JORDAN GENERALIZED DERIVATIONS ON TRIVIAL EXTENSION ALGEBRAS

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Abstract. In this paper, we investigate the problem of describing the form of Jordan generalized derivations on trivial extension algebras. One of the main results shows, under some conditions, that every Jordan generalized derivation on a trivial extension algebra is the sum of a generalized derivation and an antiderivation. This result extends the study of Jordan generalized derivations on triangular algebras (see [12]), and also it can be considered as a "generalized" counterpart of the results given on Jordan derivations of a trivial extension algebra (see [11]).

1. Introduction and preliminaries

Throughout the paper $\mathcal{R}$ will denote a commutative ring with identity, A will be a unital $\mathcal{R}$-algebra with center $Z(A)$ and M will be a unital A-bimodule.

For $a \in A$ and $m \in M$, we use $a \circ m$ (resp., $[a, m]$) to denote the Jordan product $am + ma$ (resp., the Lie product $am - ma$) of a and m.

Let $d : A \longrightarrow M$ and $f : A \longrightarrow M$ be linear maps. Recall that f is said to be a generalized d-derivation (or simply a generalized derivation) if

$$(1.1) \qquad f(ab) = f(a)b + ad(b) \quad (a, b \in A).$$

For $d = f$, a generalized d-derivation f is just the classical derivation. Following [12], f is said to be a Jordan generalized d-derivation (or simply a Jordan generalized derivation) if

$$(1.2) \qquad f(a \circ b) = f(a) \circ b + a \circ d(b) \quad (a, b \in A).$$

For $d = f$, a Jordan generalized d-derivation f is just the classical Jordan derivation.

Describing various kind of derivation on some algebra constructions has been the subject of several interesting works. It mainly helps to construct new interesting examples of algebras satisfying preassigned conditions. In particular,

the construction of trivial extension algebras, which can be seen as a generalization of triangular algebras, has been used by many authors (see for example [5, 9]) and in various contexts in order to produce new family of particular examples or to resolve some open questions (see for instance [17]). In this paper, we mainly deal with the problem of describing the form of Jordan generalized derivations on trivial extension algebras. One of our main result (Theorem 2.19) shows that, where the algebra A and the A-bimodule M are 2-torsion free, if there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$, and such that $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation. This result generalizes the study of Jordan generalized derivations on triangular algebras done by Li and Benkonič in [12]. Also, it can be considered as a "generalized" counterpart of Ghahramani's main result in [11]. To prove it, several preparatory results are given. Namely, we first characterize, in Section 2, the general form of Jordan generalized derivations, generalized derivations and antiderivations on trivial extension algebras (see Lemmas 2.1, 2.2 and 2.3). Then, we characterize in terms of the form of their components when every Jordan generalized derivation on a trivial extension algebra can be written as a sum of a generalized derivation and an antiderivation (see Theorem 2.5). This approach allows us to treat each component of a Jordan generalized derivation on a trivial extension algebra separately. For this, a few lemmas are given (see Lemmas 2.8 to 2.18). The method followed in this paper allows us to establish other new situations than those ones presented in [11], in which Jordan generalized derivations are described (see Theorems 2.7 and 2.21).

In [2, Theorem 1.3], Benkovič proved (under some conditions) that every f-derivation is a Jordan derivation. Then, naturally one can ask whether there exists a "generalized" counterpart of Benkovič's results. In Section 3, we answer this natural question positively.

For the reader's convenience we briefly recall the constructions of trivial extension algebras and triangular algebras. Recall that the direct product $A \times M$ together with the pairwise addition, scalar product and the algebra multiplication defined by

$$(a, m)(b, n) = (ab, an + mb) \qquad (a, b \in A, m, n \in M),$$

is a unital algebra which is called a trivial extension of A by M and will be denoted by $A \ltimes M$.

The class of trivial extension algebras includes a wide variety of algebras includes a triangular algebra

$$\text{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : \ a \in A, m \in M, b \in B \right\};$$

where, A and B are unital algebras and M is a unital (A, B)-module, which is faithful as a left A-module as well as a right B-module. Indeed, it can be

readily verified that $\mathrm{Tri}(A, M, B)$ is isomorphic to the trivial extension algebra $(A \oplus B) \ltimes M$, where the algebra $A \oplus B$ has its usual pointwise operations and M as an $(A \oplus B)$-module is equipped with the module operations $(a, b)m = am$ and $m(a, b) = mb$; $(a \in A, b \in B, m \in M)$. Note that $A \ltimes M$ is naturally isomorphic to the subalgebra of $\mathrm{Tri}(A, M, A)$ consisting of matrices $\left(\begin{smallmatrix} a & m \\ & a \end{smallmatrix}\right)$ where $a \in A$ and $m \in M$.

Triangular algebras introduced by Cheung [7]; see also [6]. Upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras are standard examples of triangular algebras. Following [6], an algebra A is said to have a triangular matrix representation if A is isomorphic to a triangular algebra. By [6, Theorem 5.1.4], a unital algebra A has a triangular matrix representation if there exists a nontrivial idempotent $e \in A$ such that $(1 - e)Ae = 0$. Namely, in this case, A is isomorphic to $\mathrm{Tri}(eAe, eA(1 - e), (1 - e)A(1 - e))$.

2. Jordan generalized derivations on $A \ltimes M$

Our aim is to study a Jordan generalized derivation on a trivial extension algebra. We give conditions under which it is a sum of a generalized derivation and an antiderivation. Let us start with a general description of these kind of mappings on a trivial extension algebra.

Clearly, every linear mapping $f : A \ltimes M \longrightarrow A \ltimes M$ can be presented in the form

$$(2.1) \qquad f(a, m) = (f_A(a) + h_1(m), f_M(a) + h_2(m)) \qquad ((a, m) \in A \ltimes M),$$

where the linear mappings $f_A : A \longrightarrow A$, $f_M : A \longrightarrow M$, $h_1 : M \longrightarrow A$ and $h_2 : M \longrightarrow M$ are given by $f_A(a) = (\pi_A \circ f)(a, 0)$, $f_M(a) = (\pi_M \circ f)(a, 0)$, $h_1(m) = (\pi_A \circ f)(0, m)$ and $h_2(m) = (\pi_M \circ f)(0, m)$, respectively. Here $\pi_A : A \ltimes M \longrightarrow A$ and $\pi_M : A \ltimes M \longrightarrow M$ are the natural projections given by $\pi_A(a, m) = a$ and $\pi_M(a, m) = m$, respectively.

In the sequel, we suppose that f has a presentation given as in (2.1), and a linear map d on $A \ltimes M$ with a presentation as follows

$$d(a, m) = (d_A(a) + T(m), d_M(a) + S(m)), \quad ((a, m) \in A \ltimes M).$$

The following three lemmas are obtained using standard arguments.

Lemma 2.1. *A linear map f is a Jordan generalized d-derivation if and only if the following conditions hold*:

    (1) *f_A is a Jordan generalized d_A-derivation,*

    (2) *f_M is a Jordan generalized d_M-derivation,*

    (3) *$h_1(a \circ m) = a \circ h_1(m) = a \circ T(m)$ for all $a \in A$ and $m \in M$,*

    (4) *$h_2(a \circ m) = f_A(a) \circ m + a \circ S(m) = a \circ h_2(m) + d_A(a) \circ m$ for all $a \in A$ and $m \in M$,*

    (5) *$m \circ h_1(n) + h_1(m) \circ n = 0$ for all $m, n \in M$.*

Lemma 2.2. *A linear map f is a generalized d-derivation if and only if the following conditions hold*:

(1) f_A is a generalized d_A-derivation,
(2) f_M is a generalized d_M-derivation,
(3) $h_1(am) = ah_1(m)$ and $h_1(ma) = h_1(m)a$ for all $a \in A$ and $m \in M$,
(4) $h_2(am) = f_A(a)m + aS(m)$ and $h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$ and $m \in M$,
(5) $mh_1(n) + h_1(m)n = 0$ for all $m, n \in M$.

Lemma 2.3. *A linear map f is an antiderivation if and only if the following conditions hold*:

(1) f_A and f_M are antiderivations,
(2) $h_1(am) = h_1(m)a$ and $h_1(ma) = ah_1(m)$ for all $a \in A$ and $m \in M$,
(3) $h_2(am) = h_2(m)a + mf_A(a)$ and $h_2(ma) = ah_2(m) + f_A(a)m$ for all $a \in A$ and $m \in M$,
(4) $mh_1(n) + h_1(m)n = 0$ for all $m, n \in M$.

Remark 2.4.   (1) Notice that when $A \ltimes M$ has a triangular matrix representation, $h_1 = 0$ for a Jordan generalized derivation f on $A \ltimes M$. However, in general h_1 is not zero. For this we use the example given in [1]: Consider the trivial extension $M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ where $M_2(\mathbb{Z}/2\mathbb{Z})$ is the algebra of 2×2 matrices with entries from $\mathbb{Z}/2\mathbb{Z}$. Consider the identity map $h_1 : M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z})$. Since the map h_1 verified (3) and (5) in Lemma 2.1, the linear map $f : M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z}) \to M_2(\mathbb{Z}/2\mathbb{Z}) \ltimes M_2(\mathbb{Z}/2\mathbb{Z})$ defined by $f((a,b)) = (h_1(b), 0)$ for all $a, b \in M_2(\mathbb{Z}/2\mathbb{Z})$ is a Jordan generalized derivation with $h_1 \neq 0$. However, using the equation (5) in Lemma 2.1, we can give a situation where $h_1 = 0$ (see Lemma 2.6).

(2) Note also that if $g : A \to M$ is a Jordan generalized derivation with an associated linear map d_g, then

(2.2) $$2g(a) = g(1) \circ a + 2d_g(a) \qquad (a \in A).$$

Thus, if $g(1) = 0$ and M is a 2-torsion free A-module, $f = d$ is a Jordan derivation. However, as shown in the following example, $f(1)$ is not zero in general: Let A_2 be the algebra of 2×2 upper triangular matrix on $\mathbb{R}$. Consider $\mathbb{R}$ as an A_2-module under the module operations $am = a_{22}m$ and $ma = ma_{11}$ ($a \in A_2, m \in \mathbb{R}$). Fix $0 \neq \alpha \in \mathbb{R}$ and define $g : A_2 \ltimes \mathbb{R} \longrightarrow A_2 \ltimes \mathbb{R}$ with $g(a,m) = (\alpha a, a_{12} + \alpha m)$. Then g is a Jordan generalized derivation with an associated linear map $d_g(a,m) = (0, a_{12})$ such that $f(I_{22}, 0) \neq 0$.
In Lemma 2.6 we give a situation where $g(1) = 0$ for a Jordan generalized derivation $g : A \to M$.

(3) From the proof of [12, Theorem 2.5], $f(1) \in Z(A \ltimes M)$ when $A \ltimes M$ has a triangular matrix representation and f is a Jordan generalized d-derivation. This was the key of the proof. Indeed, using [12, Theorem

2.3], this implies that the mapping d is a Jordan derivation and $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes M$. However, this does not hold for any trivial extension algebra as shown by [12, Example 2].

Now we give the first fundamental result.

Theorem 2.5. *Every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation if and only if the following conditions hold*:

(1) *Every Jordan generalized derivation $g : A \longrightarrow M$ is a sum of a generalized derivation and an antiderivation*,

(2) *Every linear map $h : M \to A$ such that, for all $a \in A, m, n \in M$, $h(a \circ m) = a \circ h(m)$ and $m \circ h(n) + h(m) \circ n = 0$, is a sum of an A-antihomomorphism δ and an A-homomorphism β which satisfy $m\delta(n) + \delta(m)n = 0 = m\beta(n) + \beta(m)n$ for all $m, n \in M$,*

(3) *Every Jordan generalized derivation f on $A \ltimes M$ of the form $f(a, m) = (f_A(a), h_2(m))$ (i.e., $h_1 = 0$ and $f_M = 0$ in the presentation (2.1)) can be written as the sum of a generalized derivation and an antiderivation.*

Proof. ($\Rightarrow$) We only need to prove (1) and (2).

(1) Let g be a Jordan generalized derivation from A into M. Clearly $(0, g)$ is a Jordan generalized derivation on $A \ltimes M$. Then, by hypothesis, there exist a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and an antiderivation $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(0, g(a)) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $a = 0$, we get $\mathcal{L}(m) + \mathcal{L}'(m) = 0$. Hence $g = D_M + \delta_M$, we are done.

(2) By hypotheses $(h, 0)$ is a Jordan generalized derivation on $A \ltimes M$. Then, by hypothesis, there exist a generalized derivation $(\delta_A + \mathcal{K}', \delta_M + \mathcal{L}')$ and an antiderivation $(D_A + \mathcal{K}, D_M + \mathcal{L})$ such that, for all $a \in A, m \in M$,

$$(h(m), 0) = (D_A(a) + \mathcal{K}(m) + \delta_A(a) + \mathcal{K}'(m), D_M(a) + \mathcal{L}(m) + \delta_M(a) + \mathcal{L}'(m))$$

Take $m = 0$, we get $D_A + \delta_A = 0$ and $D_M + \delta_M = 0$. Therefore, $h = \mathcal{K} + \mathcal{K}'$, as desired.

($\Leftarrow$) Let $f : A \ltimes M \longrightarrow A \ltimes M$ be a Jordan generalized d-derivation. By hypothesis, h_1 is a sum of an A-antihomomorphism δ and an A-homomorphism β. Also, f_M is a sum of a generalized derivation f_1 and an antiderivation f_2. On the other hand, Lemma 2.1 shows that the linear map $(a, m) \longmapsto (f_A(a), h_2(m))$ is a Jordan generalized derivation on $A \ltimes M$. Then, by (3), it can be written as the sum of a generalized derivation Θ and an antiderivation Δ. Then, $f(a, m) = ((\delta(a), f_2(a)) + \Delta(a, m)) + ((\beta(a), f_1(a)) + \Theta(a, m))$, where, using Lemmas 2.1, 2.2 and 2.3, $(a, m) \longmapsto (\delta(a), f_2(a)) + \Delta(a, m)$ is an antiderivation and $(a, m) \longmapsto (\beta(a), f_1(a)) + \Theta(a, m)$ is a generalized derivation. $\square$

From [12, Theorem 2.5], triangular algebras are examples of algebras that satisfy the conditions of Theorem 2.5. Our second main result (Theorem 2.19)

generalizes both [12, Theorem 2.5] and [11, Theorem 3.1]. Before giving this result, we treat another situation which is of independent interest. It gives new other examples of algebras outside of the prime ones on which every Jordan generalized derivation is a generalized derivation (see [12, Lemma 2.6] in which it is shown that on prime algebras every Jordan generalized derivation is a generalized derivation).

First we give the following lemma.

Lemma 2.6. *Assume that A is 2-torsion free. If a linear map $h : A \to A$ satisfies $h(a \circ b) = a \circ h(b)$ and $a \circ h(b) + h(a) \circ b = 0$ for all $a, b \in A$, then $h = 0$.*

Proof. We have, for every two elements $a, b \in A$,

$$0 = a \circ h(b) + h(a) \circ b = 2h(a \circ b).$$

Then, since A is 2-torsion free, $h(a \circ b) = 0$. This implies that $h = 0$.   □

Theorem 2.7. *Assume that A is a 2-torsion free prime algebra. Then every Jordan generalized d-derivation f on $A \ltimes A$ is a generalized d-derivation of the form $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes A$.*

Proof. Let f be a Jordan generalized d-derivation. Using [12, Lemma 2.6], the Jordan generalized derivations f_A and f_M are generalized derivations (here $M = A$). And, by Lemma 2.6, $h_1 = 0$. Now, the relation $h_2(a \circ b) = h_2(a) \circ b + a \circ d_A(b)$ shows that h_2 is a Jordan generalized derivation, and so it is a generalized derivation. Then, $h_2(ab) = h_2(a)b + ad_A(b) = ah_2(b) + d_A(a)b$. Then, $h_2(1) \in Z(A)$. Indeed,

$$h_2(1)a + d_A(a) = h_2(1.a) = h_2(a.1) = ah_2(1) + d_A(a).$$

It remains to prove that $h_2(ab) = f_A(a)b + aS(b)$. We first show that $S(b) = f_A(b) - bh_2(1) + b \circ S(1)$. We have $f_A(a) \circ b + a \circ S(b) = h_2(a \circ b) = a \circ f_A(b) + b \circ S(a)$. Then, $h_2(1) = f_A(1) + S(1)$ and $f_A(1) \circ b + 2S(b) = 2f_A(b) + b \circ S(1)$. Hence, $2(bh_2(1) - b \circ S(1)) = b \circ h_2(1) - 2b \circ S(1) = b \circ (f_A(1) - S(1)) = 2f_A(b) - 2S(b)$. So

$$S(b) = f_A(b) - bh_2(1) + b \circ S(1).$$

Now,

$$
\begin{aligned}
h_2(ab) - f_A(a)b - aS(b) &= bf_A(1a) + S(b)a - h_2(ba) \\
&= bf_A(1)a + bd_A(a) + f_A(b)a - bh_2(1)a \\
&\quad + bS(1)a + S(1)ba - h_2(b)a - bd_A(a) \\
&= f_A(b)a + S(1)ba - h_2(b)a \\
&= f_A(b)a + S(1)ba - (h_2(1)b + d_A(b))a \\
&= f_A(b)a - f_A(1)ba - d_A(b)a \\
&= f_A(b)a - f_A(b)a = 0.
\end{aligned}
$$

Finally, using [12, Proposition 2.1], f is of the form $f(x) = f(1)x + d(x)$ for all $x \in A \ltimes A$. This completes the proof. $\qquad\square$

Now we turn to our second aim. We study Jordan generalized derivations on $A \ltimes M$ when there exists a nontrivial idempotent e in A that satisfies $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). To get the second main result we need some lemmas. First recall that the existence of the above idempotent implies the following nice properties which will be used without explicit mention (see also the remark given before [14, Theorem 2.2]).

Lemma 2.8 ([5, Proposition 2.5])**.** *Consider a non-trivial idempotent e of an algebra A and set $e' = 1 - e$. For every A-bimodule M, the following assertions are equivalent:*

    (1) *For every $m \in M$, $eme' = m$.*
    (2) *For every $m \in M$, $e'm = 0 = me$.*
    (3) *For every $m \in M$, $em = m = me'$.*
    (4) *For every $m \in M$ and $a \in A$, $am = eaem$ and $ma = me'ae'$.*

We start with the following lemma which shows that the first condition of Theorem 2.5 holds when M is a 2-torsion free A-bimodule.

Lemma 2.9. *Assume that the A-bimodule M is 2-torsion free. Let $g : A \to M$ be a Jordan generalized derivation with an associated linear map d_g. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$), then $g(1) = 0$, $g = d_g$ is a Jordan derivation and g is a sum of a derivation and an antiderivation.*

Proof. Using equation (2.2) for g, we get

$$2g(e') = g(e' \circ e') = g(e') \circ e' + e' \circ d_g(e') = g(e') + d_g(e').$$

Then $g(e') = d_g(e')$. Now, replacing a by e' in equation (2.2), we get $g(1) = 0$ and so 2-torsion freeness of M implies that $g = d_g$.

Now let $g = f_1 + f_2$, where f_1 and f_2 are defined by $f_1(a) = g(e'ae)$ and $f_2(a) = g(eae + eae' + e'ae')$ for all $a \in A$. We prove that f_1 is an antiderivation. Let $a, b \in A$. We have

$$\begin{aligned}
f_1(ab) &= g(e'abe) \\
&= g(e'aebe) + g(e'ae'be) \\
&= g((e'ae) \circ (ebe)) + g((e'ae') \circ (e'be)) \\
&= bg(e'ae) + g(e'be)a \\
&= bf_1(a) + f_1(b)a.
\end{aligned}$$

It remains to prove that f_2 is a derivation. To this end, we show that $\Gamma : a \longmapsto g(eae) + g(e'ae')$ is an inner derivation (that is a derivation of the form $\Gamma(a) = ax - xa$ for a fixed $x \in A$) and $d' : a \longmapsto g(eae')$ is a derivation. Note

that, for all $a \in A$,

$$0 = g((eae) \circ (e'ae'))$$
$$= ag(e'ae') + g(eae)a.$$

Hence, replacing a by $e'ae' + e$ in the previous equation, we get

$$g(e'ae') + g(e)a = 0.$$

And replacing a by $eae + e'$ in the same equation, we get

$$ag(e') + g(eae) = 0.$$

Using these relations with the fact that $g(e) = -g(e')$, we get, for every $a \in A$,

$$\Gamma(a) = g(eae) + g(e'ae')$$
$$= -ag(e') - g(e)a$$
$$= ag(e) - g(e)a.$$

Then, Γ is an inner derivation.

Now, for every $a, b \in A$,

$$d'(ab) = g(eabe')$$
$$= g(eae' \circ e'be') + g(eae \circ ebe')$$
$$= g(eae')b + ag(ebe')$$
$$= d'(a)b + ad'(b).$$

This completes the proof. $\qquad\qquad\square$

The following lemma shows that also the second condition of Theorem 2.5 holds when M is a 2-torsion free A-bimodule.

Lemma 2.10. *Let $h : M \to A$ be a linear map such that $h(a \circ m) = a \circ h(m)$ for all $a \in A, m \in M$. If there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ and $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$, then h is a sum of an A-antihomomorphism and an A-homomorphism.*

Proof. First note that $h(m) = h(em) = h(e \circ m) = e \circ h(m) = eh(m) + h(m)e$. Then, $eh(m)e = 0$. Similarly we get $e'h(m)e' = 0$.

This shows that $h = \delta + \beta$ where δ and β are defined by $\delta(m) = e'h(m)e$ and $\beta(m) = eh(m)e'$ (for $m \in M$). We claim that δ is an A-antihomomorphism. Let $a \in A, m \in M$. We have $\delta(am) = e'h(ea \circ m)e = e'h(m)eae + e'h(m)ea = e'h(m)ea = \delta(m)a$. Similarly we prove that $\delta(ma) = a\delta(m)$.

It remains to prove that β is an A-homomorphism. We have

$$\beta(am) = eh(am)e' = eh(ae \circ m)e' = eaeh(m)e' = ea\beta(m).$$

Since $e'a\beta(m) = e'aeh(m)e' = e'(e'ae \circ h(m))e' = e'h(e'ae \circ m)e' = 0$, we get

$$\beta(am) = ea\beta(m) + e'a\beta(m) = a\beta(m).$$

Similarly, we can show that $\beta(ma) = \beta(m)a$. $\qquad\qquad\square$

Lemmas 2.9 and 2.10 show that to get the desired result one should focus on the Jordan generalized d-derivation on $A \ltimes M$ of the form $f(a, m) = (f_A(a), h_2(m))$ (i.e., $h_1 = 0$ and $f_M = 0$ in the presentation (2.1)). In the sequel, we will refer to such a particular kind of Jordan generalized d-derivations as a Jordan generalized d-derivation of type Δ. Recall that in this case, f_A is a Jordan generalized d_A-derivation and h_2 satisfies $h_2(a \circ m) = f_A(a) \circ m + a \circ S(m) = a \circ h_2(m) + d_A(a) \circ m$ for all $a \in A$ and $m \in M$ (Lemma 2.1).

We use the idea of [11] for the decomposition of a Jordan generalized derivation. First, we give the following observation (compare it with [11, Lemma 3.7]).

Lemma 2.11. *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, for a Jordan generalized d_A-derivation f_A on A, the following assertions hold for all $a \in A$:*

    (1) $ef_A(e'ae')e = 0$.
    (2) $e'f_A(eae)e' = 0$.
    (3) $ef_A(eae')e = 0$.
    (4) $e'f_A(eae')e' = 0$.
    (5) $ef_A(e'ae)e = 0$.
    (6) $e'f_A(e'ae)e' = 0$.

Proof. We prove only (1) and (3). The other assertions are proved similarly.

(1) We have $0 = f_A(e \circ (e'ae')) = e \circ f_A(e'ae') + e'ae' \circ d_A(e)$. Then, $ef_A(e'ae')e = 0$.

(3) We have $f_A(eae') = f_A(e' \circ (eae')) = f_A(e') \circ (eae') + e' \circ d_A(eae')$. This implies that $ef_A(eae')e = 0$. $\qquad\square$

Thus, using Lemma 2.11, a Jordan generalized d-derivation f of type Δ can be decomposed as follows

$$(2.3) \qquad\qquad f = J + I + D,$$

where, for all $(a, m) \in A \ltimes M$,

$$(2.4) \qquad\qquad J(a, m) = (ef_A(e'ae)e' + e'f_A(eae')e, 0),$$

$$(2.5) \qquad I(a, m) = (ef_A(eae + e'ae')e' + e'f_A(eae + e'ae')e, 0) \quad \text{and}$$

$$(2.6) \quad D(a, m) = (ef_A(eae)e + ef_A(eae')e' + e'f_A(e'ae)e + e'f_A(e'ae')e', h_2(m)).$$

We treat each map separately. For this we use the following lemma.

Lemma 2.12. *Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, for a Jordan generalized d-derivation f, the following assertions hold for all $a, b \in A$:*

    (1) $ef(e'aebe)e' = ebef(e'ae)e'$.
    (2) $ef(e'ae'be)e' = ef(e'be)e'ae'$.

(3) $e'f(eaebe')e = e'f(ebe')eae.$

(4) $e'f(eae'be')e = e'be'f(eae')e.$

Proof. We prove only the first assertion. The other ones are proved similarly. Since f is a Jordan generalized d-derivation, we have

$$f(e'aebe) = f(e'ae)ebe + ebef(e'ae) + d(ebe)e'ae + e'aed(ebe).$$

Then, $ef(e'aebe)e' = ebef(e'ae)e'.$ $\qquad\square$

Now we prove that J (defined by (2.4)) is an antiderivation. In fact this follows from Lemma 2.3 and the following lemma which is a generalization of [11, Lemma 3.3].

Lemma 2.13. *Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, the mapping $f : A \longrightarrow A$ defined by $f(a) = ef(e'ae)e' + e'f(eae')e$ is an antiderivation.*

Proof. Using Lemma 2.12 and the assumption $eAe'Ae = \{0\} = e'AeAe'$, we get

$$\begin{aligned}
f(ab) &= ef(e'ae)e' + e'f(ebe')e \\
&= ef(e'aebe)e' + ef(e'ae'be)e' + e'f(eaebe')e + e'f(eae'be')e \\
&= ebef(e'ae)e' + ef(e'be)e'ae' + e'f(ebe')eae + e'be'f(eae')e \\
&= bef(e'ae)e' + ef(e'be)e'a \\
&\quad + e'f(eye')ea + ye'f(eae')e \\
&= bf(a) + f(b)a.
\end{aligned}$$

As desired. $\qquad\square$

Recall that a map $\mathcal{H} : A \to A$ is said to be an inner derivation (resp., a generalized inner derivation) if $\mathcal{H}(x) = ax - xa$ for a fixed $a \in A$ (resp., $\mathcal{H}(x) = ax + xb$ for fixed $a, b \in A$). In [11, Lemma 3.5], it is proved that the first component of our I is an inner derivation when f_A is a Jordan derivation. Here we prove it is an inner generalized derivation when f_A is a Jordan generalized derivation. This helps to show that I is an inner generalized derivation on $A \ltimes M$ (see Lemma 2.15).

Lemma 2.14. *Suppose there is a nontrivial idempotent e such that $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, the mapping $I_A : A \longrightarrow A$ defined by $I_A(a) = ef(eae + e'ae')e' + e'f(eae + e'ae')e$ is an inner generalized derivation. Namely, $I_A(a) = aT - T'a$ for every $a \in A$, where $T = ef(e)e' - e'f(e)e$ and $T' = ed(e)e' - e'f(e)e.$*

Proof. For all $a \in A$, we have

$$\begin{aligned}
0 &= f((eae)(e'ae') + (e'ae')(eae)) \\
&= eaed(e'ae') + f(eae)e'ae' + e'ae'f(eae) + d(e'ae')eae.
\end{aligned}$$

Then, for every $a \in A$,

(2.7) $$eaed(e'ae')e' + ef(eae)e'ae' = 0$$

and

(2.8) $$e'ae'f(eae)e + e'd(e'ae')eae = 0.$$

For any $a \in A$ replace a by $a + e$ in (2.7). This gives

$$eaed(e'ae')e' + ed(e'ae')e' + ef(eae)e'ae' + ef(e)e'ae' = 0.$$

Hence, replacing a by $e'ae'$ in the previous equation, we get

$$ed(e'ae')e' + ef(e)e'ae' = 0.$$

And also we obtain

$$ef(e'ae')e' + ed(e)e'ae' = 0.$$

Taking $a = e'$, we get

$$ed(e')e' + ef(e)e' = 0.$$

Now, for any $a \in A$, replacing a by $eae + e'$ in (2.7), we obtain

$$eaed(e')e' + ef(eae)e' = 0.$$

Using these relations we obtain

$$-eaef(e)e' + ef(eae)e' = 0.$$

Similarly, we can obtain from relation (2.8) that

$$e'ae'f(e)e + e'f(e'ae')e = 0 \quad \text{and} \quad -e'f(e)eae + e'f(eae)e = 0.$$

These relations and the assumption $eAe'Ae = \{0\} = e'AeAe'$ imply that

$$\begin{aligned}
I_A(a) &= ef(eae)e' + ef(e'ae')e' + e'f(eae)e + e'f(e'ae')e \\
&= eaef(e)e' - ed(e)e'ae' + e'f(e)eae - e'ae'f(e)e \\
&= aef(e)e' - ed(e)e'a + e'f(e)ea - ae'f(e)e \\
&= a(ef(e)e' - e'f(e)e) - (ed(e)e' - e'f(e)e)a \\
&= aT - T'a.
\end{aligned}$$

As desired. □

As a consequence of the lemma above, I is an inner generalized derivation on $A \ltimes M$. Namely we have the following lemma.

Lemma 2.15. *Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, and $eAe'Ae = \{0\} = e'AeAe'$, where $e' = 1 - e$. Then, the mapping I (defined by (2.5)) is an inner generalized derivation. Namely, $I(a, m) = (a, m)(T, 0) - (T', 0)(a, m)$ for every $(a, m) \in A \ltimes M$, where $T = ef(e)e' - e'f(e)e$ and $T' = ed(e)e' - e'f(e)e$.*

Now we prove that D is a generalized derivation. For this, we use the following two lemmas. The first one can be of independent interest. It presents some properties of a Jordan generalized d-derivation of type Δ.

Recall that the left annihilator, $l.Ann_A(M)$, of M is the set of all elements r in A such that $rM = 0$. Similarly the right annihilator, $r.Ann_A(M)$, of M is defined.

Lemma 2.16. *Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). Let f be a Jordan generalized d-derivation of type Δ. Then the following assertions hold:*

(1) $h_2(am) = f_A(a)m + aS(m)$ *and* $h_2(ma) = mf_A(a) + S(m)a$ *for all $a \in A$ and $m \in M$.*
(2) $mf_A(e') = f_A(e)m$ *for all $m \in M$.*
(3) $f_A(ab) - f_A(a)b - ad_A(b) \in r.Ann_A(M) \cap l.Ann_A(M)$.

Proof. (1) We need only to prove the first equality. The second one follows immediately. First we prove that $d_A(e)m = 0 = md_A(e)$.

We have $f_A(2e) = f_A(e \circ e) = f_A(e)e + ef_A(e) + ed_A(e) + d_A(e)e$. Then, $ed_A(e)e = 0$ (since A is 2-torsion free). Then, $d_A(e)m = ed_A(e)em = 0$. On the other hand, since $h_2(m) = h_2(e \circ m) = h_2(m) + d_A(e)m + md_A(e)$, we get

$$(2.9) \qquad\qquad md_A(e) = -d_A(e)m = 0.$$

Now,

$$
\begin{aligned}
2h_2(am) &= h_2((ae + ea)m) \\
&= h_2((ae + ea) \circ m) \\
&= f_A(e \circ a) \circ m + (ae + ea) \circ S(m) \\
&= (f_A(a)e + ef_A(a) + ad_A(e) + d_A(e)a) \circ m + 2aS(m) \\
&= 2f_A(a)m + 2aS(m).
\end{aligned}
$$

Therefore, the 2-torsion freeness of M implies that $h_2(am) = f_A(a)m + aS(m)$, as desired.

(2) For every $m \in M$, $mf_A(e') + S(m)e' = h_2(me') = h_2(em) = f_A(e)m + eS(m)$. Then, $mf_A(e') = f_A(e)m$.

(3) We have $h_2(am) = f_A(a)m + aS(m)$. Then, $h_2(m) = f_A(e)m + S(m)$ which means that $S(m) = h_2(m) - f_A(e)m$, and so

$$
\begin{aligned}
S(am) &= h_2(am) - f_A(e)am \\
&= f_A(a)m + aS(m) - f_A(e)am \\
&= (f_A(a) - f_A(e)a)m + aS(m).
\end{aligned}
$$

Then,

$$
\begin{aligned}
h_2(abm) &= f_A(a)bm + aS(bm) \\
&= f_A(a)bm + a(f_A(b) - f_A(e)b)m + abS(m).
\end{aligned}
$$

On the other hand, $h_2(abm) = f_A(ab)m + abS(m)$. Then,

(2.10) $$f_A(ab)m = (f_A(a)b + a(f_A(b) - f_A(e)b))m.$$

We prove that $(f_A(b) - bf_A(e))m = d_A(b)m$.

In (2.10), for $a = e$ we get

(2.11) $$f_A(eb)m = f_A(b)m.$$

And, for $b = e$, we get

(2.12) $$f_A(ae)m = f_A(a)m.$$

Since f_A is a Jordan generalized d_A-derivation,

$$f_A(eb + be) = f_A(e)b + bf_A(e) + ed_A(e) + d_A(e)e.$$

Then, using (2) and equalities (2.11) and (2.12), we get

$$2f_A(b)m = bf_A(e)m + bf_A(e)m + 2d_A(b)m.$$

Then, $(f_A(b) - bf_A(e))m = d_A(b)m$. Hence, (2.10) becomes

(2.13) $$(f_A(ab) - f_A(a)b - ad_A(b))m = 0.$$

The same argument as above, using $h_2(mb) = mf_A(b) + S(m)b$ and taking e' instead of e, shows that

(2.14) $$m(f_A(ba) - bf_A(a) - d_A(b)a) = 0.$$

Finally, since f_A is a Jordan generalized d_A-derivation,

$$f_A(ab) - f_A(a)b - ad_A(b) = -f_A(ba) + bf_A(a) + d_A(b)a.$$

This ends the proof. $\square$

Lemma 2.17. *Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$ (where $e' = 1 - e$). Let f be a Jordan generalized d-derivation of type Δ. If f_A is a generalized d_A-derivation, then $h_2(ma) = h_2(m)a + md_A(a)$ for all $a \in A$, $m \in M$.*

Consequently, the Jordan generalized d-derivation f is a generalized d-derivation.

Proof. For all $a \in A$ and $m \in M$, we have

$$h_2(ma) - h_2(m)a - md_A(a) = ah_2(m) + d_A(a)m - h_2(am).$$

Then, using the hypothesis and (1) of Lemma 2.16, we get

$$h_2(ma) - h_2(m)a - md_A(a)$$
$$= a(f_A(e)m + eS(m)) + d_A(a)m - f_A(a)m - aS(m)$$
$$= af_A(e)m + d_A(a)m - (f_A(e'a) + f_A(ea))m$$
$$= af_A(e)m + d_A(a)m - (f_A(e')e'a + e'd_A(e'a)$$
$$\quad + f_A(e)a + ed_A(a))m.$$

Then, using the second assertion of Lemma 2.16, we get

$$h_2(ma) - h_2(m)a - md_A(a) = af_A(e)m - amf_A(e')$$
$$= af_A(e)m - af_A(e)m = 0.$$

As desired. $\qquad\square$

Lemma 2.18. *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1-e$. If $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then D (defined by (2.6)) is a generalized derivation.*

Proof. One can show easily that D is a Jordan generalized derivation of type Δ. Then, by Lemmas 2.16 and 2.17, it suffices to prove that the map $f' : A \to A$, defined by $f'(a) = ef_A(eae)e + ef_A(eae')e' + e'f_A(e'ae)e + e'f_A(e'ae')e'$ for all $a \in A$, is a generalized derivation.

By the hypothesis and the assertion (3) of Lemma 2.16, we have

$$ef_A(eaebe)e = ef_A(eae)ebe + eaed_A(ebe)e,$$
$$ef_A(eae'be)e = ef_A(eae')e'be + eae'd_A(e'be)e = 0 \text{ (since } eAe'Ae = \{0\}),$$
$$e'f_A(e'ae'be')e' = e'f_A(e'ae')e'be' + e'ae'd_A(e'be')e', \quad \text{and}$$
$$e'f_A(e'aebe')e' = e'f_A(e'ae)ebe' + e'aed_A(ebe')e' = 0 \text{ (since } e'AeAe' = \{0\}).$$

And, since f_A is a Jordan generalized d-derivation, we get, as done in Lemma 2.12, the following equalities:

$$ef_A(eaebe')e' = ef_A(eae)ebe' + eaed_A(ebe')e',$$
$$ef_A(eae'be')e' = ef_A(eae')e'be' + eae'd_A(e'be')e',$$
$$e'f_A(e'aebe)e = e'f_A(e'ae)ebe + e'aed_A(ebe)e, \quad \text{and}$$
$$e'f_A(e'ae'be)e = e'f_A(e'ae')e'be + e'ae'd_A(e'be)e.$$

These relations with the assumption $eAe'Ae = \{0\} = e'AeAe'$ give us that $f'(ab) = f'(a)b + ad'(b)$ for all $a, b \in A$, where $d'(b) = ed_A(ebe)e + e'd_A(e'be')e' + ed_A(ebe')e' + e'd_A(e'be)e'$. That is, f' is a generalized derivation. $\qquad\square$

Finally, combining the above results we get our second main result which generalizes both [12, Theorem 2.5] and [11, Theorem 3.1]. Notice that, if $A \ltimes M$ has a triangular matrix representation, then using [5, Proposition 2.1], the antiderivation f_1 in Lemma 2.9, the antihomomorphism δ in Lemma 2.10 and the antiderivation J (defined in (2.5)) are zero.

Theorem 2.19. *Assume that the algebra A and the A-bimodule M are 2-torsion free. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$. If $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation.*

Remark 2.20. Assume that the algebra A and the A-bimodule M are 2-torsion free and there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$. The following two situations present two particular cases of the trivial extension algebras which satisfy conditions of Theorem 2.19.

(1) When $r.Ann_A(M) \cap l.Ann_A(M) = \{0\}$. In fact, all of the sets

$$eAe'Ae, e'AeAe', e'r.Ann_A(M)e' \text{ and } el.Ann_A(M)e$$

are in $r.Ann_A(M) \cap l.Ann_A(M)$.

(2) When M is a loyal $(eAe, e'Ae')$-bimodule. Recall that an (A, B)-bimodule M, where A and B are algebras, is said to be loyal if, for every $(a, b) \in A \times B$, $aMb = \{0\}$ implies $a = 0$ or $b = 0$ (see for instance [4, Definition 2.1]).

Lemmas 2.16 and 2.17 show that if we assume that every Jordan generalized derivation on A is a generalized derivation (as for the case of prime algebras), we get a new other context where Jordan generalized derivations on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation. In fact, we show that under this condition we do not need the condition that A is 2-torsion free.

Theorem 2.21. *Assume that the A-bimodule M is 2-torsion free and that every Jordan generalized derivation on A is a generalized derivation. Suppose there exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, then every Jordan generalized derivation on $A \ltimes M$ can be written as the sum of a generalized derivation and an antiderivation.*

Proof. Lemmas 2.9 and 2.10 show that we only need to prove the result for the mapping $(a, m) \longmapsto (f_A(a), h_2(m))$. From the prove of Lemmas 2.16 and 2.17, we can deduce that we need only to prove that $h_2(am) = f_A(a)m + aS(m)$ for all $a \in A$ and $m \in M$.

Note that $d_A(e)m = md_A(e') = 0$. Indeed, by the hypothesis, f_A is a generalized derivation and, by [12, Proposition 2.1], d_A is a derivation. Then, for all $m \in M$, $d_A(e)m = d_A(e)em + ed_A(e)m = d_A(e)m + d_A(e)m$. Then, $d_A(e)m = 0$. Similarly, we prove that $md_A(e') = 0$. Now,

$$\begin{aligned}
h_2(am) &= h_2(ae \circ m) \\
&= f_A(ae) \circ m + ae \circ S(m) \\
&= f_A(a)m + ad_A(e) \circ m + aS(m) \\
&= f_A(a)m + aS(m).
\end{aligned}$$

As desired. $\square$

Let $M_n(\mathbb{R})$ (resp., $T_n(\mathbb{R})$) denotes the algebra of all matrix (resp., of all upper triangular matrix) on $\mathbb{R}$. As a consequence of Theorem 2.21 and [12, Theorem 2.5] we get the following result.

Corollary 2.22. *Let M be a 2-torsion free $M_n(\mathbb{R})$-bimodule (resp., $T_n(\mathbb{R})$ - bimodule). Suppose there exists a nontrivial idempotent e in $M_n(\mathbb{R})$ (resp., $e \in T_n(\mathbb{R})$) such that $eme' = m$ for all $m \in M$. Then every Jordan generalized derivation on $M_n(\mathbb{R}) \ltimes M$ (resp., $T_n(\mathbb{R}) \ltimes M$) can be written as the sum of a generalized derivation and an antiderivation.*

3. f-generalized derivations

In a recent paper [2], Benkovič introduced the notion of f-derivations which unifies several kind of derivations including the classical derivations as follows: Consider a fixed nonzero multilinear polynomial f in noncommuting indeterminates x_i over $\mathcal{R}$:

$$(3.1) \qquad f(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_\pi x_{\pi(1)} x_{\pi(2)} \ldots x_{\pi(n)} \quad (\alpha_\pi \in \mathcal{R}),$$

where S_n denotes the symmetric group of order an integer $n \geq 2$. An $\mathcal{R}$-linear map $\mathcal{D} : A \longrightarrow M$ is called an f-derivation if it satisfies

$$(3.2) \qquad \mathcal{D}(f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, \mathcal{D}(x_i), x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in A$. Thus, a derivation is an f-derivation for the polynomial $f(x_1, x_2) = x_1 x_2$, a Jordan derivation is an f-derivation for the polynomial $f(x_1, x_2) = x_1 \circ x_2 = x_1 x_2 + x_2 x_1$, a Jordan triple derivation (see for example [10]) is an f-derivation for the polynomial $f(x_1, x_2, x_3) = x_1 x_2 x_3 + x_3 x_2 x_1$, a Lie derivation (see [8]) is an f-derivation for the polynomial $f(x_1, x_2) = [x_1, x_2] = x_1 x_2 - x_1 x_2$, and a Lie triple derivation (see for example [3] and [13]) is an f-derivation for the polynomial $f(x_1, x_2, x_3) = [[x_1, x_2], x_3]$.

In [2, Theorem 1.3], Benkovič proved (under some conditions) that every f-derivation is a Jordan derivation. Then, he used this result to show that (under some conditions) that every f-derivation on a triangular algebra is a derivation [2, Theorem 1.1]. Then, naturally one can ask whether there exists a "generalized" counterpart of Benkovič's results. In this section we answer this natural question positively.

In what follows, we consider a fixed nonzero multilinear polynomial f as defined in 3.1. An $\mathcal{R}$-linear map $F : A \longrightarrow M$ is called an f-*generalized d-derivation* (or simply, an f-generalized derivation), where $d : A \longrightarrow M$ is an $\mathcal{R}$-linear map, if

$$F(f(x_1, \ldots, x_n)) = f(F(x_1), x_2, \ldots, x_n)$$
$$+ \sum_{i=2}^{n} f(x_1, \ldots, x_{i-1}, d(x_i), x_{i+1}, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in A$. Then, obviously every f-derivation F is an f-generalized F-derivation. Also, note that the f-generalized F-derivation unifies various kind of generalized derivations including the generalized derivations and the

Jordan generalized derivations (see, for instance, [15] for the notion of d-Lie derivations (a generalized counterpart of Lie derivations), [13] for the notion of generalized Lie triple derivations, and [16] for the notion of generalized Jordan triple derivations).

We say that an element $r \in \mathcal{R}$ is M-regular if, for every $m \in M$, $rm = 0$ implies that $m = 0$. Let

$$\alpha = \sum_{\pi \in S_n} \alpha_\pi \in \mathcal{R}$$

be the sum of coefficients of the polynomial f from (3.1). We start with the generalized counterpart of [2, Theorem 1.3] which needs a similar argument with some suitable modifications.

Theorem 3.1. *Let $F : A \longrightarrow M$ be an f-generalized derivation, with $\alpha \neq 0$. If M is $(n-1)$-torsion free and α is M-regular, then F is a Jordan generalized derivation.*

Consequently, as done in [2], Theorem 3.1 together with Theorems 2.7 and 2.19 lead to a characterization of a particular case of f-generalized derivation on some trivial extension algebras.

Corollary 3.2. *Assume that A is a 2-torsion free prime algebra. Let $F : A \ltimes A \longrightarrow A \ltimes A$ be an f-generalized derivation, where with $\alpha \neq 0$. If A is $2(n-1)$-torsion free and α is A-regular, then F is a generalized derivation.*

Note that the generalized derivation F has the form $F(x) = F(1)x + d(x)$ for all $x \in A \ltimes A$ (by [12, Proposition 2.1]).

Corollary 3.3. *Let $F : A \ltimes M \longrightarrow A \ltimes M$ be an f-generalized derivation, where $f \in \mathcal{R}\langle x_1, x_2, \ldots \rangle$ is a multilinear polynomial of degree $n \geq 2$ with $\alpha \neq 0$. Consider the following conditions*:

   (i) *A and M are both $2(n-1)$-torsion free.*
   (ii) *α is A-regular and M-regular.*
   (iii) *There exists a nontrivial idempotent e in A such that $eme' = m$ for all $m \in M$, where $e' = 1 - e$, and $eAe'Ae = \{0\} = e'AeAe'$ and $e'r.Ann_A(M)e' = \{0\} = el.Ann_A(M)e$.*

If (i), (ii) and (iii) hold, then F can be written as the sum of a generalized derivation and an antiderivation.

Also as a generalization of [2, Theorem 1.1], we obtain the following result which characterizes a particular case of f-generalized derivation on triangular algebras.

Corollary 3.4. *Let A and B be unital algebras over a 2-torsion free commutative ring R, and M be a unital (A, B)-bimodule that is faithful as both a left A-module and a right B-module. Let $\mathcal{A} = Tri(A, M, B)$ be the triangular algebra. Let $F : \mathcal{A} \longrightarrow \mathcal{A}$ be an f-generalized derivation, where with $\alpha \neq 0$. If $\mathcal{A}$ is $2(n-1)$-torsion free and α is $\mathcal{A}$-regular, then F is a generalized derivation of the form $F(x) = F(1)x + d(x)$ for all $x \in \mathcal{A}$.*

It is worth noting that there are interesting f-generalized derivations with $\alpha = 0$ which deserve investigating. However, even in the case of f-derivations the situation is much more unpredictable as mentioned in [2, Problem 1.2]. Thus the question for this case remains an open interesting question.

References

[1] K. I. Beidar and Y. Fong, *On additive isomorphisms of prime rings preserving polynomials*, J. Algebra **217** (1999), no. 2, 650–667.

[2] D. Benkovič, *A note on f-derivations of triangular algebras*, Aequationes Math. **89** (2015), no. 4, 1207–1211.

[3] ———, *Lie triple derivations of unital algebras with idempotents*, Linear Multilinear Algebra **63** (2015), no. 1, 141–165.

[4] D. Benkovič and D. Eremita, *Commuting traces and commutativity preserving maps on triangular algebras*, J. Algebra **280** (2004), no. 2, 797–824.

[5] D. Bennis and B. Fahid, *Derivations and the first cohomology group of trivial extension algebras*, Mediterr. J. Math. **14** (2017), no. 4, Art. 150, 18 pp.

[6] G. F. Birkenmeier, J. K. Park, and S. T. Rizvi, *Extensions of Rings and Modules*, Birkhäuser/Springer, New York, 2013.

[7] W.-S. Cheung, *Mappings on triangular algebras*, ProQuest LLC, Ann Arbor, MI, 2000.

[8] ———, *Lie derivations of triangular algebras*, Linear Multilinear Algebra **51** (2003), no. 3, 299–310.

[9] H. R. Ebrahimi Vishki, M. Mirzavaziri, and F. Moafian, *Jordan higher derivations on trivial extension algebras*, Commun. Korean Math. Soc. **31** (2016), no. 2, 247–259.

[10] M. Fošner and D. Iliševic, *On Jordan triple derivations and related mappings*, Mediterr. J. Math. **5** (2008), no. 4, 415–427.

[11] H. Ghahramani, *Jordan derivations on trivial extensions*, Bull. Iranian Math. Soc. **39** (2013), no. 4, 635–645.

[12] Y. Li and D. Benkonič, *Jordan generalized derivations on triangular algebras*, Linear Multilinear Algebra **59** (2011), no. 8, 841–849.

[13] H. Li and Y. Wang, *Generalized Lie triple derivations*, Linear Multilinear Algebra **59** (2011), no. 3, 237–247.

[14] A. H. Mokhtari, F. Moafian, and H. R. Ebrahimi Vishki, *Lie derivations on trivial extension algebras*, Ann. Math. Sil. **31** (2017), no. 1, 141–153.

[15] A. Nakajima, *On generalized higher derivations*, Turkish J. Math. **24** (2000), no. 3, 295–311.

[16] J. Wu and S. Lu, *Generalized Jordan derivations on prime rings and standard operator algebras*, Taiwanese J. Math. **7** (2003), no. 4, 605–613.

[17] Y. Zhang, *Weak amenability of module extensions of Banach algebras*, Trans. Amer. Math. Soc. **354** (2002), no. 10, 4131–4151.

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