfy the relation $x(i_l) = x(i_{l+1}) - 1$, so $x(i) < x(j)$ if $i < j$ are in the same group. If $i < j$ are in different groups, then their function values are separated by a blank box and $x(i) \neq x(j)+1$ and $x(i) \neq x(j)$. Hence x satisfies the condition (†).

Conversely any x satisfying the two conditions can be obtained via the two-step construction. (Simply partition $[n]$ into groups so that their function values follow one another immediately.) It follows that the number of such functions x equals the number of ways to group n numbers into $q - n$ groups, which is $(q - n)^n$.

3. Application to the hyperplane arrangement $\mathcal{J}_n$

The difficulty with applying the finite field method to $\mathcal{J}_n$ is two-fold. First, the presence of the hyperplance $x_i + x_j = 1$ requires a new combinatorial interpretation of $x = (x_1, \ldots, x_n)$ as a function. This can be dealt with by expanding the function domain from $[n]$ to $[n] \cup [-n] = \{\pm 1, \pm 2, \ldots, \pm n\}$, as in [1, Theorem 3.10]. Moreover, we shall see that by deforming the arrangement, $x_i + x_j = 1$ can be replaced by $x_i + x_j = 0$: This does not pose a combinatorial challenge in enumeration. Secondly, the functions are no longer injective since the arrangement does not have $x_i = x_j$: This is an intrinsic difficulty and it is the main reason why the formula in our main theorem below is complicated.

We shall denote the boxes in $\mathbb{F}_q$ by $\langle i \rangle$.

3.1. Deformation of $\mathcal{J}_n$

Replace x_i by $x_i + 1/2$. This has the following effects on the hyperplanes in $\mathcal{J}_n$:

$$\begin{aligned} x_i = 0 \quad &\mapsto \quad x_i = -1/2, \\ x_i = 1 \quad &\mapsto \quad x_i = +1/2, \\ x_i + x_j = 1 \quad &\mapsto \quad x_i = -x_j. \end{aligned}$$

Now, by doubling the coordinates, we obtain the following deformation of $\mathcal{J}_n$:

$$\mathcal{J}_n' = \{x_i = \pm 1, x_i = -x_j : 1 \le i < j \le n\}.$$

Theorem 3. *Let q be a large prime, and $m = \frac{q-1}{2} - 1$. Then $\chi_{\mathcal{J}_n}(q)$ is equal to*

$$\sum_{k+j=n} \sum_{s=0}^{k} \sum_{t=0}^{j} \sum_{(k_1,\ldots,k_s)} \sum_{(j_1,\ldots,j_t)} \binom{m}{s}\binom{m-s}{t}\binom{k}{k_1, k_2, \ldots, k_s}\binom{j}{j_1, j_2, \ldots, j_t},$$

where the subscripts $(k_1, \ldots, k_s)$ (resp. $(j_1, \ldots, j_t)$) means that the sum runs over all partitions $k = k_1 + \cdots + k_s$ (resp. of $j = j_1 + \cdots + j_t$). Also, for $s = 0$ and $t = 0$, we define $\binom{k}{k_1,k_2,\ldots,k_s} = 1$ and $\binom{j}{j_1,j_2,\ldots,j_t} = 1$.

Proof. Of course, $\mathcal{J}_n'$ shares the same characteristic polynomial with $\mathcal{J}_n$. We shall apply the finite field method to $\mathcal{J}_n'$ and obtain the desired formula.

Let $\mathbb{F}_q$ denote the finite field with q elements. We shall be counting $x : [n] \to \mathbb{F}_q$ such that $x_i \ne \pm 1$ and $x_i \ne -x_j$. We first expand the domain to $[n] \cup -[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. To satisfy the first condition, we remove the box $\langle 1 \rangle$ and $\langle q-1 \rangle$.

Note that the elements in the set

$$\{x : [n] \to \mathbb{F}_q \setminus \{1, q-1\},\ x_i \ne -x_j\}$$

are in a bijective correspondence with the functions $x : \pm[n] \to \mathbb{F}_q \setminus \{1, q-1\}$ such that

(†) $x_i = -x_{-i}$ and,

(††) no function value on $i \in [n]$ equals a function value on $-j \in -[n]$.

Again, we are conflating x_i and $x(i)$ as before. The upshot of introducing negative integer set in the domain is that it allows us to turn the condition $x_i + x_j \ne 0$ into the condition (††) that are much easier for the purpose of enumeration.

Since $x_{-i} = -x_i$, among the $2n$ numbers $x_{\pm 1}, \ldots, x_{\pm n}$, exactly n will be in the positive half $S_+ = \{\langle 0 \rangle, \langle 1 \rangle, \ldots, \langle \frac{q-1}{2} \rangle\}$. At this point we introduce the following lingo: For each $i \in [n]$, we shall call $x(i)$ a vop (value on a positive number) and $x(-i)$, a von (value on a negative number). Using this lingo, the condition (††) above can be stated as: *no vop equals a von.*

Let $m = \frac{q-1}{2} - 1$. This is the number of allowed boxes in S_+. We can construct the desired function x by following the steps:

Step 1. Partition n into two parts: $k + j = n$.
Step 2. Choose a set $I_+ \subset [n]$ of k indices: For $i \in I_+$, we shall put $x(i) \in S_+$. For $i \in [n] \setminus I_+$, we shall put $x(-i)$ in S_+.
Step 3. Place the k vops $x(i)$, $i \in I_+$, in s boxes in S_+ allowing repetition.
Step 4. Place the j vons $x(-j)$, $i \in [n] \setminus I_+$, in t boxes in S_+ allowing repetition.

Now we count the number of ways to perform each step. There are $\binom{n}{k}$ ways to choose k vops out of n indices. To place k vops in s boxes, we first choose s boxes from the m boxes: There are $\binom{m}{s}$ ways to do this. Then we place the chosen k vops into s boxes allowing repetition. This amounts to partitioning k vops into s subsets, minding the order. There are $\binom{k}{k_1, k_2, \ldots, k_s}$ ways to do this, where k_i runs over all partitions $k = k_1 + k_2 + \cdots + k_s$.

Once we are done with the vops, we choose t boxes out of the remaining $m - s$ boxes: There are $\binom{m-s}{t}$ ways to do this. And we place $j = n - k$ vons into the chosen t boxes. As with the vops, there are $\binom{j}{j_1, j_2, \ldots, j_t}$ ways to do this where j_u runs over all partitions $j = j_1 + j_2 + \cdots + j_t$.

All in all, we have

$$\sum_{k+j=n} \sum_{s=0}^{k} \sum_{t=0}^{j} \sum_{(k_1,\ldots,k_s)} \sum_{(j_1,\ldots,j_t)} \binom{m}{s} \binom{m-s}{t} \binom{k}{k_1, k_2, \ldots, k_s} \binom{j}{j_1, j_2, \ldots, j_t}. \qquad \square$$

Note that the sum incorporates the case $k = 0$: In this case, there is only one partition $0 = 0$ ($k = 0$, $s = 0$, $k_s = k_0 = 0$) and the coefficients $\binom{m}{s}$ and $\binom{k}{k_1, \ldots, k_s}$ are both 1. Likewise, $j = 0$ is also incorporated.

Remark 1. When the finite field method is directly applied to $\mathcal{J}_n$ (as opposed to application through the deformation $\mathcal{J}'_n$), the resulting formula and its derivation are somewhat more complicated. The characteristic polynomial is

$$\sum \binom{n}{k} \binom{k}{k_1, k_2, \ldots, k_{s+1}} \binom{n-k}{j_1, j_2, \ldots, j_s} \prod_{i=1}^{s+1} \binom{k_i}{p_i} \binom{j_i}{q_i} p_i^{k_i - p_i} (q_i + 1)^{j_i - q_i},$$

where the sum runs over all choices of

(1) an integer s between $m - n$ and $m - 1$,
(2) an integer k between 0 and n,
(3) all partitions $m - s = \sum_{i=1}^{s+1} (p_i + q_i)$,
(4) all partitions $k = \sum_{i=1}^{s+1} k_i$ such that $k_i \geq p_i$,
(5) all partitions $n - k = \sum_{i=1}^{s} j_i$ such that $j_i \geq q_i$.

The formula is obtained, just as in the proof of our main theorem, by enumerating functions

$$x : \pm[n] \to \left\{ \langle 2 \rangle,\ \langle 3 \rangle,\ \ldots,\ \left\langle \frac{q-1}{2} \right\rangle \right\}$$

such that

(1) $x_i = -x_{-i}$ and,
(2) no vop immediately follows a von.

Such functions x can be constructed by following a procedure similar to the four-step procedure in the proof of the main theorem. It is a little more complicated but essentially not too different in that they both require enumeration through partitions.

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