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A GENERALIZATION OF ARMENDARIZ AND NI PROPERTIES

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Abstract. Antoine showed that the properties of Armendariz and NI are independent of each other. The study of Armendariz and NI rings has been doing important roles in the research of zero-divisors in noncommutative ring theory. In this article we concern a new class of rings which generalizes both Armendariz and NI rings. The structure of such sort of ring is investigated in relation with near concepts and ordinary ring extensions. Necessary examples are examined in the procedure.

1. Introduction

Throughout this paper every ring is associative with identity unless otherwise stated. Let R be a ring. The upper nilradical (i.e., the sum of all nil ideals), the lower nilradical (i.e., the intersection of all prime ideals), the Jacobson radical, and the set of all nilpotents in R are denoted by $N^*(R)$, $N_*(R)$, $J(R)$, and $N(R)$, respectively. It is well-known that $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The set of all idempotents and group of units in R are written by $I(R)$ and $U(R)$, respectively. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$. Let $C_{f(x)}$ denote the set of all coefficients of given a polynomial $f(x)$. $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the ring of integers and the ring of integers modulo n, respectively. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$). Use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0. Let $S\langle a_1, a_2, \ldots, a_n\rangle$ denote the free algebra generated by the noncommuting indeterminates $a_1, a_2, \ldots, a_n$ over a commutative ring S. $S\{a_1, a_2, \ldots, a_n\}$ denotes the set of all polynomials of zero constant term in $S\langle a_1, a_2, \ldots, a_n\rangle$.

A ring is usually called *reduced* if it has no nonzero nilpotent elements. Following Rege and Chhawchharia [11, Definition 1.1], a ring R is called *Armendariz* if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever $f(x)g(x) = 0$ for $f(x), g(x) \in R[x]$. Reduced rings are shown to be Armendariz by [3, Lemma 1],

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but not conversely by [9, Proposition 2]. A ring is usually said to be *Abelian* if every idempotent is central. Let R be an Armendariz ring. Then R is Abelian and $N(R)$ is a subring of R by [7, Corollary 8] and [2, Corollary 3.3], respectively.

Following Marks [10], a ring R is called *NI* if $N^*(R) = N(R)$. Reduced rings are clearly NI. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. Notice that Armendariz and NI properties are independent of each other as can be seen by the Armendariz ring in [2, Example 4.8] that is not NI, and the NI ring $T_n(R)$ that is not Abelian (hence not Armendariz), where R is a reduced ring and $n \geq 2$.

The study of Armenariz and NI rings has provided many sorts of information for zero-divisors which have been doing very important roles in ring theory. We will introduce a ring property which generalizes both Armendariz and NI ring properties.

Definition 1.1. A ring R (possibly without identity) is said to be *inserting-preserves-nilpotent* (simply, *IPN*) provided that if $f(x)g(x) \in N(R[x])$ for $f(x), g(x) \in R[x]$, then $aN(R)b \subseteq N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$.

The following lemma does basic roles in this article. We use $\oplus$ for the direct sum of rings.

Lemma 1.2. (1) *A finite direct sum of IPN rings is IPN.*

(2) *Let R be a ring and I be a nil ideal of R. If R/I is an IPN ring, then R is IPN.*

(3) *The class of IPN rings is closed under subrings* (*possibly without identity*).

Proof. (1) Let R_i be IPN rings for $i = 1, 2, \ldots, n$, and $R = \oplus_{i=1}^n R_i$. Note first $N(R) = \oplus N(R_i)$. Suppose that $f(x)g(x) \in N(R[x])$ for $f(x) = \sum_{h=0}^m (a(i))_h x^h$, $g(x) = \sum_{k=0}^l (b(i))_k x^k \in R[x]$, where $(a(i))_h, (b(i))_k \in R$. Write

$$(a(i))_h = (a(1)_h, a(2)_h, \ldots, a(n)_h) \text{ and } (b(i))_k = (b(1)_k, b(2)_k, \ldots, b(n)_k).$$

Then we have $f(x)_i g(x)_i \in N(R_i[x])$ for all i with

$$f(x)_i = \sum_{h=0}^m a(i)_h x^h \text{ and } g(x)_i = \sum_{k=0}^l b(i)_k x^k,$$

noting that $f(x)g(x)$ can be rewritten by

$$(f(x)_1 g(x)_1, f(x)_2 g(x)_2, \ldots, f(x)_n g(x)_n).$$

Since R_i is IPN, we have $c_i \alpha_i d_i \in N(R_i)$ for all $c_i \in C_{f(x)_i}$, $d_i \in C_{g(x)_i}$, and $\alpha_i \in N(R_i)$. Let here $c = (c_1, c_2, \ldots, c_n)$, $d = (d_1, d_2, \ldots, d_n)$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then c, d, α are also arbitrary in $C_{f(x)}$, $C_{g(x)}$, and $N(R)$, respectively. From these results, we now have $c\alpha d \in N(R)$ and $cN(R)d \subseteq N(R)$ follows, recalling $N(R) = \oplus N(R_i)$. Therefore R is IPN.

(2) Write $\bar{R} = R/I$ and note $\bar{N}(R) \subseteq N(\bar{R})$. Assume that R/I is IPN. Let $f(x)g(x) \in N(R[x])$ for $f(x), g(x) \in R[x]$. Then $\bar{f}(x)\bar{g}(x) \in N(\bar{R}[x])$ clearly.

Since $\bar{R}$ is IPN, $\bar{a}\bar{\alpha}\bar{b} \in N(\bar{R})$ for all $a \in C_{f(x)}, b \in C_{g(x)}$, and $\alpha \in N(R)$. This implies $(a\alpha b)^h \in I$ for some $h \geq 1$. But since I is nil, $[(a\alpha b)^h]^k = 0$ for some $k \geq 1$. Thus we now $aN(R)b \subseteq N(R)$, and so R is IPN.

(3) Let R be an IPN ring and S be a subring (possibly without identity) of R. Note $N(R) \cap S = N(S)$ and $N(R[x]) \cap S[x] = N(S[x])$. Let $f(x)g(x) \in N(S[x])$ for $f(x), g(x) \in S[x]$. Then $f(x)g(x) \in N(R[x])$. Since R is IPN, $aN(R)b \subseteq N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$. This implies

$$aN(S)b \subseteq S \cap aN(R)b \subseteq S \cap N(R) = N(S),$$

concluding that S is IPN. □

From Lemma 1.2, we obtain the following elementary fact.

Corollary 1.3. *A ring R is IPN if and only if both eR and $(1-e)R$ are IPN for a central idempotent e in R.*

Proof. The proof comes from Lemma 1.2(1, 3), noting $R \cong eR \oplus (1-e)R$. □

The class of IPN rings contains both Armendariz and NI rings as we see in the following.

Proposition 1.4. (1) *Armendariz rings are IPN.*
(2) *NI rings are IPN.*

Proof. (1) Let R be an Armendariz ring. Let $f(x)g(x) \in N(R[x])$ for $f(x), g(x) \in R[x]$. Say $(f(x)g(x))^m = 0$. Then $(ab)^m = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$, by help of [1, Proposition 1], entailing $ab \in N(R)$. Note $ba \in N(R)$.

$N(R)$ forms a subring of R by [2, Corollary 3.3], and so $ba\alpha \in N(R)$ for all $\alpha \in N(R)$. This yields $a\alpha b \in N(R)$, entailing $aN(R)b \subseteq N(R)$. Thus R is IPN.

(2) Let R be an NI ring. Then $N^*(R) = N(R)$, and so $aN(R)b \subseteq N(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$, where $f(x)$ and $g(x)$ are arbitrary in $R[x]$. Thus R is IPN. □

In the following we construct an IPN ring that is neither Armendariz nor NI.

Example 1.5. (1) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by a, b over K. Let I be the ideal of A generated by b^m for $m \geq 2$ and set $R_0 = A/I$. Then R_0 is Armendariz by [2, Example 4.8]. However R_0 is not NI as can be seen by $\bar{b} \in N(R_0)$ but $\bar{b}\bar{a} \notin N(R_0)$, i.e., $\bar{b} \notin N^*(R_0)$.

Next consider $R = T_n(R_0)$ for $n \geq 2$. Then R is not Armendariz because R is not Abelian, recalling that Armendariz rings are Abelian. Moreover R is not NI by help of [8, Proposition 4.1] because R_0 is not NI.

We will see that R is IPN. Let $I = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$. Then I is a nilpotent ideal of R such that R/I is isomorphic to the product of n-copies of R_0 (i.e., $\frac{R}{I} \cong \oplus_{i=1}^n R_i$ with $R_i = R_0$ for all i). But Armendariz rings are

IPN by Proposition 1.4(1), and so R/I is IPN by Lemma 1.2(1). Furthermore, since I is nil, R is IPN by Lemma 1.2(2).

(2) Let A be any ring and consider $R = \mathrm{Mat}_n(A)$ for $n \geq 2$. R cannot be IPN because $E_{12}E_{11} = 0$ but $E_{12}E_{21}E_{11} = E_{11} \notin N(R)$, noting $E_{21} \in N(R)$ and $0 \neq E_{11} \in I(R)$.

Recall that a ring R is said to be *directly finite* (or *Dedekind finite*) if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Abelian rings are easily shown to be directly finite.

Proposition 1.6. *IPN rings are directly finite.*

Proof. Let R be an IPN ring and assume on the contrary that R is not directly finite. Then $ab = 1$ and $ba \neq 1$ for some $a, b \in R$. Consider $a + ba$ and $1 - ba$. Then $(a + ba)(1 - ba) = 0$. Let $\alpha = b(1 - ba)$. Then $\alpha^2 = 0$. Since R is IPN, we get $(a + ba)\alpha(1 - ba) \in N(R)$. But

$$\begin{aligned}(a + ba)\alpha(1 - ba) &= (a + ba)(b(1 - ba))(1 - ba)\\ &= (a + ba)b(1 - ba)\\ &= (1 - ba) + b(1 - ba) \in I(R).\end{aligned}$$

Assume $(1 - ba) + b(1 - ba) = 0$. Then $0 = a(1 - ba) + ab(1 - ba) = 1 - ba \neq 0$. This induces a contradiction. Thus $(a + ba)\alpha(1 - ba) \notin N(R)$, contrary to R being IPN. Therefore R is directly finite. □

By Propositions 1.4(2) and 1.6, we obtain the following.

Corollary 1.7 ([8, Proposition 2.7(1)])**.** *NI rings are directly finite.*

The converse of Proposition 1.6 need not hold as can be seen by $\mathrm{Mat}_n(A)$, over any Artinian ring A for $n \geq 2$, which is Artinian (hence directly finite) but not IPN by Example 1.5(2).

By Proposition 1.6, the class of directly finite rings contains both Abelian rings and IPN rings. In the following we see that the properties of Abelian and IPN are independent of each other.

Example 1.8. (1) We follow the construction of [2, Example 4.11]. Let K be a field and $A = K\langle a, a^{-1}, b\rangle$. Let I be the ideal of A generated by

$$1 - aa^{-1},\ 1 - a^{-1}a,\ \text{and } b^2.$$

Set $R = A/I$ and identify a, a^{-1}, and b with their images in R for simplicity. Then $aa^{-1} = 1 = a^{-1}a$ and $b^2 = 0$. So

$$a(a^{-1}b) = b \in N(R) \text{ but } ab(a^{-1}b) \notin N(R),$$

in spite of $b \in N(R)$. So R is not IPN. We next show that R is Abelian.

Every element of R is able to be expressed by

$$k + f + a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1},$$

where $k \in K$, $f \in K\{a, a^{-1}, b\}$, $g_i \in R$ for $i = 1, 2, 3$, every term of f neither starts nor ends by a^{-1} when nonzero, every term of g_1 does not end by a^{-1} when nonzero, and every term of g_2 does not start by a^{-1} when nonzero. Suppose $e^2 = e$ for $e = k + f + a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1} \in R$. Then $k^2 = k$, so $k = 0$ or $k = 1$.

Case 1. $k = 0$, i.e., $e = f + a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}$.

From $e^2 = e$, we obtain

$$f^2 + fa^{-1}g_1 + g_2a^{-1}f + g_2a^{-1}a^{-1}g_1 = f$$

and

$$fg_2a^{-1} + fa^{-1}g_3a^{-1} + a^{-1}g_1e + g_2a^{-1}g_2a^{-1} + g_2a^{-1}a^{-1}g_3a^{-1} + a^{-1}g_3a^{-1}e$$
$$= a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}$$

by the properties of f, g_1, g_2. From $f^2 + fa^{-1}g_1 + g_2a^{-1}f + g_2a^{-1}a^{-1}g_1 = f$, we get

$$g_2a^{-1}a^{-1}g_1 = f \text{ and } f^2 + fa^{-1}g_1 + g_2a^{-1}f = 0$$

by the property of f. Now we have $f^2 = 0$ from the equality $f^2 + fa^{-1}g_1 + g_2a^{-1}f = 0$ also by the property of f, entailing $f = 0$. Consequently we have $e = a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}$. From $e^2 = e$, we obtain

$$g_2a^{-1}a^{-1}g_1 = 0$$

and

$$a^{-1}g_1e + g_2a^{-1}g_2a^{-1} + g_2a^{-1}a^{-1}g_3a^{-1} + a^{-1}g_3a^{-1}e = a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}.$$

The equality $g_2a^{-1}a^{-1}g_1 = 0$ implies that $g_1 = 0$ or $g_2 = 0$.

Suppose $g_1 = 0$. Then $e = g_2a^{-1} + a^{-1}g_3a^{-1}$ and

$$g_2a^{-1}g_2a^{-1} + g_2a^{-1}a^{-1}g_3a^{-1} + a^{-1}g_3a^{-1}g_2a^{-1} + a^{-1}g_3a^{-1}a^{-1}g_3a^{-1}$$
$$= g_2a^{-1} + a^{-1}g_3a^{-1}.$$

The second equality yields

$$g_2a^{-1}g_2a^{-1} + g_2a^{-1}a^{-1}g_3a^{-1} = g_2a^{-1} \text{ and}$$
$$a^{-1}g_3a^{-1}g_2a^{-1} + a^{-1}g_3a^{-1}a^{-1}g_3a^{-1} = a^{-1}g_3a^{-1}$$

by the property of g_2. The first equality implies $g_2 = 0$. Hence we have $e = a^{-1}a^{-1}g_3a^{-1}$. Next, from $e^2 = e$, we get $a^{-1}a^{-1}g_3a^{-1}a^{-1}a^{-1}g_3a^{-1} = a^{-1}a^{-1}g_3a^{-1}$, entailing $g_3 = 0$. Therefore $e = 0$.

Suppose $g_2 = 0$. Then we also get $e = 0$ through a similar method.

Summarizing, $e = 0$ when $k = 0$.

Case 2. $k = 1$, i.e., $e = 1 + f + a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}$.

Let $e_0 = f + a^{-1}g_1 + g_2a^{-1} + a^{-1}g_3a^{-1}$. Form $e^2 = e$, we obtain

$$1 + e_0 = (1 + e_0)^2 = 1 + 2e_0 + e_0^2 \text{ and } e_0^2 = -e_0.$$

Through a similar method to the case 1, we can obtain $e_0 = 0$. Thus $e = 1$ when $k = 1$.

By Cases 1 and 2, we can conclude that $e = 1$ or $e = 0$, i.e., $I(R) = \{0, 1\}$. Therefore R is Abelian.

(2) $T_n(R)$ is clearly NI (hence IPN by Proposition 1.4(2)) over a reduced ring R, but $T_n(R)$ is non-Abelian when $n \geq 2$ (also see Example 1.5(1)).

Following [6], a ring R is said to be *von Neumann regular* if for every $a \in R$ there exists $b \in R$ such that $a = aba$. When given a ring is von Neumann regular, the ring properties above are equivalent as we see in the following.

Theorem 1.9. *For a von Neumann regular ring R, the following conditions are equivalent*:
(1) *R is reduced*;
(2) *R is Armendariz*;
(3) *R is NI*;
(4) *R is Abelian*;
(5) *R is IPN*

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) are stated above. The implications (2) $\Rightarrow$ (5) and (3) $\Rightarrow$ (5) come from Proposition 1.4. (4) $\Rightarrow$ (1) is shown by [6, Theorem 3.2] because R is von Neumann regular. (1) $\Rightarrow$ (3) is obvious. Since R is von Neumann regular, $J(R) = 0$ (hence $N^*(R) = 0$). So R is reduced if R is NI, showing (3) $\Rightarrow$ (1).

(5) $\Rightarrow$ (1): Let R be IPN. Assume on the contrary that there exists $0 \neq a \in R$ with $a^2 = 0$. Since R is von Neumann regular, there exists $b \in R$ such that $a = aba$. Let

$$f(x) = a + ax \text{ and } g(x) = ab + bax$$

in $R[x]$. Then $f(x)g(x) = ax + ax^2 \in N(R[x])$. In fact, $(f(x)g(x))^2 = 0$. Since R is IPN, $aN(R)ab \subseteq N(R)$. Note $(bab(1 - ba))^2 = 0$. So $a(bab(1 - ba))ab \in N(R)$. But

$$\begin{aligned} a(bab(1 - ba))ab &= (aba)b(ab - ba^2b) \\ &= ab(ab - 0) \\ &= ab \in I(R) \end{aligned}$$

and $ab \neq 0$. This induces a contradiction. Thus $a = 0$ and so R is reduced. □

Considering Proposition 1.6, one may ask whether von Neumann regular directly finite rings are IPN (hence reduced by Theorem 1.9). However the answer is negative as can be seen by $\mathrm{Mat}_n(R)$ over a division ring R for $n \geq 2$. This $\mathrm{Mat}_n(R)$ is semisimple Artinian (hence von Neumann regular and directly finite), but not reduced.

2. Examples of IPN rings

In this section we argue about the IPN property of several kinds of rings which have roles in ring theory. Let R be a ring R and $n \geq 2$. Following the literature, let

$$D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$$

and

$V_n(R)=\{(a_{ij})\in D_n(R)\,|\,a_{st}=a_{(s+1)(t+1)}$ for $s=1,\dots,n-2$ and $t=2,\dots,n-1\}$.

Note that $V_n(R)\cong\frac{R[x]}{x^nR[x]}$ via $(a_{ij})\mapsto \bar{a}_{11}+\bar{a}_{12}x+\bar{a}_{13}x^2+\cdots+\bar{a}_{1n}x^{n-1}$.

Theorem 2.1. *Let R be a ring and $n\geq 2$. The following conditions are equivalent*:

(1) *R is IPN*;
(2) *$T_n(R)$ is IPN*;
(3) *$D_n(R)$ is IPN*;
(4) *$V_n(R)$ is IPN.*

Proof. It suffices to prove (1) implying (2) by help of Lemma 1.2(3). Let R be IPN. Set

$$I=\{(a_{uv})\in T_n(R)\mid a_{ii}=0 \text{ for all } i=1,2,\dots,n\}.$$

Then I is a nil ideal of $T_n(R)$ and $T_n(R)/I$ is isomorphic to the product of n-copies of R. So $T_n(R)/I$ is IPN by Lemma 1.2(1) because R is IPN. Moreover since I is nil, $T_n(R)$ is IPN by Lemma 1.2(2). □

We can obtain the following through almost similar method in the proof of Theorem 2.1.

Proposition 2.2. *Let R,S be rings and ${}_RM_S$ be an R-S-bimodule. Then both R and S are IPN if and only if $\left(\begin{smallmatrix}R & M\\ 0 & S\end{smallmatrix}\right)$ is an IPN ring.*

Let R be an algebra (possibly without identity) over a commutative ring S. Following [5], the *Dorroh extension* of R by S is the Abelian group $D=R\oplus S$ with multiplication given by $(r_1,s_1)(r_2,s_2)=(r_1r_2+s_1r_2+s_2r_1,s_1s_2)$, where $r_i\in R$ and $s_i\in S$.

Proposition 2.3. (1) *Let R be a unitary algebra over a reduced commutative ring S. Then R is IPN if and only if so is the Dorroh extension D of R by S.*

(2) *Let R be a nil unitary algebra over a commutative ring S. Then R is IPN if and only if so is the Dorroh extension D of R by S.*

Proof. (1) It suffices to prove the necessity by help of Lemma 1.2(3). Since R has 1, every $s\in S$ is identified with $s1\in R$. So if $(r,s)(r',s')=0$ for $(r,s),(r',s')\in D$, then $(r+s)(r'+s')=0$. We will use this fact freely. Suppose that R is IPN and let $f(x)g(x)\in N(D[x])$ for $f(x)=\sum_{i=0}^m a_ix^i, g(x)=\sum_{j=0}^n b_jx^j\in D[x]$ with $a_i=(r_i,s_i)$ and $b_j=(r'_j,s'_j)$. Let

$$\alpha_i=r_i+s_i \text{ and } \beta_j=r'_j+s'_j.$$

Then we get $F(x)G(x)\in N(R[x])$ for

$$F(x)=\sum_{i=0}^m\alpha_ix^i \text{ and } G(x)=\sum_{j=0}^n\beta_jx^j \text{ in } R[x].$$

Since R is IPN, $\alpha_i u \beta_j \in N(R)$ for all α_i, β_j, and $u \in N(R)$. But since S is reduced, we have

$$N(D) = \{(u,0) \in D \mid u \in N(R)\}.$$

So we obtain

$$\begin{aligned} a_i(u,0)b_j &= (r_i, s_i)(u,0)(r'_j, s'_j) \\ &= (r_i u r'_j + r_i u s'_j + s_i u r'_j + s_i u s'_j, 0) \\ &= (\alpha_i u \beta_j, 0) \in N(D). \end{aligned}$$

This implies $a_i N(D) b_j \subseteq N(D)$, and therefore D is IPN.

(2) Let $I = \{(r,0) \in D \mid r \in R\}$. Then I is a nil ideal of D. Moreover D/I is isomorphic to S. Since S is commutative, D/I is IPN by Proposition 1.4(2). Since I is nil, D is IPN by Lemma 1.2(2). $\square$

An element u of a ring R is usually said to be *right regular* if $ur = 0$ for $r \in R$ implies $r = 0$. A *left regular* element is defined analogously, and an element is said to be *regular* if it is both left and right regular.

Proposition 2.4. *Let R be a ring and M be a multiplicatively closed subset of R which consists of central regular elements. Then R is an IPN ring if and only if so is RM^{-1}.*

Proof. Write $E = RM^{-1}$. Let $rm^{-1} \in N(E)$ with $r \in R$ and $m \in M$. Say $(rm^{-1})^l = 0$ for $l \geq 1$. Then $0 = (rm^{-1})^l = r^l m^{-l}$ and $r^l = 0$ follows. So we have

$$N(E) = \{rm^{-1} \mid r \in N(R) \text{ and } m \in M\}$$

because $r^t = 0$ for some $t \geq 1$ implies $(rm^{-1})^t = 0$ for all $m \in M$. It suffices to prove the necessity by help of Lemma 1.2(3). Let $f(x)g(x) \in N(E[x])$ for $f(x) = \sum_{i=0}^m \alpha_i x^i, g(x) = \sum_{j=0}^n \beta_j x^j \in E[x]$. There exist $u, v \in M$ such that $\alpha_i = a_i u^{-1}, \beta_j = b_j v^{-1}$ with $a_i, b_j \in R$ for all i, j. Say $[f(x)g(x)]^k = 0$ for $k \geq 1$. Let $f_1(x) = \sum_{i=0}^m a_i x^i, g_1(x) = \sum_{j=0}^n b_j x^j \in R[x]$. Then $f(x) = f_1(x)u^{-1}$ and $g(x) = g_1(x)v^{-1}$. Moreover we have

$$0 = [f(x)g(x)]^k = [f_1(x)u^{-1}g_1(x)v^{-1}]^k = [f_1(x)g_1(x)]^k(uv)^{-1}$$

and this implies $[f_1(x)g_1(x)]^k = 0$. If R is IPN, then $a_i N(R) b_j \subseteq N(R)$ for all i, j. This yields

$$\alpha_i(rm^{-1})\beta_j = a_i u^{-1}(rm^{-1})b_j v^{-1} = a_i r b_j (umv)^{-1} \in N(E)$$

for all i, j, where $rm^{-1} \in N(E)$. Therefore RM^{-1} is IPN. $\square$

The ring of *Laurent polynomials* in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^n r_i x^i$ with the usual addition and multiplication, where $r_i \in R$ and k, n are (possibly negative) integers. This ring is usually written by $R[x; x^{-1}]$.

Corollary 2.5. *Let R be a ring. $R[x]$ is an IPN ring if and only if so is $R[x; x^{-1}]$.*

Proof. Let $M = \{1, x, x^2, \ldots\}$. Then M is clearly a multiplicatively closed subset of central regular elements in $R[x]$. Moreover $R[x; x^{-1}] = R[x]M^{-1}$. So we obtain the result by help of Proposition 2.4. □

Following Birkenmeier et al. [4], a ring R is called 2-*primal* if $N_*(R) = N(R)$. 2-primal rings are clearly NI (hence IPN), but the converse need not be true as we see in [8, Exmaple 1.2] and [10, Example 2.2]. The property of 2-primal goes up to polynomial rings by [4, Proposition 2.6]. So there are many IPN rings over which polynomial rings are also IPN.

But the preceding argument is not valid for NI rings. There exist NI rings over which the polynomial rings need not be NI by the ring construction of Smoktunowicz in [12]. In fact, we do not know of any example of an IPN ring over which the polynomial ring is not IPN. But it does seem possible for the polynomial ring $R[x]$ over the NI ring R, constructed by Smoktunowicz, to be not IPN. We end our article by raising the following.

Question. Is $R[x]$ IPN when R is an IPN ring?

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References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272.

[2] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140.

[3] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[4] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, *Completely prime ideals and associated radicals*, in Ring theory (Granville, OH, 1992), 102–129, World Sci. Publ., River Edge, NJ, 1993.

[5] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), no. 2, 85–88.

[6] K. R. Goodearl, *von Neumann Regular Rings*, Monographs and Studies in Mathematics, **4**, Pitman (Advanced Publishing Program), Boston, MA, 1979.

[7] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

[8] S. U. Hwang, Y. C. Jeon, and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), no. 1, 186–199.

[9] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488.

[10] G. Marks, *On 2-primal Ore extensions*, Comm. Algebra **29** (2001), no. 5, 2113–2123.

[11] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17.

[12] A. Smoktunowicz, *Polynomial rings over nil rings need not be nil*, J. Algebra **233** (2000), no. 2, 427–436.

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