Commun. Korean Math. Soc. **33** (2018), No. 3, pp. 751–757
https://doi.org/10.4134/CKMS.c170308
pISSN: 1225-1763 / eISSN: 2234-3024

NIL-CLEAN RINGS OF NILPOTENCY INDEX AT MOST TWO WITH APPLICATION TO INVOLUTION-CLEAN RINGS

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Abstract. A ring is nil-clean if every element is a sum of a nilpotent and an idempotent, and a ring is involution-clean if every element is a sum of an involution and an idempotent. In this paper, a description of nil-clean rings of nilpotency index at most 2 is obtained, and is applied to improve a known result on involution-clean rings.

1. Introduction

Throughout this paper we assume that rings have an identity and the subrings share the same identity. For a ring R, the Jacobson radical and the set of nilpotents of a ring R are denoted by $J(R)$ and $\mathrm{Nil}(R)$, respectively. Recently, involution-clean rings were introduced in [3] where the author proved that the structure of an involution-clean ring is reduced to a nil-clean ring R such that $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$ (see Lemma 3.1). In this paper, we target this class of nil-clean rings, and relate them to nil-clean rings of nilpotency index at most 2. We prove a description of nil-clean rings of nilpotency index at most 2, and use it to further describe involution-clean rings.

As usual, $\mathbb{M}_n(R)$ stands for the $n \times n$ matrix ring over R and $\mathbb{T}_n(R)$ for the $n \times n$ (upper) triangular matrix ring over R. We write $\mathbb{Z}_n$ for the ring of integers modulo n. An element a in a ring R is called an involution if $a^2 = 1$. A reduced ring is a ring without nonzero nilpotents. A ring is said to be of nilpotency index at most n if $a^n = 0$ for all $a \in \mathrm{Nil}(R)$.

Received July 25, 2017; Accepted September 6, 2017.
2010 *Mathematics Subject Classification.* Primary 16U60.
Key words and phrases. idempotent, nilpotent, involution, nil-clean ring of nilpotency index at most 2, involution-clean ring.
This work was supported by the National Natural Science Foundation of China(11661014, 11461010, 11661013), the Guangxi Science Research and Technology Development Project(1599005-2-13), the Guangxi Natural Science Foundation(2016GXSFDA380017) and the Scientific Research Fund of Guangxi Education Department(KY2015ZD075).

2. Nil-clean rings of nilpotency index at most 2

Following Diesl [4], a ring R is nil-clean if every element of R is a sum of a nilpotent and an idempotent. One easily sees that a ring R is Boolean if and only if R is a nil-clean ring of nilpotency index 1. In this section, we describe nil-clean rings of nilpotency index at most 2. Notice that the structure of a general nil-clean ring is, so far, unknown.

Lemma 2.1. *The following are equivalent for a ring R:*

(1) *Every element of R is a sum of an idempotent and a square-zero element.*
(2) *R is nil-clean of nilpotency index ≤ 2.*
(3) *$R/J(R)$ is nil-clean of nilpotency index ≤ 2, and $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$.*

Proof. (1) $\Rightarrow$ (2). It suffices to show that $a^{n+1} = 0$ whenever $a^{n+2} = 0$ in R for $n \geq 1$. Write $1 + a = b + e$ where $b^2 = 0$ and $e^2 = e$. Let $f = 1 - e$. Then $f + a = b$, so

$$0 = (f + a)^2 = f + fa + af + a^2.$$

Thus, $0 = (f+fa+af+a^2)a^{n+1} = fa^{n+1}+afa^{n+1} = (1+a)fa^{n+1}$, so $fa^{n+1} = 0$ (as $1+a$ is a unit). Hence $0 = (f+fa+af+a^2)a^n = fa^n+afa^n = (1+a)fa^n$, so $fa^n = 0$. Thus, $0 = (f + fa + af + a^2)a^{n-1} = fa^{n-1} + afa^{n-1} + a^{n+1} = (1+a)fa^{n-1}+a^{n+1}$, so $0 = a[(1+a)fa^{n-1}+a^{n+1}] = (1+a)afa^{n-1}$, and hence $afa^{n-1} = 0$. It follows that $fa^{n-1} + a^{n+1} = 0$. So $0 = f[fa^{n-1} + a^{n+1}] = fa^{n-1} + fa^{n+1} = fa^{n-1}$. It follows that $a^{n+1} = 0$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (1). The implications are clear in view of [4, Corollary 3.17]. $\square$

Let $(r_\alpha) \in \prod\{R_\alpha : \alpha \in \Gamma\}$. The support of (r_α) is the subset $\Lambda = \{\alpha \in \Gamma : r_\alpha \neq 0\}$. We will denote (r_α) by $(r_\alpha)_\Lambda$. Here is a description of a nil-clean ring of nilpotentcy index ≤ 2.

Theorem 2.2. *A ring R is a nil-clean ring of nilpotency index ≤ 2 if and only if $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$ and $R/J(R)$ is a subdirect product of rings $\{R_\alpha : \alpha \in \Gamma\}$, where $R_\alpha = \mathbb{Z}_2$ or $\mathbb{M}_2(\mathbb{Z}_2)$, such that whenever $(x_\alpha)_\Lambda \in R/J(R)$ with $x_\alpha^3 = 1$ and $x_\alpha \neq 1$ for all $\alpha \in \Lambda$, there exists $(y_\alpha)_\Lambda \in R/J(R)$ with $y_\alpha \neq 0$ and $y_\alpha^2 = 0$ for all $\alpha \in \Lambda$.*

Proof. ($\Rightarrow$) By Lemmas 2.1, $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$. Moreover, $R/J(R)$ is nil-clean of nilpotentcy index ≤ 2. So, by [1, Theorem 1], $R/J(R)$ is a subdirect product of prime rings $\{R_\alpha : \alpha \in \Lambda\}$ of nilpotency index ≤ 2. Hence, by [2, Corollary 6], for each α, $R_\alpha \cong \mathbb{M}_n(D)$ where D is a division ring and $n \leq 2$. As $\mathbb{M}_n(D)$ is still nil-clean, $D = \mathbb{Z}_2$ by [5, Theorem 3]. So $R_\alpha \cong \mathbb{Z}_2$ or $R_\alpha \cong \mathbb{M}_2(\mathbb{Z}_2)$. Identify $R/J(R)$ as a subring of $\prod_\Gamma R_\alpha$.

If $R/J(R)$ contains an element $x := (x_\alpha)_\Lambda$ where $1 \neq x_\alpha \in R_\alpha$ with $x_\alpha^3 = 1$ for all $\alpha \in \Lambda$, then, as x is nil-clean in $R/J(R)$, there exists a nilpotent $y \in R/J(R)$ such that $x + y$ is an idempotent. Write $y = (y_\alpha)$ where $y_\alpha \in R_\alpha$. It must be that $y_\alpha = 0$ for $\alpha \in \Gamma\backslash\Lambda$ and $y_\alpha \neq 0$ for $\alpha \in \Lambda$. So $y = (y_\alpha)_\Lambda$.

($\Leftarrow$) We only need to show that R is nil-clean. As $J(R)$ is nil, it suffices to show that $R/J(R)$ is nil-clean by [4, Corollary 3.17]. Regard $R/J(R)$ as a subring of $\prod_{\Gamma} R_\alpha$.

Let $x \in R/J(R)$. Write $x = (x_\alpha)$ where $x_\alpha \in R_\alpha$. In R_α, there are four types of elements b: $b^2 = 0$; $b^2 = b$; $b^2 = 1$ with $b \neq 1$; $b^3 = 1$ with $b \neq 1$. Thus, we can write Γ as a disjoint union of $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4 such that $x_\alpha^2 = 0$ if and only if $\alpha \in \Lambda_1$; $x_\alpha^2 = x_\alpha$ if and only if $\alpha \in \Lambda_2$; $x_\alpha^2 = 1$ with $x_\alpha \neq 1$ if and only if $\alpha \in \Lambda_3$; $x_\alpha^3 = 1$ with $x_\alpha \neq 1$ if and only if $\alpha \in \Lambda_4$. Without loss of generality, we can denote $x = (x_\alpha) = \big((x_\alpha)_{\Lambda_1}, (x_\alpha)_{\Lambda_2}, (x_\alpha)_{\Lambda_3}, (x_\alpha)_{\Lambda_4}\big)$. We have

$$\begin{aligned} x + x^7 &= \big((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big), \\ x^2 + x^5 &= \big(\mathbf{0}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, \mathbf{0}\big), \\ \big(x^2 + x^5 + x^6 + x^7\big)^2 &= \big(\mathbf{0}, \mathbf{0}, \mathbf{0}, (x_\alpha)_{\Lambda_4}\big). \end{aligned}$$

So $(x_\alpha)_{\Lambda_4} \in R/J(R)$. By our assumption, there exists $(y_\alpha)_{\Lambda_4} \in R/J(R)$ with $y_\alpha \neq 0$ and $y_\alpha^2 = 0$ for all $\alpha \in \Lambda_4$. One can check that $(x_\alpha)_{\Lambda_4} + (y_\alpha)_{\Lambda_4} \in R/J(R)$ is an idempotent. We see that

$$\begin{aligned} y :=& \big((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, (y_\alpha)_{\Lambda_4}\big) \\ =& \big((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big) + \big(\mathbf{0}, \mathbf{0}, \mathbf{1} + (x_\alpha)_{\Lambda_3}, \mathbf{0}\big) + \big(\mathbf{0}, \mathbf{0}, \mathbf{0}, (y_\alpha)_{\Lambda_4}\big) \in R/J(R) \end{aligned}$$

is nilpotent, and

$$\big(\mathbf{0}, (x_\alpha)_{\Lambda_2}, \mathbf{1}, (x_\alpha)_{\Lambda_4} + (y_\alpha)_{\Lambda_4}\big) = x + y \in R/J(R)$$

is an idempotent. Therefore, $x = y + (x+y)$ is nil-clean in $R/J(R)$. So $R/J(R)$ is nil-clean. ☐

Corollary 2.3. *If $R/J(R) \cong S \bigoplus \big(\prod \mathbb{M}_2(\mathbb{Z}_2)\big)$ for a Boolean ring S with $J(R)$ nil such that $a^2 = 0$ for all $a \in \mathrm{Nil}(R)$, then R is nil-clean of nilpotency index ≤ 2.*

A subdirect product of a Boolean ring and a family of copies of $\mathbb{M}_2(\mathbb{Z}_2)$ need not be a nil-clean ring.

Example 2.4. Let $T = \prod_{n=1}^{\infty} R_i$ where $R_i = \mathbb{M}_2(\mathbb{Z}_2)$ for all $i \geq 1$. Let $z = (z_i) \in T$ where $z_i = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right) \in \mathbb{M}_2(\mathbb{Z}_2)$. Let S be the subring of T generated by z, i.e., $S = \{0, 1, z, 1+z\}$ where $z^2 = 1 + z$. Let $R = \big(\bigoplus_{i=1}^{\infty} R_i\big) + S$. Then R is a subdirect product of $\{R_i\}$, so $J(R) = 0$ and R has nilpotency index 2. However, although R contains z, R does not contain a nilpotent (y_i) with $y_i \neq 0$ for all $i \geq 1$. So R is not nil-clean by Theorem 2.2.

In general, it is unknown whether R nil-clean implies that the corner ring eRe ($e^2 = e \in R$) is nil-clean (see [4, Question 2]). But we have:

Corollary 2.5. *If R is a nil-clean ring of nilpotency index at most 2, then so is eRe for all $e^2 = e \in R$.*

Proof. Let $S = eRe$. Then $J(S) = eJ(R)e \subseteq J(R)$ and $\mathrm{Nil}(S) \subseteq \mathrm{Nil}(R)$. Since R is a nil-clean ring of nilpotency index at most 2, $a^2 = 0$ for all $a \in J(R) \cup \mathrm{Nil}(R)$ by Theorem 2.2, so $a^2 = 0$ for all $a \in J(S) \cup \mathrm{Nil}(S)$. Moreover, $\overline{R} := R/J(R)$ is a subdirect product of $\{R_\alpha : \alpha \in \Gamma\}$ where either $R_\alpha \cong \mathbb{Z}_2$ or $R_\alpha \cong \mathbb{M}_2(\mathbb{Z}_2)$. That is, $\overline{R}$ is a subring of $\prod R_\alpha$ such that $\pi_\alpha(\overline{R}) = R_\alpha$ where $\pi_\alpha : \prod R_\alpha \to R_\alpha$ is the natural projection for all $\alpha \in \Gamma$. Let $\bar{e} = e + J(R) \in \overline{R}$. Write $\bar{e} = (e_\alpha)$ where $e_\alpha \in R_\alpha$ is an idempotent. It is easily seen that $\bar{e}\overline{R}\bar{e}$ is a subring of $\prod e_\alpha R_\alpha e_\alpha$ with $\pi_\alpha(\bar{e}\overline{R}\bar{e}) = e_\alpha R_\alpha e_\alpha$ for all α. That is, $\bar{e}\overline{R}\bar{e}$ is a subdirect product of $\{e_\alpha R_\alpha e_\alpha\}$. We notice that, if $R_\alpha \cong \mathbb{Z}_2$, then $e_\alpha R_\alpha e_\alpha = 0$ or $e_\alpha R_\alpha e_\alpha \cong \mathbb{Z}_2$, and that, if $R_\alpha \cong \mathbb{M}_2(\mathbb{Z}_2)$, then $e_\alpha R_\alpha e_\alpha = 0$, or $e_\alpha R_\alpha e_\alpha \cong \mathbb{Z}_2$, or $e_\alpha R_\alpha e_\alpha \cong \mathbb{M}_2(\mathbb{Z}_2)$ (this only occurs when e_α is the identity of R_α). Suppose that $x = (x_\alpha)_\Lambda \in \bar{e}\overline{R}\bar{e}$ where $e_\alpha \neq x_\alpha \in e_\alpha R_\alpha e_\alpha$ with $x_\alpha^3 = e_\alpha$ for all $\alpha \in \Lambda$. It must be that, for each $\alpha \in \Lambda$, $R_\alpha \cong \mathbb{M}_2(\mathbb{Z}_2)$ and $e_\alpha = 1_{R_\alpha}$. Then, by Theorem 2.2, there exists $y = (y_\alpha)_\Lambda \in \overline{R}$ such that $y_\alpha \neq 0$ and $y_\alpha^2 = 0$. But $y = \bar{e}y\bar{e} \in \bar{e}\overline{R}\bar{e}$. Note that $S/J(S) = eRe/eJ(R)e = eRe/(eRe \cap J(R)) \cong (eRe + J(R))/J(R) = \bar{e}\overline{R}\bar{e}$. Hence, by Theorem 2.2, S is a nil-clean ring of nilpotency index at most 2. □

A ring R is strongly π-regular if for each $a \in R$, there exists $n \geq 1$ such that $a^n \in a^{n+1}R \cap Ra^{n+1}$. It is unknown whether every nil-clean ring is strongly π-regular (see [4, Question]). However, every nil-clean ring of nilpotency index at most 2 is certainly strongly π-regular.

Corollary 2.6. *If R is nil-clean of nilpotency index ≤ 2, then R is strongly π-regular.*

Proof. If $a \in J(R)$, then $a^2 = 0$. Suppose that $a \notin J(R)$. Let $x = \bar{a} \in R/J(R)$. As in the proof of Theorem 2.2, $x = (x_\alpha) = \big((x_\alpha)_{\Lambda_1}, (x_\alpha)_{\Lambda_2}, (x_\alpha)_{\Lambda_3}, (x_\alpha)_{\Lambda_4}\big)$. Moreover, $x + x^7 = \big((x_\alpha)_{\Lambda_1}, \mathbf{0}, \mathbf{0}, \mathbf{0}\big)$, so $(x + x^7)^2 = \bar{0}$, i.e., $(a + a^7)^2 \in J(R)$. Hence, $a^4(1+a^6)^4 = (a+a^7)^4 = ((a+a^7)^2)^2 = 0$, showing that $a^4 \in a^5R \cap Ra^5$. So R is strongly π-regular. □

3. Involution-clean rings

Following Danchev [3], a ring is an involution-clean ring if every element is a sum of an idempotent and an involution. The following result is proved in [3].

Lemma 3.1 ([3])**.** *A ring R is an involution-clean ring if and only if $R = A \times B$, where A is a nil-clean ring with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(A)$ and B is zero or a subdirect product of $\mathbb{Z}_3$'s.*

Next, we give a further description of the ring A in the decomposition in Lemma 3.1.

Lemma 3.2. *A ring R is nil-clean with $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$ if and only if $R/J(R)$ is nil-clean of nilpotency index ≤ 2, $J(R)$ nil and $a^2 + 2a = 0$ for all $a \in \mathrm{Nil}(R)$.*

Proof. In view of [4, Proposition 3.14 and Corollary 3.17], we see that

$$\begin{aligned}
& R \text{ is nil-clean with } a^2+2a=0 \text{ for all } a\in \mathrm{Nil}(R)\\
\Longleftrightarrow\ & R/J(R) \text{ is nil-clean with } J(R) \text{ nil and with } a^2+2a=0 \text{ for all } a\in \mathrm{Nil}(R)\\
\Longleftrightarrow\ & R/J(R) \text{ is nil-clean of nilpotency index} \le 2,\ J(R) \text{ nil and}\\
& a^2+2a=0 \text{ for all } a\in \mathrm{Nil}(R).
\end{aligned}$$

□

Theorem 3.3. *A ring R is an involution-clean ring if and only if $R\cong A\times B$, where*

(1) *B is zero or a subdirect product of $\mathbb{Z}_3$'s.*
(2) *$J(A)$ is nil, $a^2+2a=0$ for all $a\in \mathrm{Nil}(A)$, and $A/J(A)$ is a subdirect product of rings $\{A_\alpha : \alpha\in\Gamma\}$, where $A_\alpha=\mathbb{Z}_2$ or $\mathbb{M}_2(\mathbb{Z}_2)$, such that whenever $(x_\alpha)_\Lambda\in A/J(A)$ with $x_\alpha^3=1$ and $x_\alpha\neq 1$ for all $\alpha\in\Lambda$, there exists $(y_\alpha)_\Lambda\in A/J(A)$ with $y_\alpha\neq 0$ and $y_\alpha^2=0$ for all $\alpha\in\Lambda$.*

Proof. This is by Lemmas 3.1, 3.2 and Theorem 2.2. □

Corollary 3.4. *If $R/J(R)\cong S\bigoplus\big(\prod \mathbb{M}_2(\mathbb{Z}_2)\big)$ for a Boolean ring S with $J(R)$ nil such that $a^2+2a=0$ for all $a\in\mathrm{Nil}(R)$, then R is an involution-clean ring.*

As seen in Example 2.4, a subdirect product of a Boolean ring and a family of copies of $\mathbb{M}_2(\mathbb{Z}_2)$ need not be an involution-clean ring.

Next we determine when a (formal or triangular) matrix ring is involution-clean.

Proposition 3.5. *Let S,T be rings and M a non-trivial (S,T)-bimodule. Then the formal matrix ring $\left(\begin{smallmatrix} S & M\\ 0 & T\end{smallmatrix}\right)$ is an involution-clean ring if and only if S,T are involution-clean rings and* $\mathrm{Nil}(S)M=M\mathrm{Nil}(T)=2M=0$.

Proof. ($\Rightarrow$) If $x\in M$, then $\left(\begin{smallmatrix} 0 & x\\ 0 & 0\end{smallmatrix}\right)^2+2\left(\begin{smallmatrix} 0 & x\\ 0 & 0\end{smallmatrix}\right)=0$, and this shows that $2x=0$. Hence $2M=0$. Let $a\in\mathrm{Nil}(S)$ an $x\in M$. Then $\left(\begin{smallmatrix} a & x\\ 0 & 0\end{smallmatrix}\right)^2+2\left(\begin{smallmatrix} a & x\\ 0 & 0\end{smallmatrix}\right)=0$, and this shows that $ax=-2x=0$. So $\mathrm{Nil}(S)M=0$. Similarly $M\mathrm{Nil}(T)=0$. As images of $\left(\begin{smallmatrix} S & M\\ 0 & T\end{smallmatrix}\right)$, S and T are clearly involution-clean rings.

($\Leftarrow$) We write $S=A\oplus A'$ and $T=B\oplus B'$ where $8=0$ in A and in B, $A'\oplus B'$ is zero or a subdirect product of $\mathbb{F}_3$'s. Write $1_S=1_A+1_{A'}$ and $1_T=1_B+1_{B'}$. From $2M=0$, one deduces that $1_{A'}M=0$ and $M1_{B'}=0$, and that $1_Ax=x1_B=x$ for all $x\in M$. Therefore,

$$\begin{pmatrix} S & M\\ 0 & T\end{pmatrix}\cong\begin{pmatrix} A & M\\ 0 & B\end{pmatrix}\times A'\times B'.$$

Thus, we only need to show that $\left(\begin{smallmatrix} A & M\\ 0 & B\end{smallmatrix}\right)$ is an involution-ring. Let $\left(\begin{smallmatrix} a & x\\ 0 & b\end{smallmatrix}\right)\in\left(\begin{smallmatrix} A & M\\ 0 & B\end{smallmatrix}\right)$. Write $a=e+v$ and $b=f+w$ where $e^2=e$, $v^2=1$, $f^2=f$ and $w^2=1$. Then $(1+v)^2=2(1+v)\in J(A)$, so $(1+v)x=0$. Similarly, $x(1+w)=0$. Thus $vx+xw=(1+v)x+x(1+w)-2x=0$, so $\left(\begin{smallmatrix} a & x\\ 0 & b\end{smallmatrix}\right)=\left(\begin{smallmatrix} e & 0\\ 0 & f\end{smallmatrix}\right)+\left(\begin{smallmatrix} v & x\\ 0 & w\end{smallmatrix}\right)$ is a sum of an idempotent and an involution. □

Theorem 3.6. *Let R be a ring and $n\geq 2$. The following are equivalent*:

(1) $\mathbb{T}_n(R)$ *is an involution-clean ring.*
(2) $\mathbb{T}_n(R)$ *is a nil-clean ring of nilpotentcy index* ≤ 2.
(3) $n = 2$ *and* R *is Boolean.*
(4) $\mathbb{M}_n(R)$ *is a nil-clean ring of nilpotency index* ≤ 2.
(5) $\mathbb{M}_n(R)$ *is an involution-clean ring.*

Proof. (1) $\Rightarrow$ (3) Write $\mathbb{T}_n(R) = \left(\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right)$, where $S = \mathbb{T}_{n-1}(R)$ and $M = \mathbb{M}_{(n-1)\times 1}(R)$. By Proposition 3.5, $2 = 0$ in R and $\text{Nil}(S)M = 0$, from which we deduce that $n = 2$ and R is a reduced ring. As an image of $\mathbb{T}_2(R)$, R is involution-clean. Thus, R is a subdirect product of involution-clean domains in which 2 is zero. One easily sees that each of the domains is isomorphic to $\mathbb{Z}_2$, so R is Boolean.

(3) $\Rightarrow$ (2) Let $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in \mathbb{T}_2(R)$. Then $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & c \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ is a sum of an idempotent and a square-zero element.

(2) $\Rightarrow$ (1) As $2 \in \text{Nil}(R)$, $2E_{11} + E_{12}$ is nilpotent, so $0 = (2E_{11} + E_{12})^2 = 4E_{11} + 2E_{12}$. This shows that $2 = 0$ in R. For $A \in \mathbb{M}_n(R)$, write $A = E + B$ where $E^2 = E$ and $B^2 = 0$. Then $A = (1 + E) + (1 + B)$ is a sum of an idempotent and an involution.

(5) $\Rightarrow$ (4) By Lemma 3.1, $\mathbb{M}_n(R) \cong A \times B$, where $8 = 0$ in A and B is zero or a subdirect product of $\mathbb{Z}_3$'s. Thus, there exists a central idempotent e of R such that $A \cong \mathbb{M}_n(eR)$ and $B \cong \mathbb{M}_n((1-e)R)$. As $n \geq 2$, it follows from Lemma 3.1 that $e = 1$, so $8 = 0$ in $\mathbb{M}_n(R)$. As $E_{12} \in \mathbb{M}_n(R)$ is nilpotent, $(E_{12})^2 + 2E_{12} = 0$, showing that $2 = 0$ in R. For $A \in \mathbb{M}_n(R)$, write $A = E + V$ where $E^2 = E$ and $V^2 = 1$. Then $A = (1 + E) + (1 + V)$ is a sum of an idempotent and a square-zero element.

(4) $\Rightarrow$ (3) If $x^2 = 0$ in R, then $xE_{11} + E_{12} \in \mathbb{M}_n(R)$ is nilpotent; so $xE_{12} = (xE_{11} + E_{12})^2 = 0$, showing $x = 0$. Hence R is a reduced ring. As $\mathbb{M}_n(R)$ is nil-clean, R is Boolean by [6, Corollary 6.3]. Assume that $n > 2$. Then, as $E_{12} + E_{23} \in \mathbb{M}_n(R)$ is nilpotent, $E_{23} = (E_{12} + E_{23})^2 = 0$. This contradictions shows that $n = 2$.

(3) $\Rightarrow$ (5) By [6, Corollary 6.3], R is nil-clean. By Lemma 3.1, it suffices to show that $A^2 = 0$ for any nilpotent matrix A in $\mathbb{M}_2(R)$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be nilpotent in $\mathbb{M}_2(R)$. Then the determinant of A must be zero, so $ad = bc$. We have $A^2 = \left(\begin{smallmatrix} a+bc & ab+bd \\ ac+cd & bc+d \end{smallmatrix}\right)$, and

$$\begin{aligned} A^3 &= \begin{pmatrix} a + bc \cdot d & ab + b \cdot ad + bc + bd \\ ac + bc + c \cdot ad + cd & a \cdot bc + d \end{pmatrix} \\ &= \begin{pmatrix} a + ad & ab + bc + bc + bd \\ ac + bc + bc + cd & ad + d \end{pmatrix} \\ &= \begin{pmatrix} a + bc & ab + bd \\ ac + cd & bc + d \end{pmatrix} = A^2. \end{aligned}$$

It follows that $A^2 = 0$. $\square$

Example 3.7. $\mathbb{Z}_8$ is an involution-clean ring, but 2 is not a sum of an idempotent and a square-zero element. The trivial extension $\mathbb{Z}_4 \propto \mathbb{Z}_4$ is not an involution-clean, but is a nil-clean ring with index of nilpotency ≤ 2.

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